

Fracture Mechanics of Concrete: Concepts, Models and Determination of Material Properties

Reported by ACI Committee 446, Fracture Mechanics*

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SYNOPSIS

In the first of its series of four state-of-the-art reports under preparation, the Committee describes the basic concepts of fracture mechanics of concrete, the existing theoretical models, and the methods for determining the material fracture parameters. **Chapter 1** offers five reasons for introducing fracture mechanics into certain aspects of design of concrete structures, including some code provisions: (1) a theoretical energy argument; (2) the need to achieve objectivity of finite element solutions, i.e., eliminate spurious mesh sensitivity; (3) the progressive (propagating) nature of failure, implied whenever the load-deflection diagram lacks a yield plateau; (4) the need to rationally predict ductility and energy absorption capability; and most importantly, (5) the effect of structure size on the nominal strength (i.e., nominal stress at maximum or ultimate load) as well as on ductility and energy absorption capability. The size effect is due to stored energy release into the fracture front, and is not governed by Weibull-type statistical theory. Experimental evidence on the existence of the size effect, hitherto ignored in design practice and code provisions, is documented.

Chapter 2 gives a brief review of the necessary basic results of linear elastic fracture mechanics (LEFM). In concrete, departures from this classical theory are caused by the existence of distributed cracking (or damage) in a progressively softening fracture process zone which surrounds the tip of a continuous crack. In **Chapter 3** nonlinear fracture models characterizing the softening stress-displacement or stress-strain relations (such as those of Hillerborg's fictitious crack model, crack band model, nonlocal strain-softening models, etc.) are described and random particle simulation of aggregate microstructure is discussed. The principles of implementation of these models in finite element programs are also outlined. **Chapter 4** presents simpler nonlinear fracture models which represent adaptations of linear elastic fracture mechanics, such as Jenq and Shah's model and the R-curve, along with determination of geometry-dependent R-curves from the size effect law proposed by Bazant. This law, describing the approximate dependence of the nominal stress at maximum load on structure size, is discussed in **Chapter 5**, and structural response is characterized by the brittleness number.

Chapter 6 presents in considerable detail the current methods for experimental and analytical determination of material fracture parameters, including the quasi-LEFM methods, RILEM (work-of-fracture) method, the Jenq-Shah and Karihaloo-Nallathambi methods, and the size-effect method. Experimental determination of the characteristic length for nonlocal continuum models and the strain-softening properties is then examined, and material parameters for modes II and III, shear fractures and mixed mode fracture are also discussed. **Chapter 7** then proceeds to describe various influencing factors, such as the loading rate, humidity and temperature, as well as the effect of cyclic loading. **Chapter 8** is devoted to the effect of reinforcing bars and their bond slip on fracture propagation, and to fracture of fiber-reinforced concrete. **Chapter 9** deals with more theoretical problems of modeling systems of interacting cracks. Attention is focused on systems of parallel growing cracks. Their stability decides the spacing and width of the cracks from the mechanics viewpoint.

It is concluded that, after a decade of rapid progress in research, the time appears ripe for introducing fracture mechanics into design practice. This should not only bring about more uniform safety margins, thus improving safety and economy of design, but also pave the way for safer and more efficient use of high-performance concretes and permit design extrapolations beyond the range of previous experiments and design.

KEYWORDS: Brittleness, concrete, concrete structures, crack spacing and width, cracking, damage mechanics, design codes, ductility, failure, fiber-reinforced concrete, nonlocal continuum models, reinforced concrete, size effect, strain softening, structural design, testing methods, ultimate loads.

Introduction

Concrete structures are full of cracks. Failure of concrete structures typically involves stable growth of large cracking zones and the formation of large fractures before the maximum load is reached. Yet design is not based on fracture mechanics, even though the basic fracture mechanics theory has been available since the middle of this century. So why has not fracture mechanics been introduced into concrete design? Have concrete engineers been guilty of ignorance? Not at all. The forms of fracture mechanics which were available until recently were applicable only to homogeneous brittle materials such as glass, or to homogeneous brittle-ductile metals. The question of applicability of these classical theories to concrete was explored long ago – the idea of using the stress intensity factor appeared already in the early 1950's (e.g., Bresler and Wollack, 1952) and serious investigations started in the 1960's (e.g., Kaplan, 1961, and others). But the answer was, at that time, negative (e.g., Kesler, Naus and Lott, 1971). As is now understood, the reason was that in concrete structures one must take into account strain-softening due to distributed cracking, localization of cracking into larger fractures prior to failure, and bridging stresses at the fracture front. A form of fracture mechanics that can be applied to such structures has been developed only during the last decade.

Concrete design has already seen two revolutions. The first, which made the technology of concrete structures possible, was the development of the elastic no-tension analysis during 1900-1930. The second revolution, based on a theory conceived chiefly during the 1930's, was the introduction of plastic limit analysis, which occurred during 1940-1970. There are good reasons to believe that the introduction of fracture mechanics into the design of concrete structures, both reinforced and unreinforced, might be the third major revolution. The theory, formulated mostly during the last dozen years, finally appears to be ripe.

Fracture researchers have at the present no doubt that the introduction of fracture mechanics into the design criteria for all brittle failures of reinforced concrete structures (such as diagonal shear, punching shear, torsion or pull out, or for concrete dams), can bring about significant benefits. It will make it possible to achieve more uniform safety margins, especially for structures of different sizes. This, in turn, will improve economy as well as structural reliability. It will make it possible to introduce new designs and utilize new concrete materials. Fracture mechanics will be particularly important for high strength concrete structures, fiber-reinforced concrete structures, concrete structures of unusually large sizes, and for prestressed structures. The application of fracture mechanics is most urgent for structures such as concrete dams and nuclear reactor vessels or containments, for which the safety concerns are particularly high and the consequences of a potential disaster enormous.

Surveys of concrete fracture mechanics have recently been prepared by various committees (Wittmann, 1983, and Elfgren, 1989). However, due to the rapidly advancing research, the contents of the present state-of-the-art report are quite different. A unified, systematic presentation, rather than a compilation of all the contributions by various authors, is attempted in the present state-of-the-art report. The report is aimed primarily at researchers, not necessarily specialists in fracture mechanics. However, it should also be of interest to design

engineers because it describes a theory that is likely to profoundly influence the design practice in the near future. Subsequent reports dealing with applications in design, finite element analysis of fracture, and dynamic fracture analysis, are in preparation by ACI Committee 446.



Chapter 1. WHY FRACTURE MECHANICS?

Fracture mechanics, in a broad sense, is a failure theory which (1) uses energy criteria, possibly in conjunction with strength criteria, and (2) which takes into account failure propagation through the structure.

1.1 Five Reasons for Fracture Mechanics Approach

Since concrete structures have been designed and successfully built according to codes which totally ignore fracture mechanics theory, it might seem unnecessary to change the current practice. Nevertheless, there are five compelling reasons for doing so.

Reason 1: Energy Required for Crack Formation

From the strictly physical viewpoint, it must be recognized that while crack initiation may depend on stress, the actual formation of cracks requires a certain energy – the fracture energy – which represents the surface energy of a solid. Hence, energy criteria should be used. This argument might suffice to a physicist but not a designer. But there are other reasons.

Reason 2: Objectivity of Calculations

Any physical theory must be objective in the sense that the result of calculations made with it must not depend on subjective aspects such as the choice of coordinates, the choice of mesh, etc. If a theory is found to be unobjective, it must be rejected. There is no need to even compare it to experiments. Objectivity comes ahead of experimental verification.

A powerful approach to finite element analysis of concrete cracking is the concept of smeared cracking, introduced by Rashid (1968). According to this approach, the stress in a finite element is limited by the tensile strength of the material, f'_t , and after reaching this strength limit, the stress in the finite element must decrease. As initially practiced, the stress was assumed to decrease suddenly to zero, in a vertical drop; but soon it was realized that better and more realistic results are usually obtained if the stress is reduced gradually, i.e., the material is assumed to exhibit strain-softening (Scanlon, 1971; Lin and Scordelis, 1975); see Fig. 1.1a. The concept of strain-softening, though, proved to be a mixed blessing. After strain-softening had been implemented in large finite element programs and widely applied, it was discovered that the convergence properties are incorrect and the calculation results are not objective with regard to the analyst's choice of the mesh, i.e., the results significantly change if the mesh is refined (Bazant, 1976, 1982; Bazant and Cedolin, 1979, 1980, 1983; Bazant and Oh, 1983a; Darwin, 1985; Rots, Nauta, Kusters and Blaauwendraad, 1985). Similar problems are encountered when cracking is modeled as discrete interelement cracks,

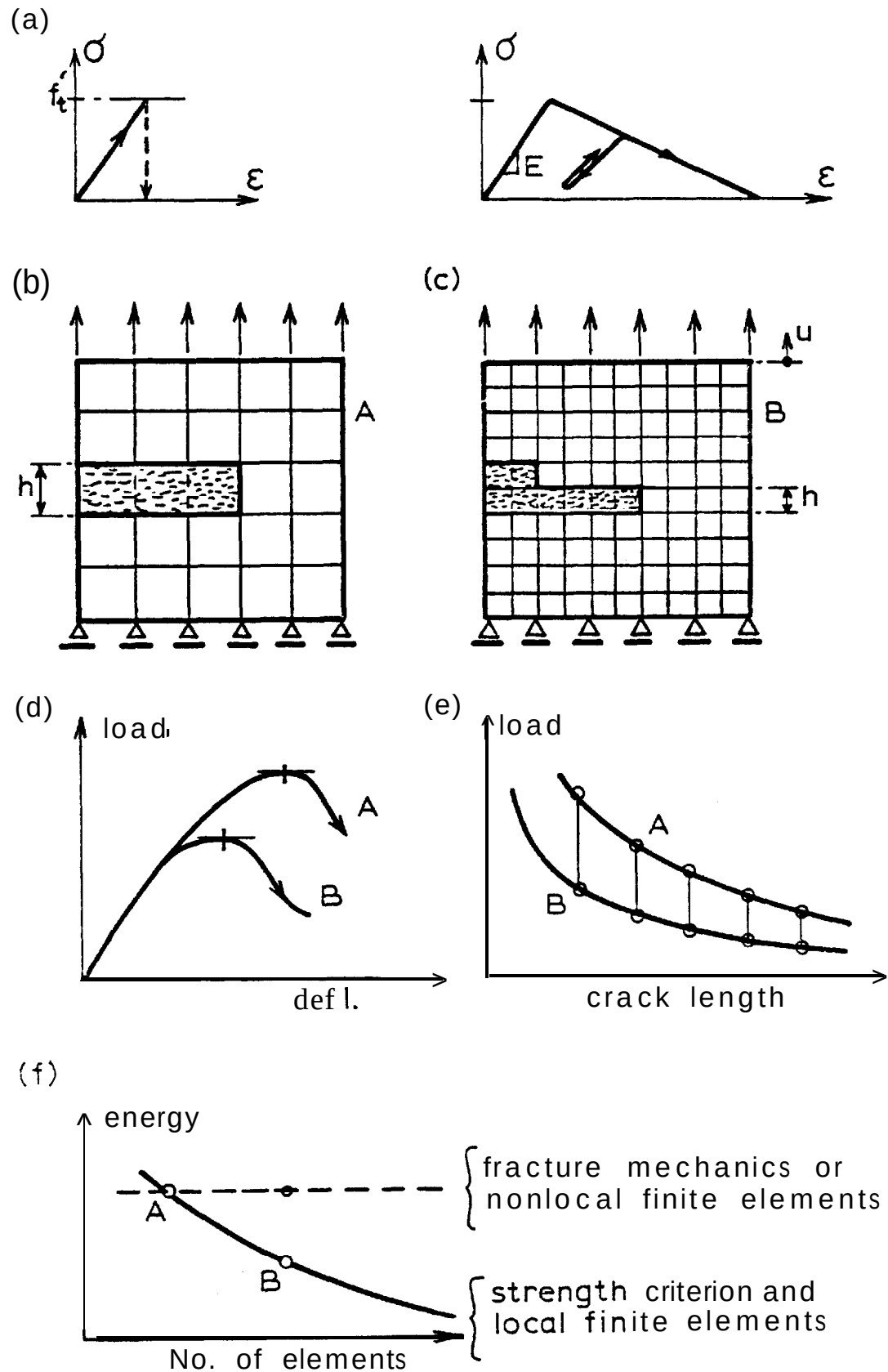


Fig. 1.1 Spurious Mesh Sensitivity

based on the strength concept (this approach was introduced into finite element analysis by Clough, 1962, and by Ngo and Scordelis, 1967).

The problem of spurious mesh sensitivity can be illustrated, for example, by the rectangular panel in Fig. 1.1b and c, which is subjected to a uniform vertical displacement at the top boundary. A small region near the center of the left side is assumed to have a slightly smaller strength than the rest of the panel, and consequently a smeared crack band starts growing from left to right. The solution is obtained by incremental loading with two finite element meshes of very different mesh sizes as shown. By stability checks it is found that the cracking must always localize into a band of single element width at the cracking front (Fig. 1.1b,c). The typical numerical results for this, as well as various other problems are illustrated in Fig. 1.1d,e,f. In the load-deflection diagram (Fig. 1.1d), it is seen that the peak load as well as the post-peak softening is strongly dependent on the mesh size, being roughly proportional to $h^{-1/2}$ where h is the element size. Plotting the load (reaction) versus the length of the crack band, large differences are again found (Fig. 1.1e).

The energy which is dissipated due to cracking decreases with the refinement of the finite element mesh (Fig. 1.1f) and converges to 0 as $h \rightarrow 0$.

The foregoing unobjectivity is physically unacceptable. The only way to avoid it is some form of fracture mechanics. By specifying the energy dissipated by cracking per unit length of the crack or the crack band, the overall energy dissipation is forced to be independent of the element subdivision (the horizontal dashed line in Fig. 1.1f), and so is the maximum load.

Reason 3: Lack of Yield Plateau

Based on load-deflection diagrams, one may distinguish two basic types of structural failure: plastic and brittle. The typical characteristic of plastic failure is that the structure develops a single-degree-of-freedom mechanism such that failure in various parts of the structure proceeds simultaneously, in proportion to a single parameter. Such failures are manifested by the existence of a long yield plateau on the load-deflection diagram (Fig. 1.2a). If the load-deflection diagram does not have such a plateau, the failure is not plastic but brittle (or brittle-ductile) (Fig. 1.2b). If there are no significant geometric effects such as the $P-\Delta$ effect in buckling, the absence of a plateau implies the existence of softening in the material due to fracture, cracking or other damage; it implies that the failure process cannot develop a single degree-of-freedom mechanism but consists of propagation of the failure zone throughout the structure. So the failure is non-simultaneous and propagating.

To illustrate this behavior, consider the punching shear failure of a slab (Fig. 1.3). The typical (approximate) distributions of tensile stress σ along the failure surface are drawn in the figure. If the material is plastic, the cross section gradually plasticizes until all its points are at the yield limit. However, if the material exhibits softening, then the stress peak moves across the failure zone, leaving a reduced stress (softening) in its wake. The stress reduction is mild only if the structure is small, in which case the plastic limit analysis is not so far off. If the structure is large, however, the stress profile develops a steep stress drop behind the peak-stress point, and therefore the limit analysis solutions grossly over-estimate the failure

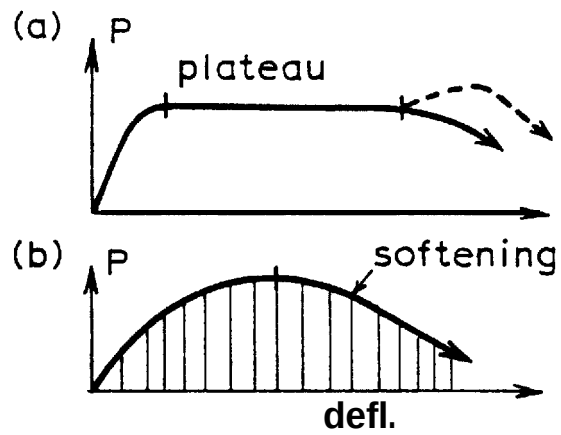


Fig.1.2 Load Deflection Diagram of Ductile and Brittle Structures

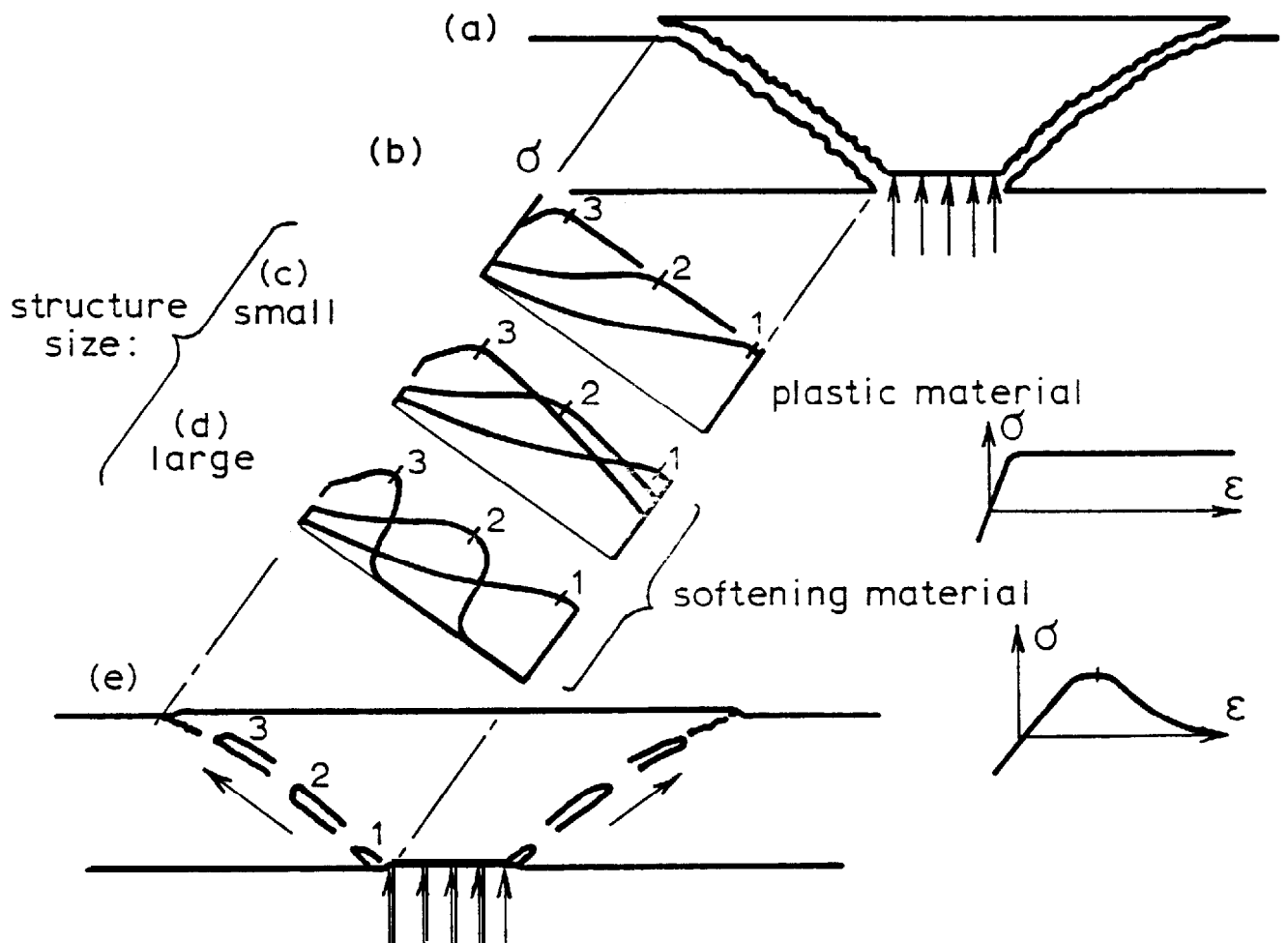


Fig.1.3 Progressive Nature of Failure Illustrated for Punching Shear of a Slab

load.

Reason 4: Energy Absorption Capability and Ductility

The area under the entire load deflection diagram represents the energy which the structure will absorb during failure and must therefore be supplied by the loads. Consideration of this energy is important especially for dynamic loading, and determines the ductility of the structure. Plastic limit analysis can give no information on the post-peak decline of the load and the energy dissipated in this process. Some form of fracture mechanics is necessary.

Reason 5: Size Effect

The size effect is, for design engineers, probably the most compelling reason for using fracture mechanics, and so a thorough discussion is in order.

The size effect is defined through a comparison of geometrically similar structures of different sizes, and is conveniently characterized in terms of the nominal stress σ_N at maximum (ultimate) load, P_u . When the σ_N -values for geometrically similar structures of different sizes are the same, we say that there is no size effect. A dependence of σ_N on the structure size (dimension) is called the size effect.

The nominal stress need not represent any actual 'stress in the structure but may be defined simply as $\sigma_N = P_u/bd$ when the similarity is two-dimensional, or as P_u/d^2 when the similarity is three-dimensional; b - thickness of the two-dimensional structure, and d characteristic dimension of the structure, which may be chosen as any dimension, e.g., the depth of the beam, or its span, since only the relative values of σ_N matter.

According to the classical theories, such as elastic analysis with allowable stress, plastic limit analysis, as well as any other theories which use some type of strength limit or failure criterion in terms of stresses (e.g., viscoelasticity, viscoplasticity), σ_N is constant, that is, independent of the structure size. This may be illustrated, e.g., by considering the elastic and plastic formulas for the strength of beams in bending, shear and torsion (regarding the definition $\sigma_N = P_u/bd$ for torsion, note that one may set $P_u = T_u/r$ where T_u = ultimate torque, P_u = force acting on an arm, r , such that r/H or r/a is constant for similar structures of different sizes; H = cross section depth, a = crack length). It is seen that these formulas are of the same form except for a factor. Thus, if we plot $\log \sigma_N$ vs. $\log d$, the failure states according to a strength or yield criterion are always given by a horizontal line (dashed line in Fig. 1.4). So failures according to strength or yield criteria exhibit no size effect.

By contrast, failures governed by linear elastic fracture mechanics exhibit a rather strong size effect which in Fig. 1.4 is described by the inclined dashed line of slope $-1/2$. The reality for concrete structures is a transitional behavior illustrated by the solid curve in Fig. 1.4. This curve approaches the horizontal line for the strength criterion if the structure is very small, and the inclined straight line for linear elastic fracture mechanics if the structure is very large (the precise meaning of "very small" and "very large" will be clarified by Eq. 5.11). This size effect, which is generally ignored by current codes (with a few exceptions), is obviously important in design.

Another size effect which calls for the use of fracture mechanics is effect of size on ductility.

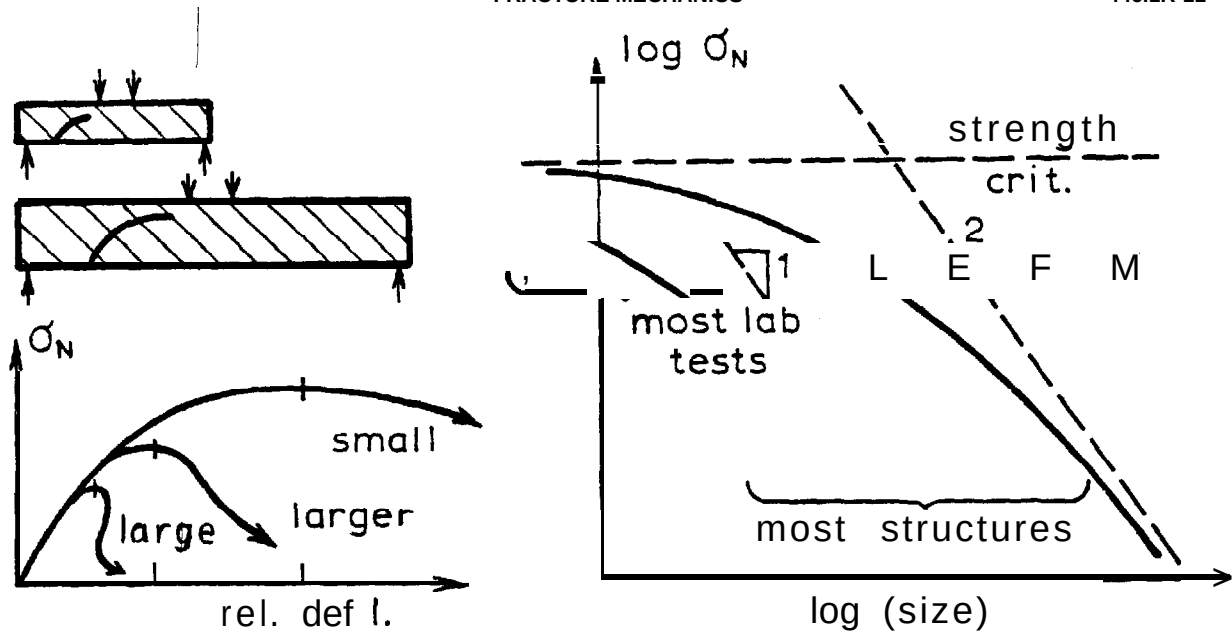


Fig.1.4 Fracture Mechanics Size Effect for Geometrically Similar Structures of Different Sizes

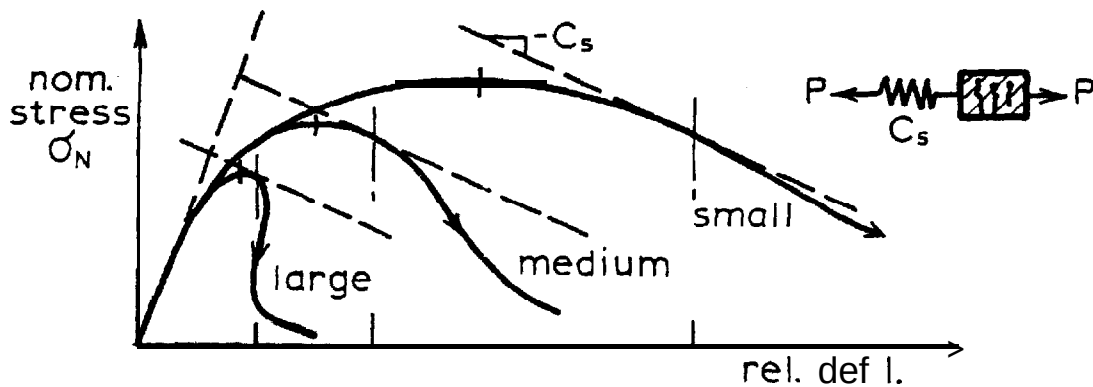


Fig.1.5 Load-Deflection Diagrams of Geometrically Similar Structures of Different Sizes

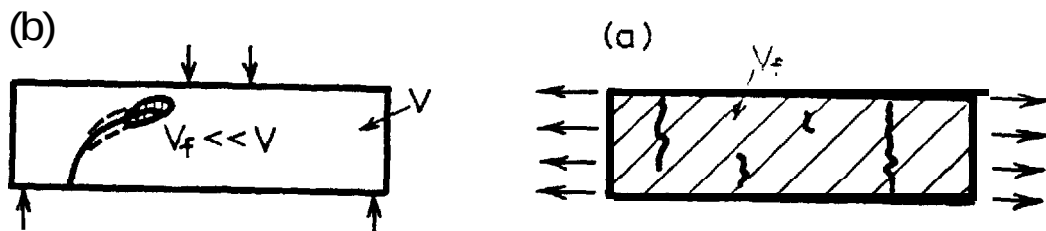


Fig.1.6 Possible Final Fracture Locations Caused by Randomness of Strength

Ductility of a structure may be characterized by the deformation at which the structure fails under a given type of loading. For loading in which the load is controlled, structures fail (become unstable) at their maximum load, while structural elements that are loaded under displacement control (i.e. imposed displacement) or through displacement-controlled elastic devices fail in their post-peak, strain-softening range. In a plot of σ_N versus deflection, the failure point is characterized by a tangent (dashed line in Fig. 1.5) of a certain constant slope $-C_s$, where C_s is the stiffness of the loading device (e.g. Baiant and Cedolin, 1990, Sec. 13.2). Geometrically similar structures of different sizes typically yield curves of the type shown in Fig. 1.5. As illustrated, failure occurs closer to the peak as size increases. This effect is again generally predicted by fracture mechanics, due to the fact that in a larger structure more strain energy is available to drive the propagation of the failure zone. A decrease of ductility of a structure represents an increase in its brittleness.

The well-known effect of structure size or member size on crack spacing and crack width is also explicable by fracture mechanics. The spurious effect of mesh size (Reason 2) can be regarded as a consequence of the structural size effect (this can be shown by considering structures of different size but with the same mesh size, and then scaling all structures to the same size along with the meshes).

1.2 Is Weibull's Statistical Theory of Size Effect Applicable?

Traditionally, the size effect has been explained statistically, by randomness of the strength value. The failure load of a chain is determined by the minimum value of the strength of the links in the chain, and the statistical size effect is due to the fact that the longer the chain, the smaller is the strength value that is likely to be encountered in the chain. This explanation, which is certainly correct for the size effect, observed in the failure of a long uniformly stressed concrete bar in tension (Fig. 1a), is described by Weibull's weakest link statistics. However, applications of this theory to concrete structures have been overdone and are questionable for the following reason.

According to Weibull-type theory of failure (Weibull, 1939; Zaitsev and Wittmann, 1973; Mihashi and Zaitsev, 1981; and Carpinteri, 1986), the probability of failure of a structure under load P and the mean nominal stress at failure are:

$$\text{Prob}(P) = 1 - \exp \left\{ - \int_V \left[\frac{\sigma(P, \mathbf{x})}{\sigma_0} \right]^m \frac{dV(\mathbf{x})}{V_r} \right\} \quad (1.1)$$

$$\bar{\sigma}_N = \frac{\bar{P}}{b d'} = \frac{1}{b d'} \int_0^\infty [1 - \text{Prob}(P)] dP \quad (1.2)$$

where $\bar{P}$ = mean load, x = coordinate vectors, V = volume of structure, V_r = representative volume of material (the smallest volume for which a material with discrete microstructure can be treated as a continuum), m = Weibull modulus of the material, σ_0 = scale parameter.

The key to applications of the Weibull theory is function $\sigma(P, x)$, representing the stress caused by load P at point x (for three-dimensional stresses, σ may be regarded as the maximum principal stress). For some structures, such as aircraft wings and metal structures

in general, failure occurs right at the initiation of the macroscopic crack growth. For those structures, function $\sigma(P, \mathbf{x})$ is known; $\sigma(P, \mathbf{x}) = Ps_e(\xi)$ where $\xi = x/d =$ relative coordinate vectors, $d =$ characteristic dimension of the structure, and $s_e(\xi)$ is the elastic stress distribution due to unit load ($P = 1$). The same assumptions have often been implied in various Weibull-type studies of concrete structures.

Concrete structures, however, behave differently. Due to reinforcement as well as the existence of strain-softening in a large zone of microcracking ahead of the tip of a continuous crack, concrete structures do not fail at crack initiation. In fact, design codes require the failure load to be significantly higher than the crack initiation load; for bending, at least 1.25 times higher for unprestressed beams and 1.2 times higher for prestressed beams (according to ACI 318), but in practice this ratio is usually much higher. Consequently, a concrete structure undergoes pronounced inelastic deformation and macroscopic crack growth prior to reaching the failure load. This causes stress redistributions, such that the stress distribution $\sigma(P, \mathbf{x})$ at failure is very different from the elastic stress distribution $s_e(\xi)$.

This distribution is difficult to determine, but the near-tip asymptotic elastic stress field, $Ps_e(\xi)$, may be used as an approximation at distances not too far from and not too close to the tip of the macrocrack at the moment of failure. Now, due to singularity of this field, the stress values farther away from the tip of the macrocrack are relatively small and make a negligible contribution to $\bar{\sigma}_N$ (Eq. 1.2) compared to the stresses in the volume V_f of the fracture process zone around the tip. This of course reflects the fact that the volume in which the macrocrack tip at failure might be located (as dictated by the laws of mechanics) is very small (e.g. the diagonal shear crack in Fig. 1.6b cannot grow toward the lower midspan region or toward the upper left corner of the beam, regardless of the strength values there).

Consequently, the statistical size effect must be smaller than in a uniformly stressed tensile bar, where failure is precipitated by a macrocrack anywhere within the volume of the bar. Thus, as pointed out by Bazant (1986, 1987a), if one calibrates Weibull parameters m and σ_0 on the basis of uniaxial strength tests and then uses the same parameters to predict P_f and $\bar{\sigma}_N$ for diagonal shear failure, one must find the statistical size effect to be rather small. Thus, even though limited test results for diagonal shear failures have been successfully fitted by formulas based on the Weibull distribution, the size effect data for both the diagonal shear failures and the uniaxial failures cannot be successfully fitted using the same material parameters (unless an incorrect elastic stress field $Ps_e(\xi)$ is used).

Thus, the principal fault of the Weibull-type statistical explanations of the size effect in concrete structures is that they ignore the size effect caused by the redistribution of stress $\sigma(P, \mathbf{x})$ prior to failure. This size effect, which is of the fracture mechanics type, is associated with the energy release into the front of a large crack and would exist even if the material behavior were deterministic.

Therefore, the proper approach is to fit the size effect data for a concrete structure first by a fracture mechanics theory, and only if some part of the observed size effect remains unaccounted for it may be attributed to Weibull-type statistical phenomena (Bazant, 1986, 1987a).

Weibull theory then describes a size effect of the volume of the structure, expressing the

fact that the larger the volume, the greater is the chance of encountering a critical microscopic flaw that triggers failure. Thus, as far as the load at initiation of cracking (or damage) is concerned, Weibull-type theory is, of course, applicable. Saouridis (1989) demonstrated that by analysis of L'Hermite's tests of size effect in unnotched beams. Indeed, as long as there are no stress redistributions, i.e., all stresses are fixed in their proportion to the load, the interaction of the elements of the structure is mathematically equivalent to a series coupling of elements, the same as in a uniaxially stressed bar of variable cross section. When there are stress redistributions, however, the structure behaves as a combination of series and parallel couplings. But Weibull's theory is valid only for series coupling, as in a chain of elements (hence the term "weakest-link statistics").

1.3 Simple Energy Explanation of Size Effect

The fracture mechanics type size effect, which is due to energy release, can be simply explained by considering the uniformly stressed panel with a crack or crack band of initial length $a = a_0$, shown in Fig. 1.7. It may be imagined that the formation of a crack band of thickness h reduces the strain energy density $\sigma_N^2/2E$ in the cross-hatched area to zero (E = elastic modulus of concrete). When the crack band extends by a , the additional strain energy that is released comes from the densely cross-hatched strip of horizontal dimension Au (Fig. 1.7a). If the failure modes are geometrically similar, as is usually the case, then the larger the panel, the longer is the crack band at failure. Consequently, the area of the densely cross-hatched strip is also larger, being given by $h\Delta a + 2ka\Delta a$ where k = empirical constant depending on the shape of the structure. This illustrates that, in a larger structure, more energy is released by the same extension of the crack band. The energy released from the strip is $-\delta\Pi/\delta a = G_f b$, i.e.,

$$\frac{1}{\Delta a}(h\Delta a + 2ka\Delta a)\frac{\sigma_N^2}{2E} = G_f b \quad (1.3)$$

where Π = potential energy stored in the structure, b = thickness and G_f = fracture energy (dimension J/m^2) = energy needed to create a fracture or crack band of unit length and (in the third direction) unit width. The value of G_f is approximately constant and represents a material property. Solving from Eq. 1.2 for the nominal stress, one obtains the size effect law proposed by Bazant (1984a):

$$\sigma_N = B f'_t (1 + \beta)^{-1/2}, \quad \beta = d/d_0 \quad (1.4)$$

where $B = (2G_f E b/h)^{1/2}/f'_t$, $d_0 = (h/2k)(d/a)$ = reference size which depends on the shape of the structure but is independent of structure size if the structures are geometrically similar (because d/a = constant); f'_t = tensile strength, introduced for convenience; and h = width of the crack band front, which may be treated approximately as constant, independent of structure size. Empirically, $d_0 \simeq n d_a$ where d_a = maximum aggregate size and n is approximately constant when d or d_a is varied, but depends on structure geometry. Eq. 1.4 will be discussed more in Chapter 5 and derived in Appendix I.

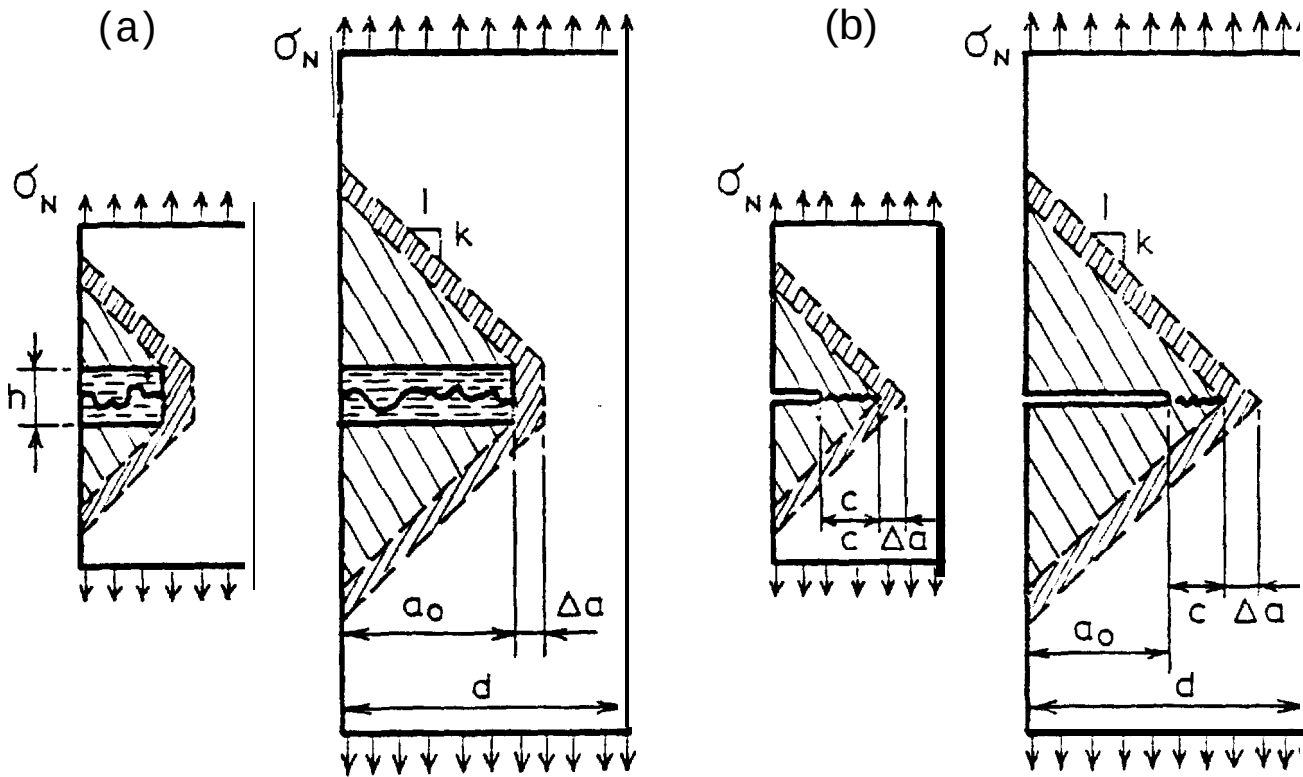


Fig.1.7 Areas of Energy Release in Similar Small and Large Specimens

Lest one might get the impression that this explanation of the size effect works only for a crack band but not for a sharp line crack, consider the same panel with a line crack of length a_0 shown in Fig. 1.7b for similar panels of different sizes. In concrete, there is always a sizeable fracture process zone ahead of the tip of a continuous crack, of some finite length which may, in the crudest approximation, be considered constant. Over the length of this zone, the transverse normal stress gradually drops from f'_t to 0. Because of this zone, the elastically equivalent crack length which causes the release of strain energy from the adjacent material is longer than the continuous crack length, a , by a distance c which is approximately a material constant. (Strictly speaking, c varies, but much less than in proportion to the size. Anyway, a more involved derivation in which c is variable yields the same result, and will be indicated just before Eq. 5.5.)

When the crack extends by length Δa , the fracture process zone travels with the crack-tip, and the area from which additional strain energy is released consists of the strips of horizontal dimension Δa which are densely cross-hatched in Fig. 1.7b. Their area is $k(a_0 + c)\Delta a$ where k is approximately a constant. The energy released per unit crack advance, $-\delta\Pi/\delta a$, must equal to $G_f b$ where G_f = the fracture energy of the material, and so

$$\frac{1}{\Delta a} 2k(a_0 + c)\Delta a \frac{\sigma_N^2}{2E} = G_f b \quad (1.5)$$

Solving this equation for σ_N , one again obtains the size effect law in Eq. 1.4 in which now $B = (EG_f b/kc)^{1/2}/f'_t$, $d_0 = c(d/a) = \text{constant}$.

Eq. 1.4, which describes the transitional size effect given by the solid curve in Fig. 1.4, is also obtained for various other structural geometries. For large sizes, the curve of Eq. 1.4 in Fig. 1.4 approaches a straight line of slope $-1/2$, which represents the size effect of linear elastic fracture mechanics (see Sec. 1.2).

Eq. 1.4 can also be derived, in a completely general way, by dimensional analysis and similitude arguments (Bazant, 1984a). This general derivation rests on two basic hypotheses: (1) the propagation of a fracture or crack band requires an approximately constant energy supply (the fracture energy, G_f) per unit area of fracture plane, and (2) the potential energy released by the structure due to the propagation of the fracture or crack band is a function of both the fracture length and the size of the fracture process zone at the fracture front.

It must be kept in mind that Eq. 1.4 is approximate, valid only within a size range of about 1:20 (i.e., the largest structure for which Eq. 1.4 can be applied is about 20 times larger than the smallest structure). For a broader size range, a more complicated formula would be required. Nevertheless, the aforementioned size range is sufficient for most practical purposes.

The main problem with the Weibull-type statistical theory for the size effect is that the existing works ignore stress redistributions and the consequent energy release from the structure (as illustrated in Fig. 1.7 and manifested by the size effect law). The statistical size effect should properly appear only as an addition to the fracture mechanics type size effect (which is deterministic), and would have to describe only that part of the size effect which is not explained by the fracture mechanics size effect. So far, such comparisons have

not indicated any large systematic deviations which would require some other explanation, such as statistical.

Applications of Eq. 1.4 to brittle failures of concrete structures rest on two additional hypotheses: (3) the failure modes of geometrically similar structures of different sizes are also geometrically similar (e.g., a diagonal shear crack has at failure about the same slope and the same relative length), and (4) the structure does not fail at crack initiation (which is a requirement of good design). These hypotheses are usually applicable, but not always over the entire size range of interest. A sufficiently large change of size may alter the failure mode and thus render Eq. 1.4 inapplicable beyond that size. (This apparently is the case for the brazilian split cylinder tests.)

1.4 Experimental Evidence for Size Effect in Structures

Extensive tests have been carried out to verify Eq. 1.4 for various types of failure of concrete structures (using microconcrete specimens). Good agreement of Eq. 1.4 with test results has been demonstrated for:

1. Diagonal shear failure of beams (Bazant and Cao, 1986b; Bazant and Kazemi, 1989a).
2. Punching shear failure of slabs (Bazant and Cao, 1987).
3. Torsional failure of beams (Bazant, Sener and Prat, 1987).
4. Pullout failure of reinforcing bars (Bazant and Sener, 1988).
5. Double-punch tests of cylinders (Marti, 1989).

Typical experimental results, which can be regarded as a verification of the applicability of fracture mechanics to the brittle failures of concrete structures, are shown in Fig. 1.8-1.11 (tests made at Northwestern University on microconcrete specimens with aggregate of maximum diameter 3/8 in. or 1/4 in.). As further evidence of applicability of fracture mechanics, Fig. 1.12 shows, for the punching shear failure, that the post- peak load drop becomes steeper and larger as the size increases. This is because in a larger specimen there is (for the same σ_N) more energy to be released into a unit crack extension, but since the fracture extension dissipates the same amount of energy, the value of δ_N (and thus the load) must be reduced as the structure size increases.

Note also that the shape of the measured size effect curves (Fig. 1.8-1.11) does not really agree with the Weibull-type statistical model, for which the slope of the curve would have to diminish rather than increase with increasing size and approach a horizontal asymptote.

The existing test data on concrete specimens with regular-size aggregate reported in the literature also offer evidence of size effect, and the need of a fracture mechanics based explanation has been pointed out by various researchers, e.g., Reinhardt (1981a,1981b) or Bazant and Kim (1984). The data from the literature are generally found to agree with Fig. 1.4, but this evidence is not very strong because the data exhibit very large statistical

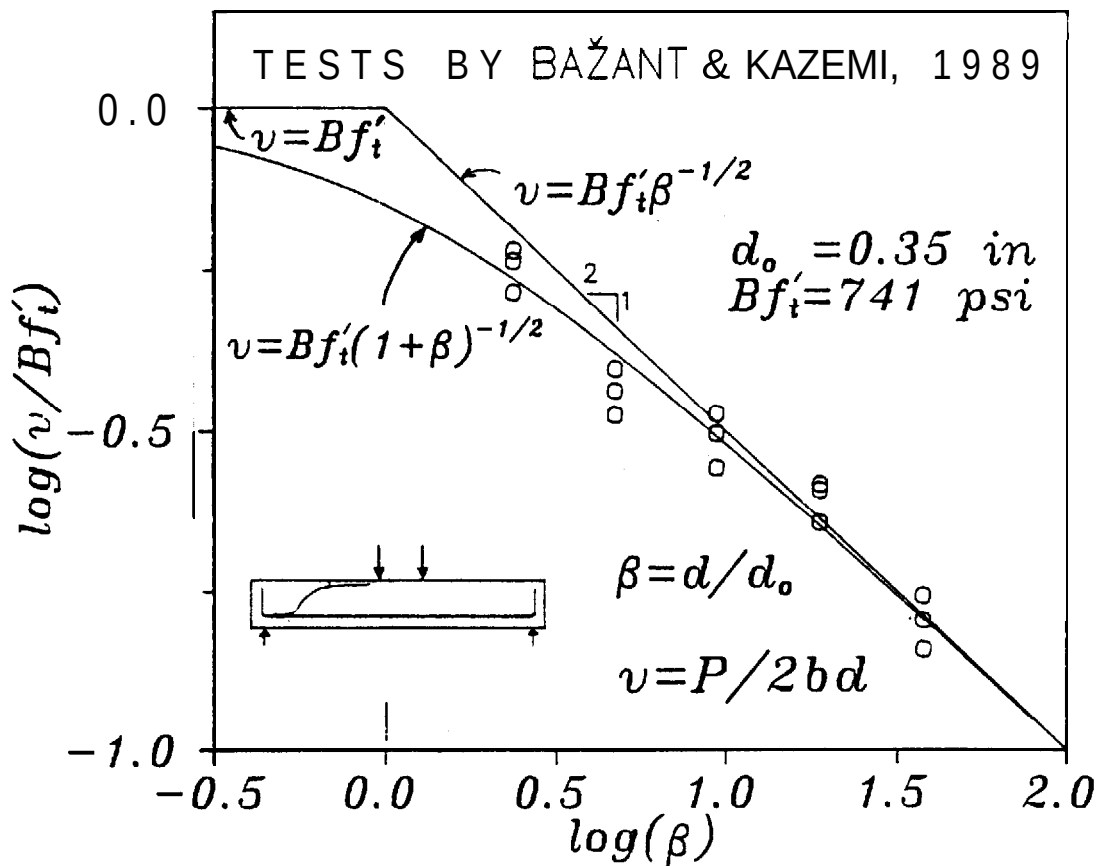


Fig.1.8 Test Results (Bazant and Kazemi, 1989a) on Size Effect in Diagonal Shear Failure; v is the nominal shear strength

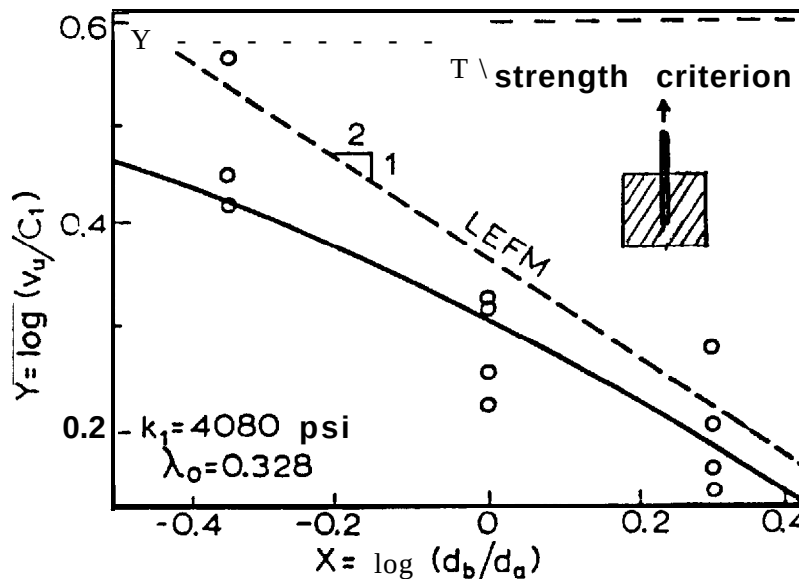


Fig.1.9 Test Results (Bazant and Sener, 1988) on Size Effect in Pull-out Failure of Bars (d_b = bar diameter, v_u = nominal pullout strength, C_i , K_i and λ_0 = empirical constants)

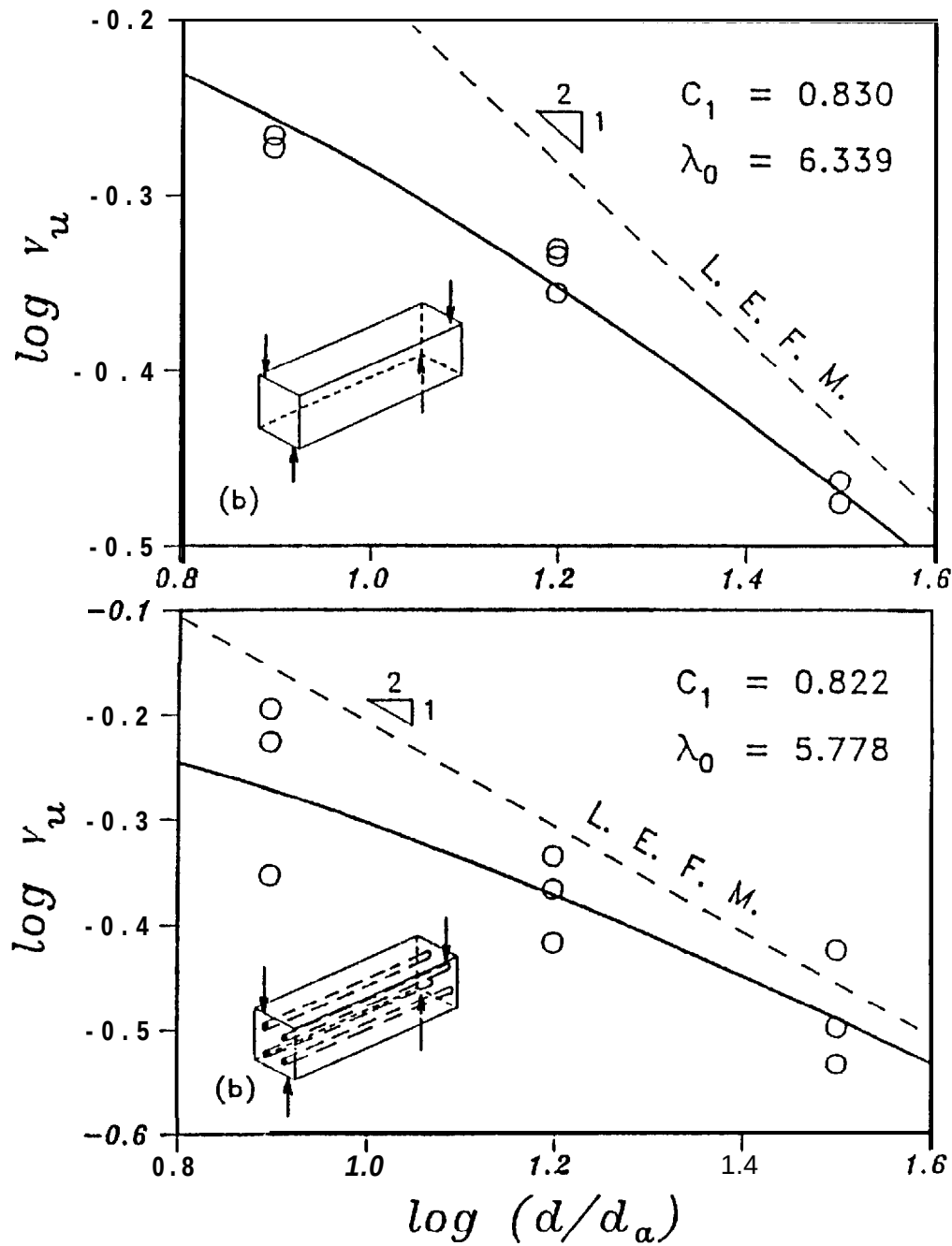


Fig.1.10 Test Results (Bazant, Sener and Prat, 1989) on Size Effect in Torsional Failure of Plain and Reinforced Beams (d = depth of beam, v_u = nominal torsional strength, c_i, λ_0 = empirical constants)

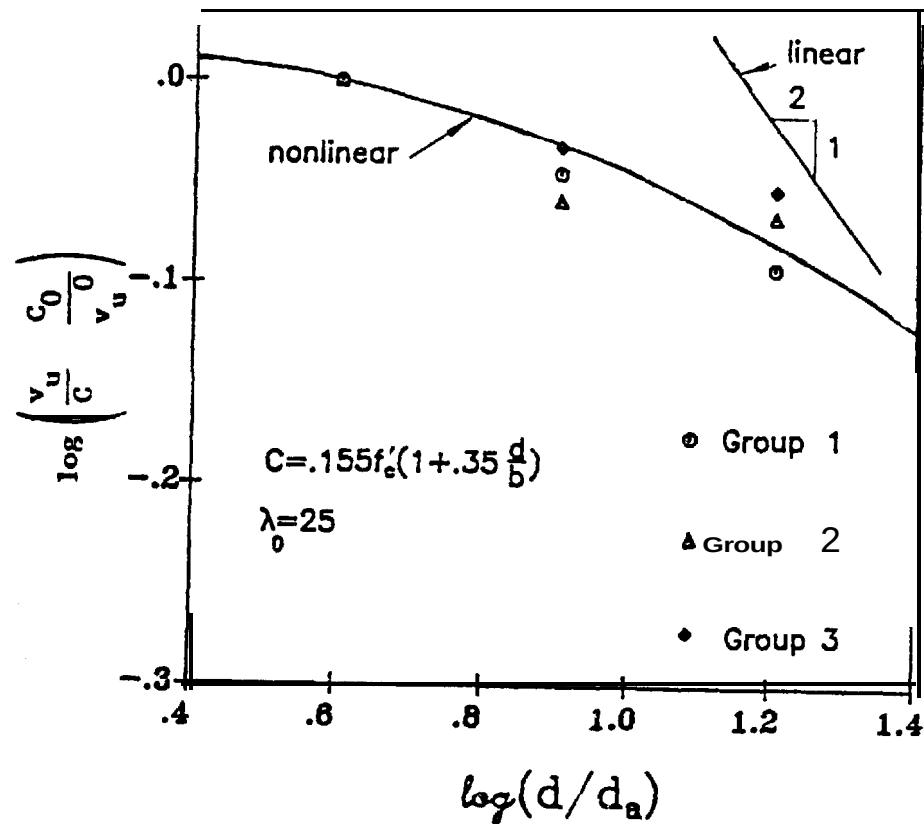


Fig.1.11 Test Results (Bazant and Cao, 1987) on Size Effect in Diagonal Shear Failure in Slabs (v_u = nominal shear strength, d = slab thickness, b = punch diameter, v_u^0 and $c_0 = v_u$ and C for smallest slab, λ_0 = empirical constants)

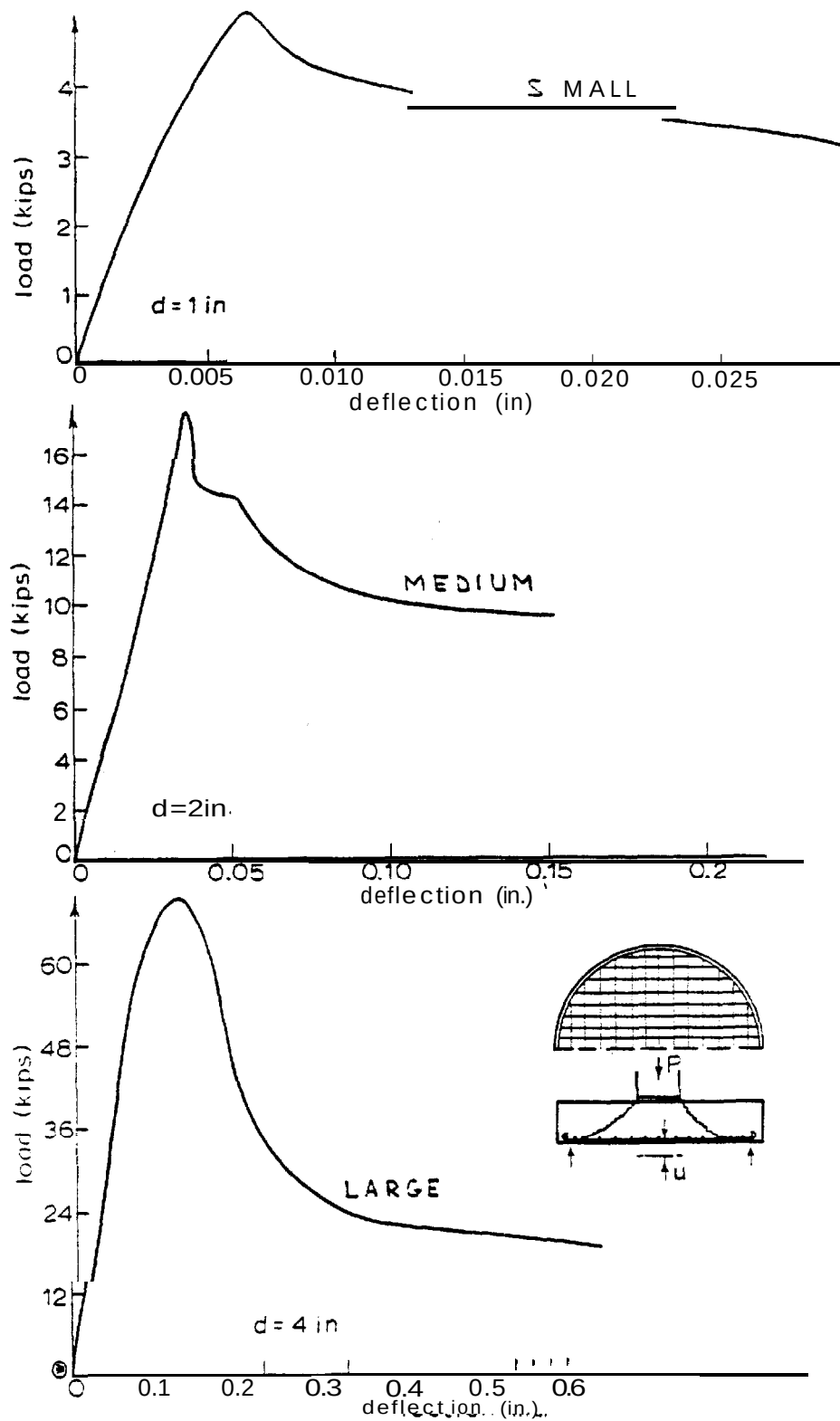


Fig.1.12 Measured Load-Deflection Diagrams of Small, Medium and Large Slabs (Bazant and Cao, 1987)

scatter. Due to scatter, about equally good fits can be obtained with other theories of size effect, e.g., Weibull's statistical theory. Although part of the scatter is inevitable and random, most of the huge scatter observed in brittle failures such as diagonal shear probably stems from the fact that the test specimens of various sizes were not geometrically similar, and so theoretical adjustments must be made for the factors of shape before a comparison with Eq. 1.4 can be made. Since the exact theory is not known, such adjustments introduce additional errors. In addition to the structures listed above, available comparisons with test data also include the beam and ring failures of unreinforced pipes (Gustafson and Hillerborg, 1984; Bazant and Cao, 1986a). Comparisons with the bulk of test data available in the literature were made for diagonal shear failures of both unprestressed and prestressed beams, and beams without and with stirrups (Bazant and Kim, 1984; Bazant and Sun, 1987; Bazant and Cao, 1986b), as well as torsional failures (Bazant and Sener 1987). Statistical analysis was included in these studies.

1.5 Explanation of Size Effect on Ductility

Structural action is normally a combination of series and parallel couplings of the cracking zones and the uncracked (elastic) zones. The size effect on ductility is explained by the series coupling aspect. Consider a cracking element coupled with an elastic element as shown in Fig. 1.13. The load-displacement diagrams of these elements are also shown. Since the force in both elements is the same and the deformations are superimposed, the response of both elements combined is obtained by passing a horizontal line at each level P and summing the corresponding deformations a and b , as shown in Fig. 1.13. If the elastic element is sufficiently soft, this can obviously produce a load-displacement diagram which exhibits the so-called snapback, in which the displacement diagram turns back at a positive slope. The snapback behavior is unstable even under displacement control, and the structure fails at the maximum displacement, labeled as u_{cr} . This displacement represents a ductility limit for the system. Since the addition of an elastic element is equivalent to increasing the size of the structure, it is clear that an increase of size tends to decrease ductility. Ductility is not a material property but a structure property which is governed by fracture mechanics and depends on structure size (as well as the type of concrete). It is worth noting that Hawkins (1984) identified twenty-nine provisions in the ACI Code which seem to be empirical but could probably be explained by fracture mechanics. They include various ductility limitations, conditions for minimum reinforcement, crack spacing and crack width, etc.

To sum up, the experimentally observed structural size effect, as well as the related spurious effect of the mesh size, presents the most potent argument for the application of fracture mechanics to concrete structures.

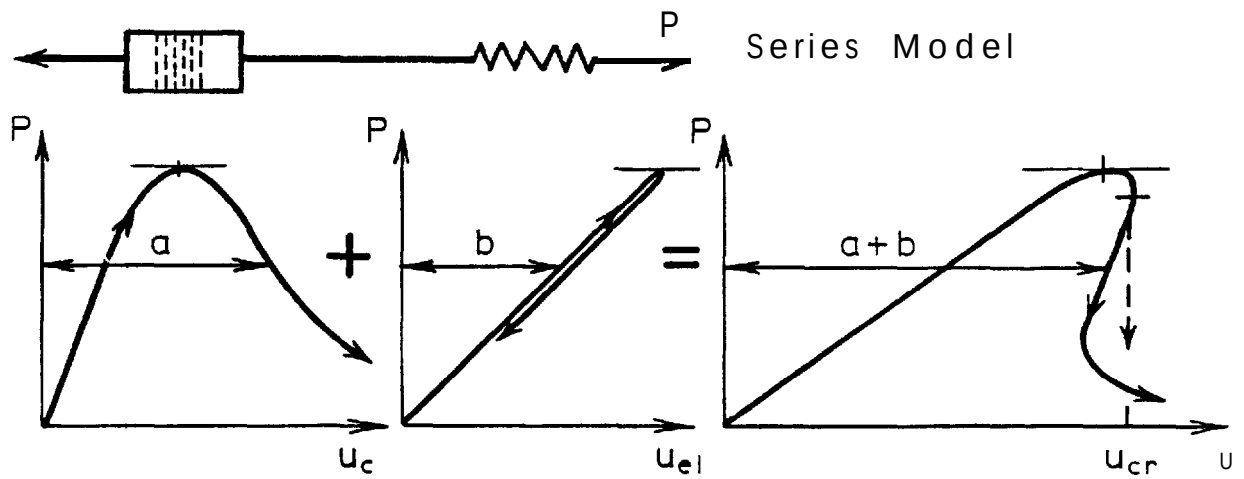


Fig.1.13 Change in Post-Peak Load-Deflection Diagram Due to Series Coupling with Elastic Element

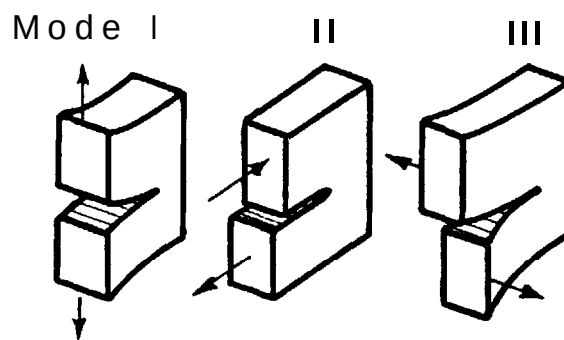


Fig.2.1 Modes I, II and III (Opening, Plane Shear, Anti-Plane Shear Fractures)

Chapter 2. ESSENTIAL RESULTS FROM LINEAR ELASTIC FRACTURE MECHANICS

In linear elastic fracture mechanics (LEFM) it is assumed that all of the fracture process happens at the crack tip and that the entire volume of the body remains elastic. Under this assumption, the questions of crack propagation and structural failure can be solved by methods of linear elasticity.

It is convenient to distinguish three elementary fracture modes, Modes I, II and III, also called the opening mode, the planar shear mode and the antiplane shear mode; see Fig. 2.1. Modes I and II are planar symmetric and antisymmetric, while Mode III is three-dimensional. General fracture is a linear combination of these three modes.

2.1 Stress Singularity

Introduction of a crack into a linear elastic body produces stress concentrations near the crack tips. This may be illustrated by the perturbation of the trajectories of the maximum principal stress shown in Fig. 2.2. The stress field is singular at the crack tip, with ALL the nonzero stress components approaching infinity as the radial distance r from the crack tip tends to zero (Fig. 2.3). In a sufficiently close neighborhood of the sharp crack tip, the stress components σ_{ij} are the same regardless of the shape of the body and the manner of loading, and may be expressed as:

$$\sigma_{ij}^I = K_I f_{ij}^I(\theta)(2\pi r)^{-1/2}, \sigma_{ij}^{II} = K_{II} f_{ij}^{II}(\theta)(2\pi r)^{-1/2}, \sigma_{ij}^{III} = K_{III} f_{ij}^{III}(\theta)(2\pi r)^{-1/2} \quad (2.1)$$

Here the subscripts and superscripts I, II and III refer to the elementary modes, θ is the polar angle, K_I , K_{II} and K_{III} are parameters called the stress intensity factors, and functions f_{ij} are the same regardless of the body geometry and the manner of loading. For example, $f_{11}^I(\theta) = \cos \alpha(1 - \sin \alpha \sin 3\alpha)$, $f_{22}^I(\theta) = \cos \alpha(1 + \sin \alpha \sin 3\alpha)$, $f_{12}^I(\theta) = \cos \alpha \sin 2\alpha \cos 3\alpha$, where $\alpha = \theta/2$; see e.g., Knott (1973), Broek (1974) Owen and Fawkes (1983), Hellan (1984), and Kanninen and Popelar (1985).

2.2 Energy Criterion

The fact that, according to the theory of elasticity, the stress near the crack tip approaches infinity, no matter how small the load, was noted by Griffith (1921, 1924) on the basis of the previous solution for elliptical holes by Inglis (1913). He concluded that, if linear elasticity is used, one cannot introduce a strength criterion as a condition of failure, but must instead decide failure on the basis of an energy criterion. As the crack tip propagates, energy flows into the crack tip where it is dissipated by the fracture process. The energy flow is characterized by the energy release rate which is expressed as:

$$Gb = -\frac{\partial \Pi(a)}{\partial a} \simeq -\frac{1}{\Delta a} \left[\Pi\left(a + \frac{\Delta a}{2}\right) - \Pi\left(a - \frac{\Delta a}{2}\right) \right] \quad (2.2)$$

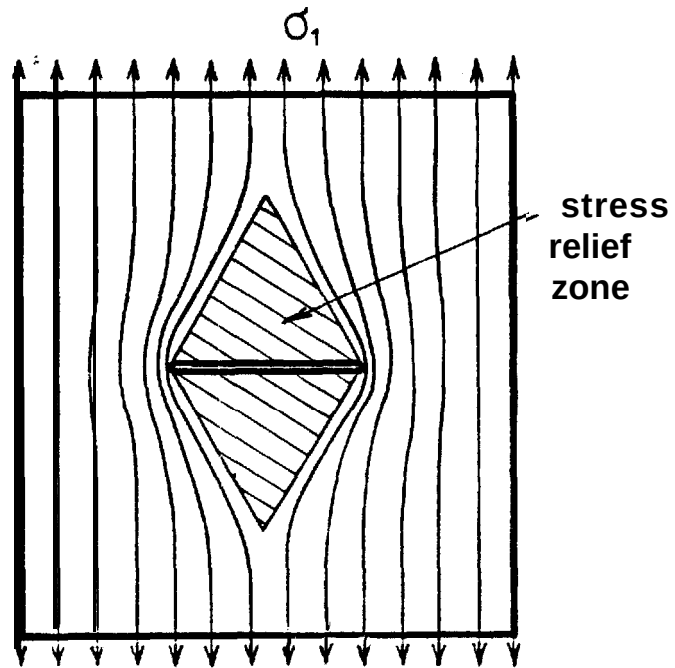


Fig.2.2 Principal Stress Trajectories in a Cracked Specimen

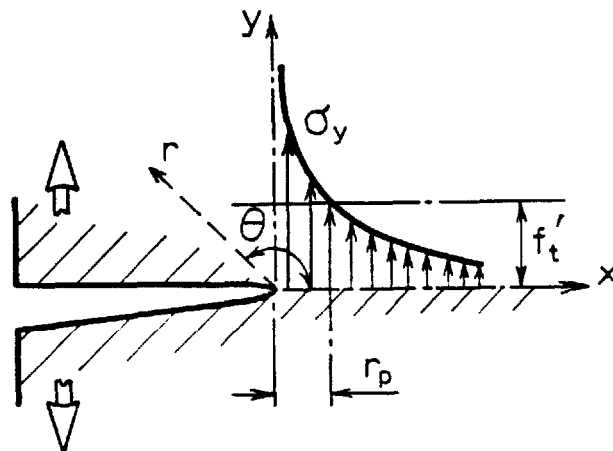


Fig.2.3 Stress Distribution Near Crack Tip

in which $\Pi = U - W$ = potential energy of the structure, W = work of loads, and U = strain energy of the structure as a function of the crack length a . Eq. 2.2 also gives a finite difference approximation which may be used to calculate G by the finite element method. For that purpose one may model the crack as a line gap between adjacent elements and calculate the strain energy stored in the mesh for the crack tip displaced by either $-\Delta a/2$ or $\Delta a/2$. Instead of using an interelement line crack, one may, for the sake of convenience, model the crack by considering a band of elements to have zero stiffness (if one uses a mesh of square elements whose size is not more than about 0.1 of the cross section dimension, the error of considering a crack band instead of a line crack is usually less than 1%; see Bazant and Cedolin, 1979).

According to Griffith, the condition of crack propagation (critical state) is

$$G = G_f \quad (2.3)$$

G is the fracture energy, which has the dimension of J/m^2 (or N/m) and represents a basic material property. For $G < G_f$, the crack cannot propagate, and for the case $G > G_f$ equilibrium is impossible. If $G = G_f$ and $\partial G/\partial a > 0$, which is normally the case, the crack is unstable under load control (i.e., the structure fails). There exist, however, some cases in which $G = G_f$ and $\partial G/\partial a < 0$, and then the crack can grow under load control in a stable manner.

The energy release rate for modes I, II and III may be expressed on the basis of the stress intensity factors as follows:

$$G_I = K_I^2/E', \quad G_{II} = K_{II}^2/E', \quad G_{III} = K_{III}^2/\mu \quad (2.4)$$

in which μ = elastic shear modulus, for the case of plane stress $E' = E$ = Young's elastic modulus, for the case of plane strain $E' = E/(1-\nu^2)$, and ν = Poisson's ratio. For general loading, the total energy release rate is:

$$G = G_I + G_{II} + G_{III} \quad (2.5)$$

The stress intensity factors are proportional to the applied load, and may generally be expressed in the form:

$$K_I = \frac{P}{bd} \sqrt{\pi a} f(\alpha) = \frac{P}{bd} \sqrt{d} \varphi(\alpha), \quad \alpha = a/d \quad (2.6)$$

in which f is a certain nondimensional function of the relative crack length α (d = characteristic structure dimension), and $\varphi(\alpha) = f(\alpha)\sqrt{\pi\alpha}$ = another non-dimensional function. For various simple geometries of notched fracture specimens, accurate expressions for function f are available in textbooks and handbooks (e.g., Tada et al., 1985, and Murakami, 1987). For other geometries, the function f can always be calculated by linear elastic analysis; e.g., through a finite element program. For the special case of a single line crack of length a in an infinite solid subjected at infinity to nominal stress σ_N in the direction normal to the crack plane, one has $f(\alpha) = 1$. Eq. 2.6 shows that, for geometrically similar structures of different



sizes, the stress intensity factor is proportional to the square root of the size, and the energy release rate is proportional to the size of the structure.

Instead of Eq. 2.3, the condition of mode I crack propagation (critical state) can be expressed in terms of the stress intensity factor as:

$$K_I = K_{Ic} \quad (2.7)$$

in which K_{Ic} = critical value of K_I , which is also called fracture toughness and represents a material property; $K_{Ic} = G_f E'$. If Eq. 2.7 is substituted into Eq. 2.6, the nominal stress at failure (crack propagation) is obtained as:

$$\sigma_N = \frac{K_{Ic}}{\sqrt{\pi a f(\alpha)}} = \frac{K_{Ic}}{\sqrt{d \varphi(\alpha)}} \quad (2.8)$$

It may be noted that, according to Eq. 2.8,

$$\log \sigma_N = -\frac{1}{2} \log d + \text{const.} \quad (2.9)$$

This relation shows that the size effect plot according to linear elastic fracture mechanics is an inclined straight line of slope $-1/2$ (Fig. 1.4).

2.3 Limits of Applicability

In reality, the fracture process cannot take place at a point. The fracture process zone must have some finite size. According to Irwin (1958) a crude estimate of the length r_f of the fracture process zone may be obtained by setting the transverse normal stress in Eq. 2.1 to be equal to the tensile strength f'_t . This yields

$$r_f = \frac{1}{2\pi} \frac{K_{Ic}^2}{f'_t} = \frac{\ell_0}{2\pi}, \quad \ell_0 = \frac{K_{Ic}^2}{f_t'^2} = \frac{E' G_f}{f_t'^2} \quad (2.10)$$

Note that this length is expressed only in terms of material properties, and therefore is a material property, too. An alternative estimate of the size of the fracture process zone of concrete can be based on the maximum aggregate size d_a . Bazant and Oh (1983a) concluded that the length and effective width of the fracture process zone of concrete in three-point bend specimens are roughly $12d_a$ and $3d_a$, respectively.

Linear elastic fracture mechanics is applicable when r_f is much smaller than the cross section dimension of the structure. This condition is not satisfied for most concrete structures, with the possible exception of some very large structures such as concrete dams. However, a more precise criterion for the applicability of linear elastic fracture mechanics, which also takes into account the structure shape and the manner of loading, can be given in terms of the so-called brittleness number, β , which will be explained in Sec. 5.2.

The inapplicability of linear elastic fracture mechanics to brittle failures of typical concrete structures is clearly apparent from the test results shown in Fig. 1.8-1.11. The data points indicate a milder size effect than the straight line of slope $-1/2$.

Chapter 3. NONLINEAR FRACTURE MODELS WITH SOFTENING ZONE

The reason for deviations of concrete behavior from linear elastic fracture mechanics is the development of a relatively large fracture process zone which undergoes progressive softening damage due to microcracking. The effect of this microcracking is: (1) to reduce the flux of energy that can be released into the crack tip; and (2) at the same time to increase the combined surface area of cracking, and thus enhance the energy absorption capability of the fracture process zone.

Therefore, a relation describing the softening damage needs to be included in the fracture model. This can be done basically in two ways: (1) in the form of a stress-displacement relation for the frontal zone of a line crack, or (2) a stress-strain relation for the strain-softening (microcracking) region in front of the main crack. We will first describe these approaches and then discuss their relative merits (For further discussions, see e.g. Planas and Elices, 1988a; Rots, 1988).

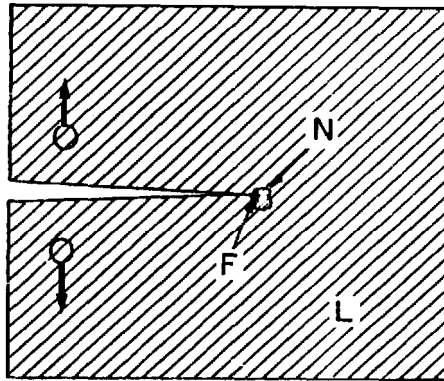
In general, one may distinguish two types of nonlinear fracture mechanics: (1) ductile (metals), and (2) nonductile (concrete, as well as ceramics); see Fig. 3.1. In contrast to linear elastic fracture mechanics, the nonlinear zone is large for both types of nonlinear fracture mechanics. For ductile fracture mechanics, most of the nonlinear zone undergoes plastic hardening or perfect plasticity, and the fracture process zone is still a very small part of the nonlinear zone. By contrast, for nonductile fracture mechanics, which is the case for concrete, the fracture process zone is large and occupies nearly the entire nonlinear zone. Thus, although many results of the fracture theory of metals which evolved earlier are useful, most of them cannot be directly transplanted.

Remarks: The fracture process zone is defined as the zone in which the material undergoes strain-softening, i.e., the stress normal to the crack-plane decreases with increasing strain. The stress can be understood as the macrostress or average stress σ_{ij} that is calculated as $(1/\Omega_1)$ -times the force resultant transmitted across area Ω_1 on which the heterogeneous material with aggregates and microcracks can be approximated as a continuum (on the macroscale). Alternatively, (and customarily), $\sigma_{ij} = \int_{V_r} \sigma_{ij}^m dV / V_r$ where σ_{ij}^m are the microstresses (actual stresses in the aggregates, matrix and interfaces, which show high random local scatter), and V_r is for the representative volume of the material, defined below Eq. 1.2 (its size is the characteristic length ℓ). These concepts are developed more precisely in the statistical theory of heterogeneous materials (e.g. Kröner, 1967; Krumhansl, 1968).

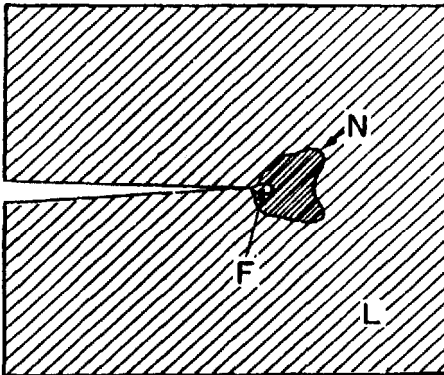
3.1 Softening Stress-Displacement Relations

This approach developed as a modification of a similar approach previously formulated for metals. Dugdale (1960) and Barenblatt (1959, 1962) proposed that a plastic (yielding cohesive) zone of a certain finite length must exist at the front of a crack (Fig. 3.2a). The length of this zone must be such that the stresses from the fracture process zone cancel the stress singularity caused at the tip of the equivalent elastic crack by the applied load

(a) Linear Fracture



(b) Metals



(c) Concrete

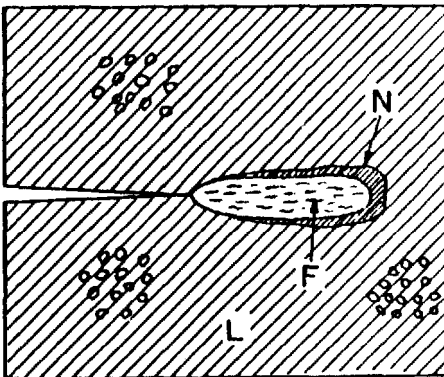


Fig.3.1 Linear Zone (L), Non-Linear Zone (N) and Fracture Process Zone (S) in Fracture of Different Materials

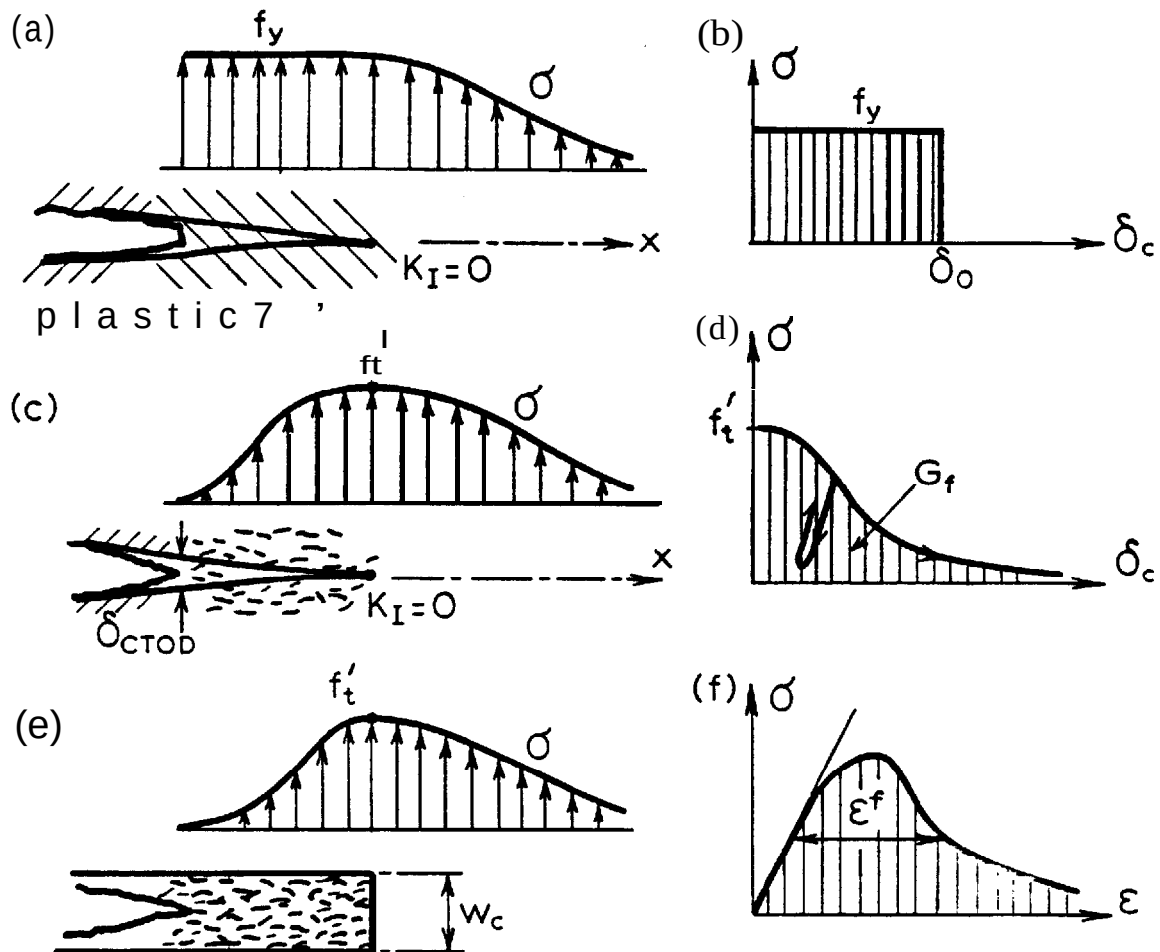


Fig.3.2 Stress Distributions In and Near the Fracture Process Zone in Different Models

(i.e., $K_I = 0$). The crack opening at the beginning of the plastic zone, where the stress suddenly drops to zero, may be regarded as a material property which controls propagation (Fig. 3.2b).

For some metals and other materials it was later noted that the cohesive zone should exhibit a gradual rather than sudden stress drop, characterized by a softening relation of the normal stress σ across the crack vs. the crack-opening displacement δ_c (Fig. 3.2d); see e.g. Knauss (1974), Kfouri and Miller (1974), and Wnuk (1974). For concrete, this type of model was proposed by Hillerborg, Mod  er and Petersson (1976) under the name of fictitious crack model; see also Petersson (1980a, 1981), Hillerborg (1983, 1985a, 1985b). (The term “fictitious” refers to the fact that the portion of a crack which transmits tensile stress cannot be a continuous crack with full separation of the surfaces; the real crack ends at the point where the stress is reduced to zero; Fig. 3.2d.) In this model, which has been widely applied in finite element analysis of concrete fracture, the material fracture properties are defined by the softening stress- displacement relation:

$$\sigma_y = f(\delta_c) \quad (3.1)$$

(see Fig. 3.2d), where σ_y is the stress in the direction normal to the crack. The area under the curve represents the fracture energy of the material, i.e.:

$$G_f = \int_0^\infty \sigma_y d\delta_c \quad (3.2)$$

The crack begins to open when the stress at the tip reaches the tensile strength limit f'_t . If the shape of the softening curve is fixed, then the fracture properties are completely characterized by two parameters: f'_t and G_f . The precise shape of the softening stress-displacement diagram has a considerable influence on the calculation results. In various works this shape is considered triangular or bilinear, in which case the stress is reduced to zero at a finite displacement δ_0 . An exponential shape has also been used. When the softening zone is unloaded and reloaded (Fig. 3.2d), the behavior is as sketched in Fig. 3.2d.

Hillerborg’s fictitious crack model was verified and calibrated by various comparisons with test data. However, it seems that an exhaustive comparison with all the important concrete fracture data from the literature has not yet been presented. But due to equivalence with the crack band model, the extensive comparisons of the latter with test data (Bazant and Oh, 1983a) indirectly validate the fictitious crack model.

The shape of the stress-displacement curve was studied on the basis of micromechanics of microcracks by Horii (1988), Horii et al. (1987, 1988), and Bazant (1987b), and on ceramics with similar behavior by Ortiz (1988). From these models it transpired that the microcrack coalescence generally tends to produce snapback instability of the stress-displacement curve. However, the existing measurements, albeit limited in scope, do not show any snapback. It may well be that some other mechanism, such as frictional pullout of aggregate and fragments

from the crack faces, eliminates the snapback.

3.2 Softening Stress-Strain Relations

Since the cracks in concrete are not straight but tortuous, and the microcracking zone in front of the continuous fracture is not likely to develop along a straight line, the behavior of the fracture process zone can equally well be described by stress-strain relations with strain-softening, i.e., declining stress at increasing strain. This approach is quite convenient for computer programming since no separation of the nodes of two adjacent elements needs to be introduced and fracture is handled by adjustments of the incremental stiffness of finite elements, basically in the same way as any inelastic behavior.

Strain-softening in the form of a sudden vertical drop was introduced in finite element analysis by Rashid (1968). The need to consider progressive strain-softening in tension was recognized by Scanlan (1971) who introduced a sequence of small stress drops. The fact that concrete exhibits strain-softening in tension was experimentally observed by L'Hermite (1959), Rusch and Hilsdorf (1963), Hughes and Chapman (1966), and Evans and Marathe (1968). Extensive and carefully controlled measurements were recently reported by Petersson (1981), Reinhardt and Cornellisen (1984), Gopalaratnam and Shah (1985), and others.

From the continuum mechanics viewpoint, the concept of strain-softening involves certain severe mathematical difficulties, such as imaginary wave speed (or change of type of the partial differential equation of motion from hyperbolic to elliptic) and ill-posedness of the boundary value problem, which were pointed out and analyzed by Hadamard (1903), Thomas (1961), Bazant (1976), Sandler (1984), Read and Hegemier (1984), Wu and Freund (1984), Bazant, Belytschko and Chang (1984) and others. The chief problem is that the zone of energy dissipation tends to localize to a zone of zero volume (a surface) so that the total energy dissipation at failure is incorrectly indicated to be zero (cf. review by Bazant, 1986). These difficulties are circumvented if strain-softening in finite element models is introduced through fracture concepts in one of two forms: (1) the crack band model and (2) the nonlocal model. The latter requires a finer mesh but allows better resolution of localized strain fields.

3.2.1 Crack Band Model

The basic idea of the crack band model, which was proposed by Bazant (1976), is:

- (1) to characterize the material behavior in the fracture process zone in a smeared manner through a strain-softening constitutive relation, and
- (2) to impose a fixed width w_c of the front of the strain-softening zone (crack band), representing a material property.

The imposition of constant w_c is required in order to avoid spurious mesh sensitivity and achieve objectivity, assuring that the energy dissipation due to fracture per unit length (and unit width) is a constant, equal to the fracture energy of the material, G_f . In keeping with the classical approach to smeared cracking, the detailed formulation of the crack band model (Bazant and Cedolin, 1979, 1980, 1983) first employed a sudden stress drop instead

of a gradual softening. Later comparisons with numerous test results, however, indicated that the reduction of stress to zero must be gradual, thus creating a relatively long fracture process zone ahead of the front of the fracture (Fig. 3.2e). This formulation, which was given in detail by Bazant (1982) and Bazant and Oh (1983a) has been shown to agree with essentially all the basic fracture test data available in the literature, particularly those on the effect of specimen size on the maximum load, the R-curve (see Chapter 4), and the differences between various specimen geometries. At the same time it was shown that the crack band model and Hillerborg's fictitious crack model give essentially the same results, except when closely spaced parallel cracks occur. (Thus, the extensive experimental justification of the crack band model also indirectly provided justification for the fictitious crack model.)

Softening is caused by fracturing strain, ϵ^f , which is superimposed on the elastic strain. Assuming all the cracks to be parallel and smeared (continuously distributed), and choosing axis x_2 to be normal to the cracks, we have in two dimensions ($x_1 - x_2$ plane):

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & 0 \\ C_{2211} & C_{2222} & 0 \\ 0 & 0 & C_{1212}/\beta \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} 0 \\ \epsilon^f \\ 0 \end{Bmatrix} \quad (3.3)$$

The column matrices on the left hand and right hand sides consist of the strain and stress components ($\gamma = 2\epsilon_{12}$ = shear angle), $C_{1111}, C_{1122} = C_{2211}, C_{2222}$ and C_{1212} are the elastic compliances, and β is an empirical factor, ($0 < \beta \leq 1$), called the shear retention factor (introduced by Suidan and Schnobrich, 1972, and Yuzugullu and Schnobrich, 1973, Phillips, 1973, and Phillips and Zienkiewicz, 1976). If the material is isotropic, $C_{1111} = C_{2222} = 1/E'$, $C_{1122} = C_{2211} = \nu'/E'$, $C_{1212} = 2(1+\nu)/E$, in which $E' = E$, $\nu' = \nu$ for plane stress, and $E' = E/(1 - \nu)$, $\nu' = \nu/(1 - \nu)$ for plane strain. The fracturing strain can be incorporated into the compliance, in which the expression $\epsilon = [\omega C_{2222}/(1 - \omega)]\sigma_{22}$ has been introduced; ω can be regarded as damage and can be considered to be a function of the strain normal to the crack, ϵ_{22} , or the maximum principal strain; $\omega/(1 - \omega) = \phi(\epsilon_{22})$. Initially we have $w = 0$ (no damage), and complete damage (continuous fracture) occurs for $w = 1$. Always $0 \leq w \leq 1$. The fracture energy is obtained as

$$G_f = w_c \int_0^\infty \sigma_{22} d\epsilon^f \quad (3.4)$$

The fully cracked state is obtained for $w \rightarrow 1$. It was shown (Bazant and Oh, 1983a) that if Eq. 3.4 is inverted and the limit of the ensuing stiffness matrix for $w \rightarrow 0$ is calculated, the result coincides with the well-known stiffness matrix for a material that is cracked unidirectionally and continuously, as introduced by Rashid (1968).

Eq. 3.4 reflects only a very special form of damage. In general, damage needs to be introduced through the formulation of the internal variable theory (for concrete, see e.g. Pijaudier-Cabot and Mazars, 1989, and Pijaudier-Cabot and Bazant, 1988).

The uniaxial softening stress-strain relation underlying the crack band model is characterized by function $\phi(\epsilon_{22})$, which defines damage w . The results are as sensitive to the shape of the softening stress-strain relation as are those for the fictitious crack model. It appears

that the simple formula $\phi(\epsilon) = (E/f'_t) \exp a(\epsilon - \epsilon_p)$, with empirical constants a and ϵ_p , is generally adequate. A straight line softening, i.e., a triangular stress-strain diagram, has also been used successfully.

There are two simple variants to the crack band model. The original one (Bazant and Oh, 1983a) presumes that the smeared parallel cracks start to form in the direction normal to the maximum principal stress but subsequently the crack orientation is fixed in the material even if the principal stress direction rotates. More recent research (e.g., Gupta and Akbar, 1984; Cope, 1984) seems to indicate that it is better to assume that the crack orientation rotates with the direction of the maximum principal stress, which means that shear stresses on the crack plane can never arise. For this variant, the general triaxial stress-strain relation for the microcracked material can be written in the form (Bazant and Lin, 1988b):

$$\epsilon_{ij} = \left(C_{ijk m} + \frac{\omega}{(1 - \omega)E'} n_i n_j n_k n_m \right) \sigma_{km} \quad (3.5)$$

in which n_i = direction cosines of the current maximum principal stress direction (repeated tensorial subscripts imply summation) and $C_{ijk m}$ is the elastic (undamaged) compliance tensor.

When the principal stress directions rotate significantly, the nonrotating crack method must be generalized to allow for the formation of secondary and tertiary cracking of different orientations. Such multidirectional smeared cracking models were especially perfected by de Borst (1984).

In reality, the microcracks prior to the final continuous fracture are distributed over all orientations, with different frequencies for various orientations. This feature is captured by the microplane model (Bazant and Oh, 1985) which seems to be physically the most realistic as well as conceptually simplest model of damage due to cracking but is very demanding for computer time. In this model (which will be considered further in [Section 3.2.5](#)), the hypothesis that the cracking strain tensor is additive to the elastic strain tensor is abandoned and cracking is modelled as strain-softening separately on planes of all orientations (called microplanes), subject to the hypothesis that the strain components on each microplane are the resolved components of the (macroscopic) strain. The stresses after cracking on the planes of various orientations are not exactly in equilibrium, but overall equilibrium is enforced by using the principle of virtual work.

It may be pointed out that linear elastic stability analysis in which the microcracks are assumed to grow in a homogeneous elastic continuum indicates that the softening stress-displacement or stress-strain relation should exhibit a maximum displacement or strain (snapback), after which the stress suddenly drops to zero (Bazant, 1987b). But it is not yet known whether the prediction of snapback also results from models taking into account inhomogeneities, inelastic behavior, and friction.

The width of the crack band front can be assumed to be approximately $w_c \simeq 3d_a$ (d_a = maximum aggregate size). This conclusion was drawn on the basis of optimum fitting of numerous test data (Bazant and Oh, 1983a). However, the optimum was not sharp; w_c - values ranging from d_a to $6d_a$ gave almost equally good results, provided, of course, the

post-peak function $\phi(\epsilon_{22})$ was adjusted for each different value of w_c .

Once the shape of the softening stress-strain relation is fixed, the crack band model is fully characterized by three material parameters: f'_t , G and w_c (although the influence of the w_c -value is rather weak for situations with isolated cracks). By contrast, the fictitious crack model has only two basic parameters, f'_t and G_f . Why the extra parameter in the crack band model? The extra parameter, w_c , is important only in situations when there are parallel cracks (e.g., in the presence of tension reinforcement); then w_c basically determines the minimum possible crack spacing, as a material property. It should be pointed out that the fictitious crack model, because of its lack of the extra parameter w_c , can give results that are not objective in situations with parallel closely spaced cracks. (A certain length has also been defined as an additional material parameter in the fictitious crack model; however, in contrast to w_c , it is not an independent parameter.)

The finite element size $h = w_c$, required by the crack band model, may be too small in the case of very large structures. In that case, it is possible to enlarge the element size (i.e. use $h > w_c$) provided that the softening stress-strain relation is adjusted so as to assure the same energy dissipation, G_f . This is illustrated in Fig. 3.3. In view of the series coupling model, already discussed in connection with Fig. 1.8, the given stress-strain diagram OPA for the strain-softening crack band needs to be replaced, at increasing element size h , by diagrams OPB, OPC, OPD, etc., such that when the areas under any of these diagrams is multiplied by h the same fracture energy value G_f is obtained (Fig. 3.3c). In terms of the stress-displacement diagrams for length h of finite element, a change in h requires that the actual stress-displacement diagram 012 in Fig. 3.3b be replaced by diagrams 032, 042, 052, etc. These stress-displacement diagrams have the property that the area under them is constant, thus assuring constant fracture energy G_f .

One can, of course, also use elements with $h < w_c$ (say $0.1w_c$), provided the post-peak slope is decreased as shown by the $\sigma - \epsilon$ diagram OPF in Fig. 3.3c so that the $\sigma - \delta$ diagram 082 (Fig. 3.3b) would again have the same area as 012. In this case, the row of cracked elements can be narrower than the adjacent elements and the mesh may look as shown in Fig. 3.4a but with w_c replaced by h . Obviously, when $h \rightarrow 0$, the crack band model in the limit becomes identical to Hillerborg's model (which in this sense is a special case of the crack band model).

As the element size is increased, the softening slope gets steeper, until for a certain element size h_0 a vertical stress drop OPC or 42 is obtained. For a still larger element size, the diagram OPD or 052 would exhibit snapback, which would cause numerical difficulties. In that case, one may replace the snapback segment 52 of the element stress-displacement diagram by a vertical stress drop, 67 in Fig. 3.3b. The point of vertical drop is determined again from the condition that the area under the diagram 067 must be the same as the area under the diagram 012 or 042. This consideration indicates that the equivalent tensile strength f_{eq} of the large finite element of size $h > w_c$ is given by

$$f_{eq} = f'_t \sqrt{\frac{h_0}{h}}, \quad h_0 = \frac{2EG_f}{f_t'^2} = \frac{K_{Ic}^2}{f_t'^2} \quad (\text{for } h \geq h_0) \quad (3.6)$$

(e.g., Bazant and Cedolin, 1979, 1980) in which h_0 is the element size for which a vertical stress drop (diagram 042) is obtained. Note that the expression for h_0 is similar to Irwin's Eq. 2.10 for the size of the yielding zone.

The foregoing adjustments are ideally defined for square finite elements subjected to tension or shear (normal or parallel to the sides). However, extensions to non-square elements are possible. In this case, if the dimension of the element in each direction is about the same, one may again use Eq. 3.7 in which $h = \sqrt{A}$, where A = element area (in 2D).

Eq. 3.7 follows from the relation $h f_{eq}^2/2E = h_0 f_t'^2/2E = G_f$ where h represents either the actual width of the cracking element in the direction normal to the cracks, provided the element is square or rectangular, or the effective width of the element defined as $h = w_c A_{el}/A_b$ where A_{el}, A_b are the areas of the element and of the crack band within the element. For element width $h < h_0$, for which the post-peak slope is reduced but the strength limit is kept as f_t' , the stress-strain diagram may be expressed, according to the series model (Fig. 1.8), as follows:

$$\epsilon = \frac{w_c}{h} \epsilon_{soft} + \left(1 - \frac{w_c}{h}\right) \epsilon_{unl}, \quad h = w_c \frac{A_{el}}{A_b} \quad (\text{for } h < h_0) \quad (3.7)$$

$\epsilon_{soft}, \epsilon_{unl}$ = the strains corresponding to the same stress according to the strain-softening stress-strain diagram and an unloading stress-strain diagram starting from the peak stress point.

If larger elements need to be used, another possibility is to keep the size of the elements on the line of crack band constant, equal to size w_c , and enlarge all the remaining elements as shown in Fig. 3.4a. If the crack band is much thinner than the adjacent finite elements, the model, of course, becomes practically identical to the fictitious crack model.

In the case of crack propagation in an arbitrary direction, the crack band model as well as the fictitious crack model requires remeshing so that the boundary lines of the crack band or the crack line would conform to the interelement boundaries (for the fictitious crack model, these techniques were perfected by Ingraffea (1985), Ingraffea and Gerstle (1985), and Ingraffea and Saouma (1985). If the remeshing is not done, it is still possible to approximately represent fracture running in an arbitrary direction by allowing the crack band to have a zig-zag form as shown in Fig. 3.4b. Some adjustments of the fracture energy are then necessary to take into account the average width of such a zig-zag band (Bazant, 1985a).

Even with such adjustments, however, there is a certain bias imposed by the mesh orientation. Moreover, if shear stresses parallel to the overall crack band direction arise, the zig-zag band can introduce spurious shear locking of the opposite faces. These problems can be overcome either by introducing a nonlocal version of the model, described later, or by enriching the finite element either with a strain field that is discontinuous along the boundaries of an arbitrarily oriented band, as proposed by Droz (1987) for tension, (similar to the models reported by Ortiz et al., 1987; Leroy and Ortiz, 1989, and Belytschko et al., 1988 for shear bands) or with displacement field discontinuous along a line, as proposed by Dvorkin et al. (1989).

The crack band model offers the possibility of introducing the influence of nonsingular

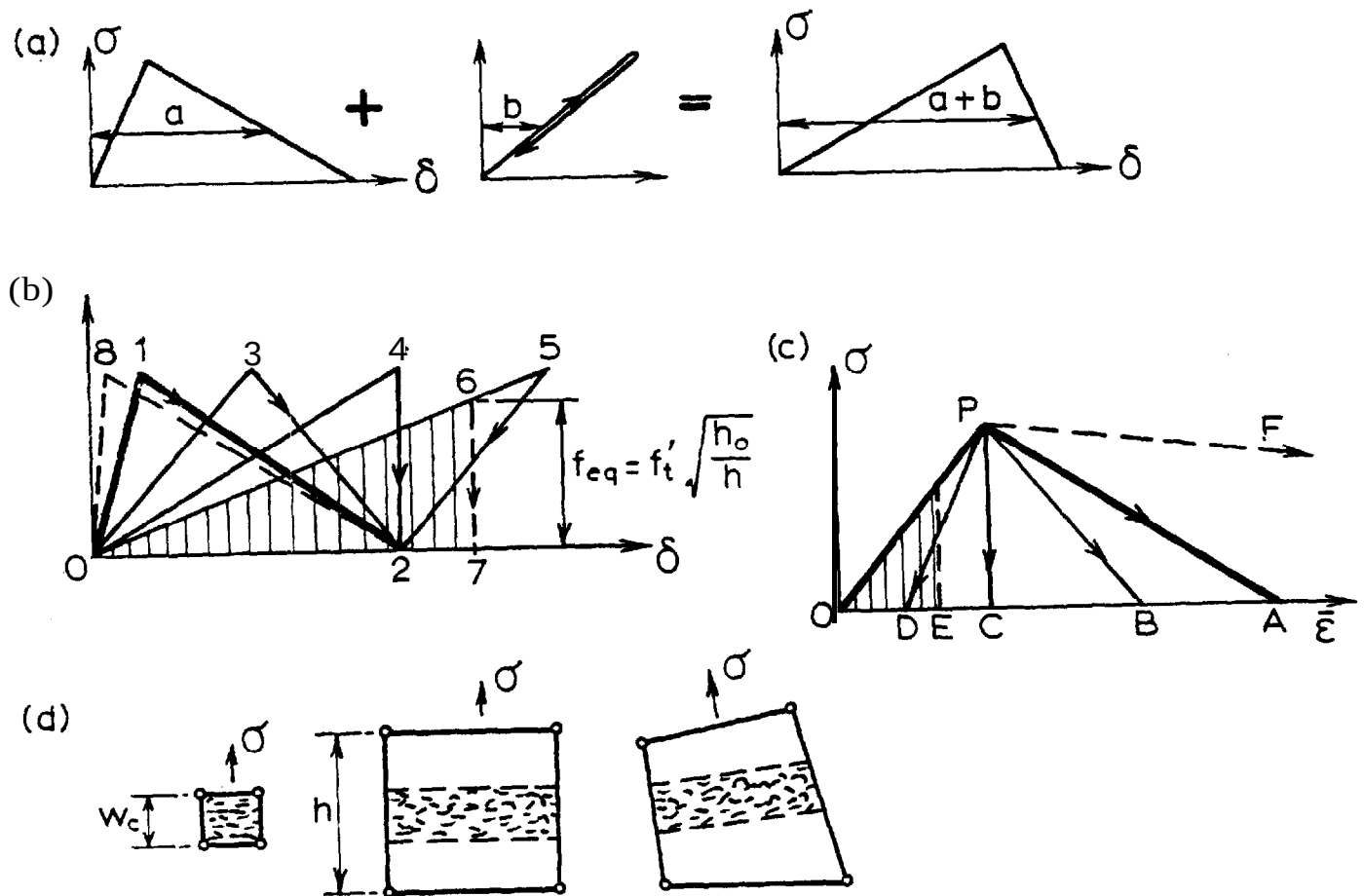


Fig.3.3 Effect of Size on the Steepness of Post-Peak Softening

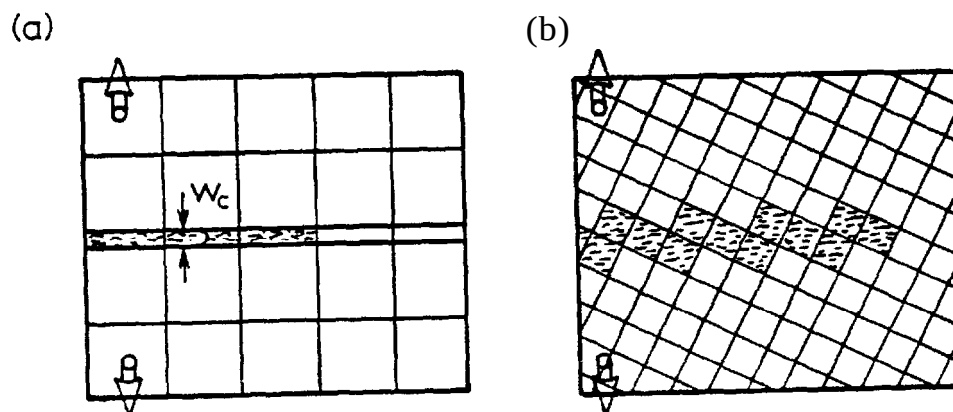


Fig.3.4 Meshes to Represent a Very Narrow Crack Band and a Zig-Zag Band



three-dimensional stresses on fracture. For example, it is known that a compressive stress σ_x parallel to the crack plane promotes cracking. Based on the known shape of the biaxial failure envelope of concrete, this influence (for $\sigma_x < 0$) may be taken into account by replacing f'_t with

$$f'_t = f_t^* (1 + \sigma_x / f'_c) \quad (3.8)$$

(Bazant, 1985a) where f'_c = compression strength. For $\sigma = -f'_c$ (compression failure), one must have $f_t^* = 0$, which agrees with Fig. 3.9. Note also that, in contrast, σ_x has no effect if linear elastic fracture mechanics is used; nor when line crack models with softening zones are used.

The crack band model has been implemented in some large general purpose finite element codes (e.g. DIANA, TEMP-STRESS, NONSAP).

3.2.2 Crack Layer Model

A variant of the crack band model is represented by Chudnovsky's crack layer model (Bessendorff and Chudnovsky, 1984; Chudnovsky, 1986). This model is more general than the crack band model in that the width of the crack band front is not considered to be a constant. Rather it is allowed to expand or shrink based on a material property. This property links a certain path-independent integral that characterizes the energy change due to the expansion to another path-independent integral around the crack front which characterizes the energy flow into the crack front due to crack propagation (Rice's J-integral).

This model, however, has not been developed for application in concrete and has not been compared to typical test data for concrete. The properties governing the band expansion might be unidentifiable from the experimental data at the current degree of sophistication of testing methods.

3.2.3 Composite Fracture Model

The composite fracture model is essentially an adaptation of the crack band model to finite elements of a larger size than the crack band width w_c , thus permitting large structures to be analyzed with a cruder mesh. The model assumes that a crack band of width w_c , which is a material property (or an equivalent line crack) is embedded within a larger finite element, approximately as illustrated in Fig. 3.3d. This idea, which is similar to Eq. 3.8, was proposed first for plastic shear bands by Pietruszczak and Mroz (1979) and later developed for tensile cracking of concrete by Willam et al. (1985, 1986). If the element size shrinks to w_c , this model becomes essentially identical to the crack band model. The crack band within the finite element is introduced by enriching the smooth distribution function with a discontinuous strain field function which characterizes cracking. The approach is similar to those by Droz (1987), Ortiz et al. (1987), Belytschko et al. (1988), Leroy and Ortiz (1989)

and Dvorkin et al. (1989) already mentioned at the end of [Sec. 3.2.1](#).

3.2.4 Nonlocal Continuum with Local Strain (Nonlocal Damage)

A nonlocal continuum is a continuum in which some state variable depends not only on the stresses or strains at the same point but also on the stress or strain field in the neighborhood of the point. As shown by Kröner (1967), Krumhansl (1968), Kunin (1968), Eringen and Edelen (1972) and others (cf. Bazant, 1986), the nonlocal concept is appropriate for statistically heterogeneous materials and follows logically from statistical micromechanical considerations. Recently, the nonlocal approach has been shown to be very effective for fracture mechanics of distributed cracking.

The nonlocal concept represents a general approach which makes it possible to use stress-strain relations with strain-softening. As mentioned before, if the strain-softening concept is introduced in the local continuum, softening zones localize to a vanishing volume, causing improper convergence on mesh refinement, spurious mesh sensitivity and physically incorrect predictions such as failure with vanishing energy dissipation as the mesh size is refined to zero. To avoid such behavior, the computational model must include some mathematical device that limits the localization (Bazant, Belytschko and Chang, 1984). The simplest, albeit crudest, way to limit localization is to impose a fixed size, w_c , on the frontal finite element, as done in the crack band model (unless the post-peak stress-strain relation is artificially modified). However, this approach makes it impossible to resolve the field in the fracture process zone itself, or determine how the fracture process zone width could vary during the fracture growth (no modification of stress-strain relations could be permitted for those purposes).

A more general way to limit localization, in order to cope with such problems, is the nonlocal continuum. The concept may be introduced in various forms, such as averaging of strains or strain-related quantities over a neighborhood of a point (Bazant, 1984b; Bazant, Belytschko and Chang, 1984) or introduction of first and second spatial derivatives of strains or other quantities into the constitutive relation (e.g., Schreyer and Chen, 1986; and for the Cosserat medium formulation for rocks by Sulem and Vardoulakis, 1988). The spatial derivative formulation can be obtained by expanding the spatial averaging integral in Taylor series. Only the spatial averaging approach has so far been worked out in detail for the multidimensional analysis of fracture of materials such as concrete.

In the early formulation, spatial averaging was applied through the total strains used in the strain-softening constitutive relation. However, that approach, which resulted in a so-called imbricate continuum model (Bazant 1984b) proved to be relatively cumbersome and have some computationally inconvenient properties. Recently (Bazant and Pijaudier-Cabot, 1987, 1988a; Pijaudier-Cabot and Bazant, 1987; Bazant and Lin, 1988a, 1988b), it was found that a more effective approach is to consider the elastic strains as local (no averaging), and apply the nonlocal averaging only to the softening part of the strains which represent distributed microcracking. This new approach has the advantage that the continuum field equations of equilibrium as well as the boundary conditions remain the same as for the



classical local continuum, and that there is no possibility of zero-energy instability modes (previously shown to exist for fully nonlocal models). It has also been mathematically proven that the averaging of the fracturing part of the strains is sufficient to prevent localization of strain-softening damage into a zone of zero volume. The unloading-reloading behavior is always local in this approach.

The spatial averaging operator, denoted by an overbar, may be defined in terms of strain ϵ_{11} as follows:

$$\bar{\epsilon}_{11}(\mathbf{x}) = \frac{1}{V_r(\mathbf{x})} \int_V \alpha(\mathbf{s} - \mathbf{x}) \langle \epsilon_{11}(\mathbf{s}) \rangle dV = \int_V \alpha'(\mathbf{x} - \mathbf{s}) \langle \epsilon_{11}(\mathbf{s}) \rangle dV \quad (3.9)$$

in which

$$V_r(\mathbf{x}) = \int_V \alpha'(\mathbf{s} - \mathbf{x}) dV, \quad \alpha(\mathbf{x} - \mathbf{s}) = \frac{\alpha(\mathbf{s} - \mathbf{x})}{V_r(\mathbf{x})} \quad (3.10)$$

$$\langle \epsilon_{11} \rangle = \epsilon_{11} \text{ if } \epsilon_{11} > 0, \quad \langle \epsilon_{11} \rangle = 0 \text{ if } \epsilon_{11} \leq 0 \quad (3.11)$$

Here V = volume of the structure, $\mathbf{x}$ = coordinate vector, $\mathbf{s}$ = coordinate vectors of adjacent points, α = given weighting function of the distance $|\mathbf{s} - \mathbf{x}|$, considered to be a material property (Fig. 3.5). V_r has approximately, but not exactly, the same meaning as the representative volume in the statistical theory of heterogeneous materials.

The weighting function could be defined as uniform ($\alpha = 1$) over volume V_r representing a circle in two dimensions, a bar segment in one dimension, or a sphere in three dimensions, with a zero value outside V_r . However, it has been experienced that the calculations converge better if the weighting function is smooth. One possible choice (Bazant and Pijaudier-Cabot, 1987,1988a; Bazant and Lin, 1988b) is a Gaussian distribution function, which, however, has nonzero values over the entire structure. It seems preferable to use a polynomial bell-shaped function which is exactly zero beyond a certain distance, e.g. (Fig. 3.5)

$$\alpha(\mathbf{s} - \mathbf{x}) = \left(1 - \frac{|\mathbf{s} - \mathbf{x}|}{k\ell}\right)^2 \text{ if } |\mathbf{s} - \mathbf{x}| < k\ell, \text{ otherwise } \alpha(\mathbf{s} - \mathbf{x}) = 0 \quad (3.12)$$

Here k is a normalizing constant determined so that the integral of α over a line segment, or a circle, or a sphere of size $k\ell$ would be the same as the integral of $\alpha = 1$ over the same region; $k = 15/16 = 0.9375$ for 1D, $(3/4)^{1/2} = 0.9086$ for 2D, $k = (105/192)^{1/3} = 0.8178$ for 3D. The parameter ℓ is the so-called characteristic length of the nonlocal continuum, which is a material property that defines the size of the averaging volume and is determined by the size of the inhomogeneities in the microstructure. For one quite typical concrete, measurements (Bazant and Pijaudier-Cabot, 1988b) have shown that $\ell = 2.7d_a$ where d_a = maximum aggregate size (Sec. 6.9).

If point $\mathbf{x}$ is too close to the surface of the body, part of the averaging zone protrudes beyond the surface. This is handled by deleting the protruding part from the integration region. This deletion causes $V_r(\mathbf{x})$ to depend on location $\mathbf{x}$.

In finite element calculations, the integrals in Eq. 3.10 and 3.11a are approximately evaluated as finite sums over all the integration points of all the elements in the structure.

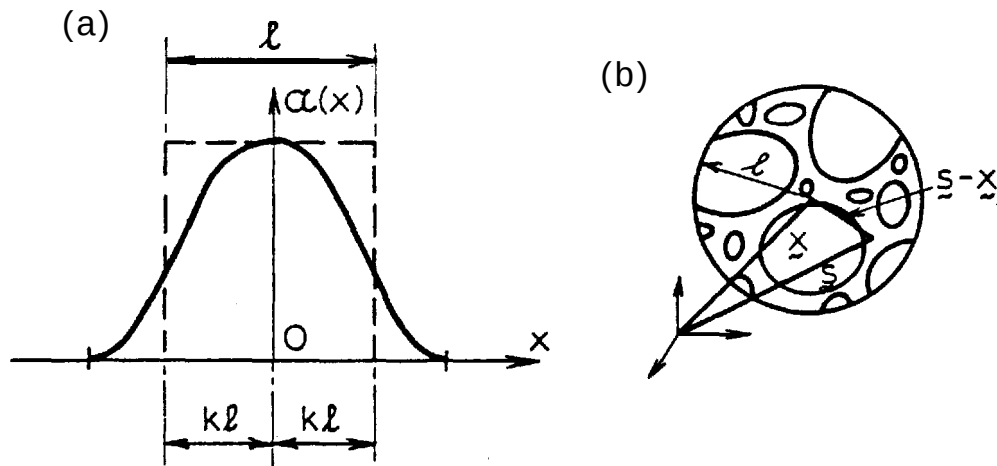


Fig.3.5 Weighting Function for Non-Local Continuum Theory and Representative Volume of the Material

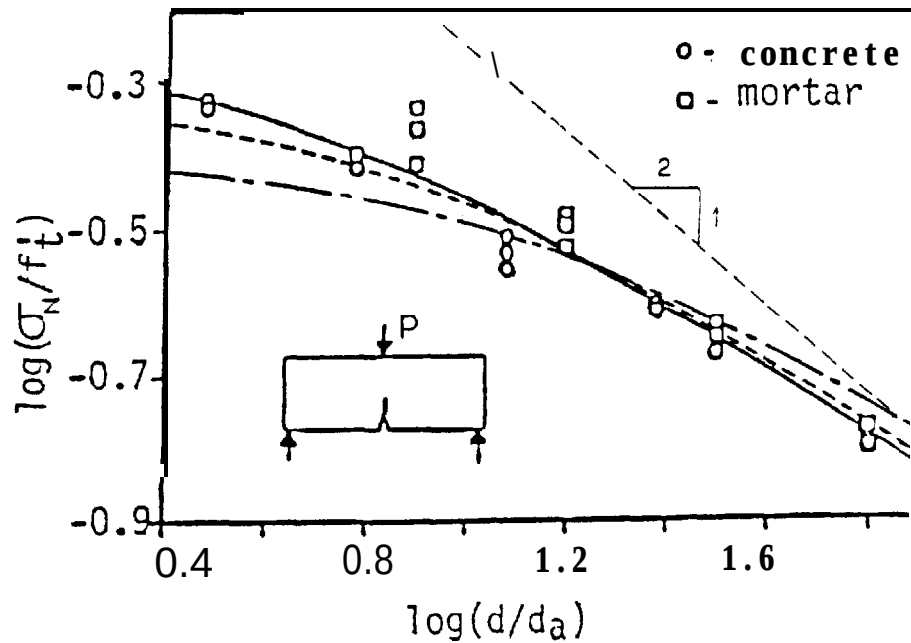


Fig.3.6 Size Effect Calculated by Non-Local Continuum Theory (Bazant and Lin, 1988B) for Exponential [dashed curve) and Linear (dash-dot curve) Tensile Strain Softening Curves, Compared with Size Effect Law of Bazant (Eq. 1.4, solid curve) and with test results of Bazant and Pfeiffer (1987) (data points)

However, the integration points whose distances from point $\mathbf{x}$ exceed $k\ell$ may be omitted from the sum since for them $\alpha = 0$.

Ideally, the form of function $\alpha(\mathbf{s} - \mathbf{x})$ should be derivable from micromechanical modeling of the microstructure with aggregates and microcracks, same as the form of function $\phi(\epsilon)$ in Eq. 3.4 and 3.6. However, this goal is too difficult at present, while at the same time the precise form of function $\alpha(\mathbf{s} - \mathbf{x})$ has only little influence on computation results (this fact makes identification of function α from test data problematic). What is important is the value of ℓ (provided the body is not too large compared to ℓ).

The simplest nonlocal constitutive model, appropriate for unidirectional cracking, is obtained by generalizing the crack band model (Bazant and Lin, 1988b). One alternative is to assume the crack direction to be fixed at the time the cracks start to form. In this case, the nonlocal generalization is obtained by replacing ω in Eq. 3.4 with nonlocal damage $\bar{\omega}$, such that

$$(1 - \bar{\omega})^{-1} = \varphi(\bar{\epsilon}_I) \quad (3.13)$$

where $\bar{\epsilon}_I$ is the maximum nonlocal principal strain calculated from the nonlocal (averaged) strain tensor $\bar{\epsilon}_{ij}$.

The second alternative of the crack band model, in which the cracking direction is assumed to rotate with the maximum principal stress, is obtained by replacing ω with $\bar{\omega}$ in Eq. 3.6.

It has been verified (Bazant and Lin, 1988b) that this nonlocal generalization of the crack band model permits the use of finite elements of any size less than ℓ , without causing spurious mesh sensitivity and improper convergence. For element sizes no smaller than ℓ , the nonlocal model becomes identical to the crack band model, and the stress-strain relation for the finite element which undergoes cracking must be adjusted according to the element size, as already explained.

It has also been shown that the nonlocal smeared cracking model provides an excellent fit of various fracture test data. See e.g., Fig. 3.6, in which the test results from Bazant and Pfeiffer (1987) are compared with nonlocal finite element results of Bazant and Lin (1988b) (dashed and dash-dot curves) and with the size effect law (Eq. 1.4, solid curve) (the dashed curve corresponds to a softening tensile stress-strain relation in an exponential form, which fits the data better than the dash-dot curve which corresponds to a linear stress-strain relation; still better fits could probably be obtained with other shapes of this relation). From Fig. 3.6 it should be especially noted that a nonlocal finite element code exhibits the correct transitional size effect of nonlinear fracture mechanics, as approximately described by the size effect law proposed by Bazant. By contrast, the local finite element analysis exhibits no size effect, as observed on geometrically similar specimens of different sizes, with similar meshes.

One practical advantage of the nonlocal formulation is that if the finite element sizes are less than about 1/3 of the characteristic length ℓ , then there is no directional bias of the mesh with regard to crack propagation. This has been verified (Bazant and Lin, 1988b) by analyzing the fracture of the same rectangular specimen with an aligned square mesh and a slanted square mesh. Thus, the nonlocal model can be used in general fracture situations.

An interesting, and potentially useful property of the nonlocal formulation is that the width of the cracking zone is not fixed but varies during the course of loading. The consequence is that the nonlocal fracture model does not correspond to a unique stress-displacement diagram for the line crack model (fictitious crack). This fact might explain why it has proved rather difficult to obtain unique values of the fracture energy on the basis of the fictitious crack model.

It should also be noted that a large finite element program for the nonlocal fracture model usually runs faster than the corresponding local program, despite the fact that additional computer time is needed to calculate the spatial averages. The reason apparently is that the nonlocal averaging stabilizes the response, thus causing the iterations in the loading steps to converge faster.

3.2.5 Multidirectional Cracking and Nontensile Fracture

For concrete structures subjected to complicated loading histories, it may be necessary to analyze fracture taking into account the existence of multidirectional cracking (Bazant 1983; de Borst, 1984, 1987a, 1987b; de Borst and Nauta, 1985; Ottosen and Dahlblom, 1986; Willam et al., 1987; Rots, 1988; Rots and Blaauwendraad, 1989; Crisfield and Wills, 1989). An effective approach to such behavior appears to be the microplane model, which was already mentioned below Eq. 3.6 in connection with the damage law. Instead of a macroscopic stress-strain relation, the material behavior in the microplane model is characterized independently on planes of various orientations, called the microplanes. In practice, only a certain number of discrete microplanes is considered, according to a numerical integration formula for spatial directions. The macroscopic response is determined as a certain average of the responses from the individual microplanes, obtained according to the principle of virtual work. This model, whose basic concept is similar to the slip theory of plasticity (Taylor, 1938; Batdorf and Budianski, 1949) was first developed in a local form (Bazant and Oh, 1985). It was also shown that this model can represent well not only the existing test data on tensile fracture but also those on crack shear, including the effects of the transverse normal stress and shear-induced expansion across the crack (Bazant and Gambarova, 1984). Recently (Bazant and Ozbolt, 1989), the microplane model has been generalized to a nonlocal form.

There are other types of fracture which need to be modeled for concrete structures. A new theory will have to be developed for Brazilian tensile splitting fracture of very large cylinders, which cannot be adequately analyzed with the existing tensile fracture models. On the other hand, shear fracture seems, at least partially, amenable to analysis by the existing tensile cracking models. In those models, the shear fracture is described as a band of inclined microcracks, governed by a softening stress-strain relation.

3.3 Stress-Displacement vs. Stress-Strain Softening Relations

There has been an incessant debate on the relative merits and deficiencies of the line

crack and crack band representations of fracture. There are three viewpoints to mention:

3.3.1 Isolated Cracks: Moot Point Computationally

First, one must realize that the line crack model (i.e. the fictitious crack model of Hillerborg) and the crack band model yield about the same results (with differences of about 1%) if the stress displacement relation in the first model and the stress-strain relation in the second model are calibrated in such a way that

$$\delta_c = w_c \epsilon^f \quad (3.14)$$

i.e., the crack opening displacement is taken as the fracture strain, ϵ^f , accumulated over the width w_c of the crack band. This equivalence is already suggested by the fact that, in linear elastic fracture mechanics, an interelement line crack and a single-element wide crack band with a symmetry line that coincides with the line crack give essentially the same results in the calculation of the energy release rate, with differences of only about 1%, provided the element sizes near the fracture front do not exceed about 1/10 of the cross section dimension (Bazant and Cedolin, 1979). In fact, the difference between these two methods of calculation of the energy release rate for linear elastic fracture mechanics is not larger than the error of approximating the continuum by finite elements in the first place.

Therefore, the question “line crack or a crack band?” is moot from the viewpoint of approximating reality by computational modeling. The only point worthy of debate is computational effectiveness and convenience. But even in that regard, the two models appear to be equal.

Various numerical modeling aspects, however, deserve attention. Leibengood, Darwin and Dodds (1986) showed that the results for stress-displacement and stress-strain relations match closely if the cracking directions at the integration points within the finite elements are forced to be parallel to each other (and to the actual crack). If the cracking direction is not known a priori, as in the general case, it is difficult to achieve this parallelness. But then, if cracks are allowed to form at different orientations at each integration point, the response of the finite element model with smeared cracking is somewhat stiffer than that with discrete interelement cracking, even if the element sides are parallel to the true crack.

3.3.2 Parallel Cracks: Third Parameter

As already mentioned, if the shape of the tensile softening curve is fixed, then the line crack (fictitious crack) model is defined by two parameters, G_f and f_t' , while the crack band model is defined by three parameters G_f , f_t' and w_c . For the fictitious crack model, too, a length parameter (called the “characteristic length”) has been defined; $\ell_0 = EG_f/f_t'^2$ (which is similar to Eq. 3.7 as well as Irwin’s Eq. 2.10). However, this parameter is a derived parameter, not an independent one, while w_c is an independent parameter. Why this difference?

For isolated cracks, it turns out that the effect of w_c on the results is almost negligible,