

Finite Element Analysis of Fracture in Concrete Structures

Reported by ACI Committee 446

Vellore Gopalaratnam¹
Chairman

Walter Gerstle^{1,2}
Secretary and Subcommittee Co-Chairman

David Darwin^{1,2}
Subcommittee Co-Chairman

Farhad Ansari
Zdenek P. Bazant^{1,3}
Oral Buyukozturk^{1,3}
Ignacio Carol¹
Rolf Eligehausen
Shu-Jin Fang^{1,3}
Ravindra Gettu
Toshiaki Hasegawa
Neil M. Hawkins
Anthony R. Ingraffea^{1,3}
Jeremy Isenberg

Yeou-Sheng Jenq
Mohammad T. Kazemi
Neven Krstulovic
Victor C. Li
Jacky Mazars
Steven L. McCabe^{1,3}
Christian Meyer
Hirozo Mihashi¹
Richard A. Miller
Sidney Mindess
C. Dean Norman

Philip C. Perdikaris
Gilles Pijaudier-Cabot
Victor E. Saouma^{1,3}
Wimal Suaris
Stuart E. Swartz^{1,3}
Tianxi Tang
Tatsuya Tsubaki
Cumaraswamy Vipulanandan
Methi Wecharatana
Yunping Xi
Former member: Sheng-Taur Mao^{1,3}

¹Members of the Subcommittee that prepared this report

²Principal authors

³Contributing authors

Fracture is an important mode of deformation and damage in both plain and reinforced concrete structures. To accurately predict fracture behavior, it is often necessary to use finite element analysis. This report describes the state-of-the-art of finite element analysis of fracture in concrete. The two dominant techniques used in finite element modeling of fracture—the discrete and the smeared approaches—are described. Examples of finite element analysis of cracking and fracture of plain and reinforced concrete structures are summarized. While almost all concrete structures crack, some structures are fracture sensitive, while others are not. Therefore, in some instances it is necessary to use a consistent and accurate fracture model in the finite element analysis of a structure. For the most general and predictive finite element analyses, it is desirable to allow cracking to be represented using both the discrete and the smeared approaches.

Keywords: concrete; crack; cracking; damage; discrete cracking; finite element analysis; fracture; fracture mechanics; reinforced concrete; size effect; smeared cracking; structures.

ACI Committee Reports, Guides, Standard Practices, and Commentaries are intended for guidance in planning, designing, executing, and inspecting construction. This document is intended for the use of individuals who are competent to evaluate the significance and limitations of its content and recommendations and who will accept responsibility for the application of the material it contains. The American Concrete Institute disclaims any and all responsibility for the stated principles. The Institute shall not be liable for any loss or damage arising therefrom.

Reference to this document shall not be made in contract documents. If items found in this document are desired by the Architect/Engineer to be a part of the contract documents, they shall be restated in mandatory language for incorporation by the Architect/Engineer.

CONTENTS

Chapter 1—Introduction, p. 446.3R-2

- 1.1—Background
- 1.2—Scope of report

Chapter 2—Discrete crack models, p. 446.3R-3

- 2.1—Historical background
- 2.2—Linear elastic fracture mechanics (LEFM)
- 2.3—Fictitious crack model (FCM)
- 2.4—Automatic remeshing algorithms

Chapter 3—Smeared crack models, p. 446.3R-11

- 3.1—Reasons for using smeared crack models
- 3.2—Types of localization limiters

Chapter 4—Literature review of FEM fracture mechanics analyses, p. 446.3R-14

- 4.1—General
- 4.2—Plain concrete
- 4.3—Reinforced concrete
- 4.4—Closure

ACI 446.3R-97 became effective October 16, 1997.

Copyright © 1998, American Concrete Institute.

All rights reserved including rights of reproduction and use in any form or by any means, including the making of copies by any photo process, or by electronic or mechanical device, printed, written, or oral, or recording for sound or visual reproduction or for use in any knowledge or retrieval system or database, without permission in writing is obtained from the copyright proprietor.

Chapter 5—Conclusions, p. 446.3R-24

- 5.1—General summary
- 5.2—Future work

Chapter 6—References, p. 446.3R-25

- 6.1—Recommended references
- 6.2—Cited references

Appendix—Glossary, p. 446.3R-30**CHAPTER 1—INTRODUCTION**

In this report, the state-of-the-art in finite element modeling of concrete is viewed from a fracture mechanics perspective. Although finite element methods for modeling fracture are undergoing considerable change, the reader is presented with a snapshot of current thinking and selected literature on the topic.

1.1—Background

As early as the turn of the 19th century, engineers realized that certain aspects of concrete behavior could not be described or predicted based upon classical strength of materials techniques. As the discipline of fracture mechanics has developed over the course of this century (and indeed, is still developing), it has become clear that a correct analysis of many concrete structures must include the ideas of fracture mechanics.

The need to apply fracture mechanics results from the fact that classical mechanics of materials techniques are inadequate to handle cases in which severe discontinuities, such as cracks, exist in a material. For example, in a tension field, the stress at the tip of a crack tends to infinity if the material is assumed to be elastic. Since no material can sustain infinite stress, a region of inelastic behavior must therefore surround the crack tip. Classical techniques cannot, however, handle such complex phenomena. The discipline of fracture mechanics was developed to provide techniques for predicting crack propagation behavior.

Westergaard (1934) appears to have been the first to apply the concepts of fracture mechanics to concrete beams. With the advent of computers in the 1940s, and the subsequent rapid development of the finite element method (FEM) in the 1950s, it did not take long before engineers attempted to analyze concrete structures using the FEM (Clough 1962, Ngo and Scordelis 1967, Nilson 1968, Rashid 1968, Cervenka and Gerstle 1971, Cervenka and Gerstle 1972). However, even with the power of the FEM, engineers faced certain problems in trying to model concrete structures. It became apparent that concrete structures usually do not behave in a way consistent with the assumptions of classical continuum mechanics (Bažant 1976).

Fortunately, the FEM is sufficiently general that it can model continuum mechanical phenomena as well as discrete phenomena (such as cracks and interfaces). Engineers performing finite element analysis of reinforced concrete structures over the past thirty years have gradually begun to recognize the importance of discrete mechanical behavior of concrete. Fracture mechanics may be defined as that set of ideas or concepts that describe the transition from continuous to discrete behavior as separation of a material occurs. The two main approaches used in FEM analysis to represent cracking in concrete structures have been to 1) model cracks discretely (discrete crack approach); and 2) model cracks in

a smeared fashion by applying an equivalent theory of continuum mechanics (smeared crack approach). A third approach involves modeling the heterogeneous constituents of concrete at the size scale of the aggregate (discrete particle approach) (Bažant et al. 1990).

Kaplan (1961) seems to have been the first to have performed physical experiments regarding the fracture mechanics of concrete structures. He applied the Griffith (1920) fracture theory (modified in the middle of this century to become the theory of linear elastic fracture mechanics, or LEFM) to evaluate experiments on concrete beams with crack-simulating notches. Kaplan concluded, with some reservations, that the Griffith concept (of a critical potential energy release rate or critical stress intensity factor being a condition for crack propagation) is applicable to concrete. His reservations seem to have been justified, since more recently it has been demonstrated that LEFM is not applicable to typical concrete structures. In 1976, Hillerborg, Modeer and Petersson studied the fracture process zone (FPZ) in front of a crack in a concrete structure, and found that it is long and narrow. This led to the development of the fictitious crack model (FCM) (Hillerborg et al. 1976), which is one of the simplest nonlinear discrete fracture mechanics models applicable to concrete structures.

Finite element analysis was first applied to the cracking of concrete structures by Clough (1962) and Scordelis and his coworkers Nilson and Ngo (Nilson 1967, Ngo and Scordelis 1967, Nilson 1968). Ngo and Scordelis (1967) modeled discrete cracks, as shown in Fig. 1.1, but did not address the problem of crack propagation. Nilson (1967) modeled progressive discrete cracking, not by using fracture mechanics techniques, but rather by using a strength-based criterion. The stress singularity that occurs at the crack tip was not modeled. Thus, since the maximum calculated stress near the tip of a crack depends upon element size, the results were mesh-dependent (nonobjective). Since then, much of the research and development in discrete numerical modeling of fracture of concrete structures has been carried out by Ingraffea and his coworkers (Ingraffea 1977, Ingraffea and Manu 1980, Saouma 1981, Gerstle 1982, Ingraffea 1983, Gerstle 1986, Wawrzynek and Ingraffea 1987, Swenson and Ingraffea 1988, Wawrzynek and Ingraffea 1989, Ingraffea 1990, Martha et al. 1991) and by Hillerborg and coworkers (Hillerborg et al. 1976, Petersson 1981, Gustafsson 1985).

Another important approach to modeling of fracture in concrete structures is called the smeared crack model (Rashid 1968). In the smeared crack model, cracks are modeled by changing the constitutive (stress-strain) relations of the solid continuum in the vicinity of the crack. This approach has been used by many investigators (Cervenka and Gerstle 1972, Darwin and Pecknold 1976, Bažant 1976, Meyer and Bathe 1982, Chen 1982, Balakrishnan and Murray 1988). Bažant (1976) seems to have been the first to realize that, because of its strain-softening nature, concrete cannot be modeled as a pure continuum. Zones of damage tend to localize to a size scale that is of the order of the size of the aggregate. Therefore, for concrete to be modeled as a continuum, account must be taken of the size of the heterogeneous structure of the material.

This implies that the maximum size of finite elements used to model strain softening behavior should be linked to the aggregate size. If the scale of the structure is small, this presents no particular problem. However, if the scale of the structure is large compared to the size of its internal structure (aggregate size), stress intensity factors (fundamental parameters in LEFM) may provide a more efficient method for modeling crack propagation than the smeared crack approach (Griffith 1920, Bažant 1976). Most structures of interest are of a size between these two extremes, and controversy currently exists as to which of these approaches (discrete fracture mechanics or smeared cracking continuum mechanics) is more effective. This report describes both the discrete and smeared cracking methods. These two approaches, however, are not mutually exclusive, as shown, for example, by Elices and Planas (1989).

When first used to model concrete structures, it was expected that the FEM could be used to solve many problems for which classical solutions were not available. However, even this powerful numerical tool has proven to be difficult to apply when the strength of a structure or structural element is controlled by cracking. When some of the early finite element analyses are studied critically in light of recent developments, they are found to be nonobjective or incorrect in terms of the current understanding of fracture mechanics, although many produced a close match with experimental results. It is now clear that any lack of success in these models was not due to a weakness in the FEM, but rather due to incorrect approaches used to model cracks. In many cases, success can be achieved only if the principles of fracture mechanics are brought to bear on the problem of cracking in plain and reinforced concrete. These techniques have not only proven to be powerful, but have begun to provide explanations for material behavior and predictions of structural response that have previously been poorly or incorrectly understood.

While some preliminary research has been performed in the finite element modeling of cracking in three-dimensional structures (Gerstle et al. 1987, Wawrzynek and Ingraffea 1987, Ingraffea 1990, Martha et al. 1991), the state-of-the-art in the fracture analysis of concrete structures seems currently to be generally limited to two-dimensional models of structures.

1.2—Scope of report

Several previous state-of-the-art reports and symposium proceedings discuss finite element modeling of concrete structures (ASCE Task Committee 1982, Elfgren 1989, *Computer-Aided* 1984, 1990, *Fracture Mechanics* 1989, Firrao 1990, van Mier et al. 1991, *Concrete Design* 1992, *Fracture* 1992, *Finite Element* 1993, *Computational Modeling* 1994, *Fracture and Damage* 1994, *Fracture* 1995). This report provides an overview of the topic, with emphasis on the application of fracture mechanics techniques. The two most commonly applied approaches to the FEM analysis of fracture in concrete structures are emphasized. The first approach, described in Chapter 2, is the discrete crack model. The second approach, described in Chapter 3, is the smeared crack model. Chapter 4 presents a review of the literature of applications of the finite element technique to problems involving cracking of concrete. Finally, some general

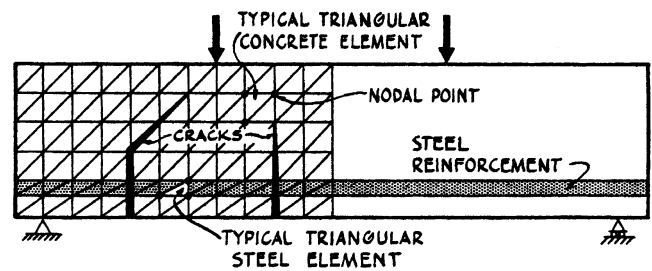


Fig. 1.1—The first finite element model of a cracked reinforced concrete beam (Ngo and Scordelis 1967).

conclusions and recommendations for future research are given in Chapter 5.

No attempt is made to summarize all of the literature in the area of FEM modeling of fracture in concrete. There are several thousand references dealing simultaneously with the FEM, fracture, and concrete. An effort is made to crystallize the confusing array of approaches. The most important approaches are described in detail sufficient to enable the reader to develop an overview of the field. References to the literature are provided so that the reader can obtain further details, as desired. The reader is referred to ACI 446.1R for an introduction to the basic concepts of fracture mechanics, with special emphasis on the application of the field to concrete.

CHAPTER 2—DISCRETE CRACK MODELS

A discrete crack model treats a crack as a geometrical entity. In the FEM, unless the crack path is known in advance, discrete cracks are usually modeled by altering the mesh to accommodate propagating cracks. In the past, this remeshing process has been a tedious and difficult job, relegated to the analyst. However, newer software techniques now enable the remeshing process, at least in two-dimensional problems, to be accomplished automatically by the computer. A zone of inelastic material behavior, called the fracture process zone (FPZ), exists at the tip of a discrete crack, in which the two sides of the crack may apply tractions to each other. These tractions are generally thought of as nonlinear functions of the relative displacements between the sides of the crack.

2.1—Historical background

Finite element modeling of discrete cracks in concrete beams was first attempted by Ngo and Scordelis (1967) by introducing cracks into the finite element mesh by separating elements along the crack trajectory, as shown in Fig. 1.1. They did not, however, attempt to model crack propagation. Had they done so, they would have found many problems, starting with the fact that the stresses at the tips of the cracks increase without bound as the element size is reduced, and no convergence (of crack tip stresses) to a solution would have been obtained. Also, in light of the findings of Hillerborg et al. (1976) that a crack in concrete has a gradually softening region of significant length at its tip, it was inaccurate to model cracks with traction-free surfaces. It is notable that Ngo and Scordelis also grappled with the theoretically difficult issue of connecting the reinforcing concrete with the concrete elements via “bond-link” elements.

Nilson (1967, 1968) was the first to consider a FEM model to represent *propagation* of discrete cracks in concrete structures. Quoting from his thesis (Nilson 1967):

The present analysis includes consideration of progressive cracking. The uncracked member is loaded incrementally until previously defined cracking criteria are exceeded at one or more locations in the member. Execution terminates, and the computer output is subjected to visual inspection. If the average value of the principal tensile stress in two adjacent elements exceeds the ultimate tensile strength of the concrete, then a crack is defined between those two elements along their common edge. This is done by establishing two disconnected nodal points at their common corner or corners where there formerly was only one. When the principal tension acts at an angle to the boundaries of the element, then the crack is defined along the side most nearly normal to the principal tension direction.

The newly defined member, with cracks (and perhaps partial bond failure), is then re-loaded from zero in a second loading stage, also incrementally applied to account for the nonlinearities involved. Once again the execution is terminated if cracking criteria are exceeded. The incremental extension of the crack is recorded, and the member loaded incrementally in the third stage, and so on. In this way, crack propagation may be studied and the extent of cracking at any stage of loading is obtained.

The problems associated with this approach to discrete crack propagation analysis are three-fold: (1) cracks in concrete structures of typical size scale develop gradually (Hillerborg et al. 1976), rather than abruptly; (2) the procedure forces the cracks to coincide with the predefined element boundaries; and (3) the energy dissipated upon crack propagation is unlikely to match that in the actual structure, resulting in a spurious solution.

In the 1970s, great strides were made in modeling of LEFM using the FEM. Chan, Tuba, and Wilson (1970) pointed out that a large number of triangular constant stress finite elements are required to obtain accurate stress intensity factor solutions using a displacement correlation technique (about 2000 degrees of freedom are required to obtain 5 percent accuracy in the stress intensity factor solution). At this time, singular finite elements had not yet been developed (singular elements exactly model the stress state at the tip of a crack). In their paper, Chan et al. pointed out that there were then three ways to obtain stress intensity factors from a finite element solution: (1) displacement correlation; (2) stress correlation; and (3) energy release rate methods (line integral or potential energy derivative approaches).

Wilson (1969) appears to have been the first to have developed a singular crack tip element. Shortly thereafter, Tracey (1971) developed a triangular singular crack tip finite element that required far fewer degrees of freedom than analysis with regular elements to obtain accurate stress intensity factor. At about the same time, Tong, Pian, and Lasry (1973) developed and experimented with hybrid singular crack tip elements (including stress-intensity factors, as well as displacement components, as degrees of freedom).

Jordan (1970) noticed that shifting the midside nodes along adjacent sides of an eight-noded quadrilateral toward the corner node by one-quarter of the element's side length caused the Jacobian of transformation to become zero at the corner node of the element. This led to the discovery by Henshell and Shaw (1975) and Barsoum (1976) that the shift allowed the singular stress field to be modeled exactly for an elastic material. Thus, standard quadratic element with midside nodes shifted to the quarter-points can be used as a $r^{-1/2}$ singularity element for modeling stresses at the tip of a crack in a linear elastic medium.

The virtual crack extension method for calculating Mode I stress intensity factors was developed independently by Hellen (1975) and Parks (1974). In this method, G , the rate of change of potential energy per unit crack extension, is calculated by a finite difference approach. This approach does not require the use of singular elements to obtain Mode I (opening mode) stress-intensity factors. Recently, it has been found that by decomposing the displacement field into symmetric and antisymmetric components with respect to the crack tip, the method may also be extended to calculate Mode II (sliding) and Mode III (tearing) energy release rates and stress-intensity factors (Sha and Yang 1990, Shumin and Xing 1990, Rahul Kumar 1992).

Having developed the capability to compute stress intensity factors using the FEM, the next big step was to model linear elastic crack *propagation* using fracture mechanics principles. This was started for concrete by Ingraffea (1977), and continued by Ingraffea and Manu (1980), Saouma (1981), Gerstle (1982, 1986), Wawrzynek and Ingraffea (1986), and Swenson and Ingraffea (1988). These attempts were primarily aimed at facilitating the process of discrete crack propagation through automatic crack trajectory computations and semi-automatic remeshing to allow discrete crack propagation to be modeled. Currently, the main technical difficulties involved in modeling of discrete LEFM crack propagation are in the 3D regime. In 2D applications, automatic propagation and remeshing algorithms have been reasonably successful and are improving. In three-dimensional modeling, automatic remeshing algorithms are on the verge of being sufficiently developed to model general crack propagation, and computers are just becoming powerful enough to accurately solve problems with complex geometries caused by the propagation of a number of discrete cracks.

Another development in discrete crack modeling of concrete structures has been the realization that LEFM does not apply to structural members of normal size, because the FPZ in concrete is relatively large compared to size of the member. This has led to the development of finite element modeling of nonlinear discrete fracture—usually as the implementation of the fictitious crack model (FCM) (Hillerborg et al. 1976), in which the crack is considered to be a strain softening zone modeled by cohesive nodal forces or by interface elements [first developed by Goodman, Taylor, and Brekke (1968)].

Finally, there appear to be situations in which even the FCM seems inadequate to model realistic concrete behavior in the FPZ. In this case, a smeared crack model, as described in Chapter 3, becomes necessary.

2.2—Linear elastic fracture mechanics (LEFM)

Linear elastic fracture mechanics (LEFM) is an important approach to the fracture modeling of concrete structures, even though it is only applicable to very large (say several meters in length) cracks. For cracks that are smaller than this, LEFM over-predicts the load at which the crack will propagate. To determine whether LEFM may be used or whether nonlinear fracture mechanics is necessary for a particular problem, one must determine the size of the steady state fracture process zone (FPZ) compared to the least dimension associated with the crack tip (ACI 446.1R). The FPZ size and the crack tip least dimension are discussed next.

The FPZ may be defined as the area surrounding a crack tip within which inelastic material behavior occurs. The FPZ size grows as load is applied to a crack, until it has developed to the point that the (traction-free) crack begins to propagate. If the size of the FPZ is small compared to other dimensions in the structure, then the assumptions of LEFM lead to the conclusion that the FPZ will exhibit nonchanging characteristics as the crack propagates. This is called the steady state FPZ. The size of the steady state FPZ depends only upon the material properties. In concrete, as opposed to metals, the FPZ can often be thought of as an interface separation phenomenon, with little accompanying volumetric damage. The characteristics of the steady state FPZ depend upon the aggregate size, shape and strength, and upon microstructural details of the particular concrete under consideration. The FPZ was first studied in detail by Hillerborg, Modeer, and Petersson (1976). The size of the FPZ depends on the model used in the study. For example, in the analysis carried out by Ingraffea and Gerstle (1985) for normal strength concrete, the steady state FPZ ranged from 6 in. (150 mm) to 3 ft (1 m) in length.

The least dimension (L.D.) associated with a crack tip is best defined with the aid of Fig. 2.1 (Gerstle and Abdalla 1990). The least dimension is used to calculate an approximate radius surrounding the crack tip within which the singular stress field can be guaranteed to dominate the solution. The least dimension can be defined as the distance from the crack tip to the nearest discontinuity that might cause a local disturbance in the stress field. Fig. 2.1(a) shows the case where the crack tip L.D. is controlled by the proximity to the crack tip of a free surface. Fig. 2.1(b) shows the case where the least dimension is the crack length itself. Fig. 2.1(c) shows the case where the least dimension is controlled by the crack tip passing close by a reinforcing bar. [Of course, if the reinforcement is considered as a smeared (rather than discrete) constituent of the reinforced concrete composite, then it need not be modeled discretely, and the constitutive relations and the FPZ must correspondingly include the effect of the smeared reinforcing bars.] Fig. 2.1(d) shows the case where the least dimension is controlled by the size of the ligament (the remaining uncracked dimension of the member). In Fig. 2.1(e), the least dimension is governed by a kink in the crack. Finally, Fig. 2.1(f) shows an example of the least dimension being controlled by the radius of curvature of the crack.

As explained in Chapter 2 of ACI 446.1R, one of the fundamental assumptions of LEFM is that the size of the FPZ is negligible (say, no more than one percent of the least dimension

associated with the crack tip). It is this assumption that allows for a theoretical stress distribution near the crack tip in linear elastic materials, in which the stress varies with $r^{-1/2}$, in which r is the distance from the crack tip. Stress-intensity factors K_I , K_{II} , and K_{III} are defined as the magnitudes of the singular stress fields for Mode I, Mode II, and Mode III cracks, respectively. If the FPZ is not small compared to the least dimension, then singular stress fields may not be assumed to exist, and consequently, K_I , K_{II} , and K_{III} are not defined for such a crack tip. In such a case, the FPZ must be modeled explicitly and a nonlinear fracture model is required.

As mentioned earlier, fracture process zones in concrete can be on the order of 1 ft (0.3 m) or more in length. For the great majority of concrete structures, least dimensions are less than several feet. Therefore, fracture in these types of structures must be modeled using nonlinear fracture mechanics. Only in very large concrete structures, for example, dams, is it possible to apply LEFM appropriately. For dams with large aggregate, possibly on the size scale of meters, LEFM may not be applicable because of the correspondingly larger size of the FPZ.

Even though it is recognized that LEFM is not applicable to typical concrete structures, it is appropriate to review the details of the finite element analysis of LEFM. Then, in Section 2.3, the finite element analysis of nonlinear discrete fracture mechanics will be presented.

2.2.1 Fracture criteria: K , G , mixed-mode models—Stress-intensity factors K_I , K_{II} , and K_{III} or energy release rates G_I , G_{II} , and G_{III} may be used in LEFM to predict crack equilibrium conditions and propagation trajectories. There are several theories that can be used to predict the direction of crack propagation. These include, for quasistatic problems, the maximum circumferential tensile stress theory (Erdogan and Sih 1963), the maximum energy release rate theory (Hussain et al. 1974), and the minimum strain energy density theory (Sih 1974). These theories all give practically the same crack trajectories and loads at which crack extension takes place, and therefore the theory of choice depends primarily upon convenience of implementation. Each of these theories may also be applied to dynamic fracture propagation problems (Swenson 1986). As in metals, cyclic fatigue crack propagation in concrete may be modeled with the Paris Model (Barsom and Rolfe 1987) in conjunction with the mixed-mode crack propagation theories just mentioned. However, it is rare that an unreinforced concrete structure is both (1) large enough to merit LEFM treatment and (2) subject to fatigue loading.

In most of the literature on discrete crack propagation in concrete structures, it has been considered necessary to model the stress singularity at a crack tip using singular elements. However, accurate results can also be obtained without modeling the stress singularity, but rather by calculating the energy release rates directly (Sha and Yang 1990, Rahul Kumar 1992). However, for a comprehensive treatment, we discuss modeling of stress singularities next.

2.2.2 FEM modeling of singularities and stress intensity factors—Special-purpose singular finite elements have been created with stress-intensity factors included explicitly as

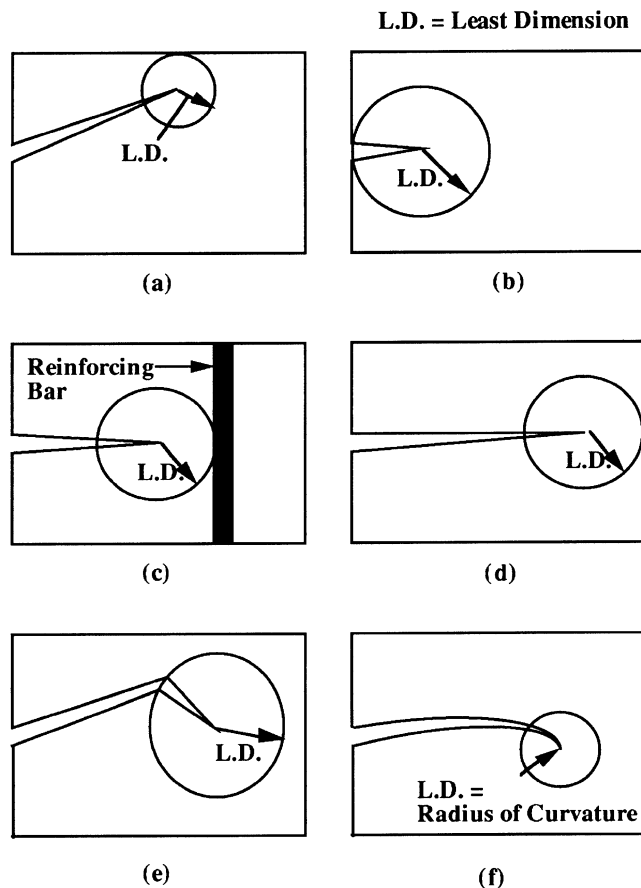


Fig. 2.1—Examples illustrating the concept of “least dimension (L.D.)” associated with a crack tip (Gerstle and Abdalla 1990).

degrees-of-freedom (Byskov 1970, Tong and Pian 1973, Atluri et al. 1975, Mau and Yang 1977). However, these are special-purpose hybrid elements that are not usually included in standard displacement-based finite element codes, and will not be discussed in further detail here. The most successful displacement-based elements are the Tracey element (Tracey 1971) and the quarterpoint quadratic triangular isoparametric element (Henshell and Shaw 1975, Barsoum 1976, Saouma 1981, Saouma and Schwemmer 1984). Most general purpose finite element codes unfortunately do not include the Tracey element, but they do include six noded triangular elements, which can then be used as singular quarterpoint crack tip elements.

After a finite element analysis has been completed, stress-intensity factors can be extracted by several approaches. The most accurate methods are the energy approaches: the J -integral, virtual crack extension, or stiffness derivative methods. However, these approaches are not as easy to apply for the case of mixed-mode crack propagation, and have been applied only rarely to three-dimensional problems (Shivakumar et al. 1988). Simpler to apply (for mixed-mode fracture mechanics) are the displacement correlation techniques. Because these techniques sample local displacements at various points, and correlate these with the theoretical displacement field associated with a crack tip, they are

generally not as accurate as the energy approaches, which use integrated information. The displacement correlation techniques are usually used only when singular elements are employed, while the energy approaches are used for determining energy release rates for cracks that may or may not be discretized with the help of singular elements.

The displacement and stress correlation techniques assume that the finite element solution near the crack tip is of the same form as the singular near-field solution predicted by LEFM (Broek 1986). By matching the (known) finite element solution with the (known, except for K_I , K_{II} , and K_{III}) theoretical near-field LEFM solution, it is possible to calculate the stress-intensity factors. Since only three equations are needed to obtain the three stress-intensity factors, while many points that can be matched, there are many possible schemes for correlation. These include matching nodal responses only on the crack surfaces and least-squares fitting of all of the nodal responses associated with the singular elements.

The displacement correlation approach is more accurate than the stress correlation approach because displacements converge more rapidly than stresses using the FEM. Therefore only the displacement correlation approach is discussed in detail here (Shih et al. 1976).

Consider a linear elastic isotropic material with Young's modulus E and Poisson's ratio ν . For the case of plane strain, the near-field displacements (u, v), in terms of polar coordinates r and θ , shown in Fig. 2.2, are given by:

$$u = \frac{2K_I(1+\nu)}{E} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left[1 - 2\nu + \sin^2 \frac{\theta}{2} \right] + \quad (2.1)$$

$$\frac{2K_{II}(1+\nu)}{E} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left[2 - 2\nu + \cos^2 \frac{\theta}{2} \right]$$

$$v = \frac{2K_I(1+\nu)}{E} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left[2 - 2\nu - \cos^2 \frac{\theta}{2} \right] + \quad (2.2)$$

$$\frac{2K_{II}(1+\nu)}{E} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left[-1 + 2\nu + \sin^2 \frac{\theta}{2} \right]$$

in which u and v are parallel and perpendicular to the crack face, respectively.

Now consider a crack tip node surrounded by quarter point triangular elements shown in Fig. 2.2. Interpolating the radial coordinate, r , along the side AC , by using quadratic shape functions associated with nodes A , B , and C , and solving for the natural triangular area coordinate ξ_1 in terms of r , we obtain:

$$\xi_1 = 1 - \sqrt{\frac{r}{L}} \quad (2.3)$$

where L is the length of the side AC . Now interpolating the displacements along the side AC by using the computed displacement components at nodes A , B , and C , and using Eq. (2.3), we obtain the displacements along the surface AC in terms of r . These are given by:

$$u = u_A + (-3u_A + 4u_B - u_C)\sqrt{\frac{r}{L}} + (2u_A - 4u_B + 2u_C)\frac{r}{L} \quad (2.4)$$

$$v = v_A + (-3v_A + 4v_B - v_C)\sqrt{\frac{r}{L}} + (2v_A - 4v_B + 2v_C)\frac{r}{L} \quad (2.5)$$

Similarly the displacements alongside AE can be written as:

$$u = u_A + (-3u_A + 4u_D - u_E)\sqrt{\frac{r}{L}} + (2u_A - 4u_D + 2u_E)\frac{r}{L} \quad (2.6)$$

$$v = v_A + (-3v_A + 4v_D - v_E)\sqrt{\frac{r}{L}} + (2v_A - 4v_D + 2v_E)\frac{r}{L} \quad (2.7)$$

Subtracting Eq. 2.6 from 2.4 and subtracting 2.7 from 2.5, the crack opening displacement (COD) and crack sliding displacement (CSD) are computed as:

$$COD = (4v_B - v_C - 4v_D + v_E)\sqrt{\frac{r}{L}} + (-4v_B + 2v_C + 4v_D - 2v_E)\frac{r}{L} \quad (2.8)$$

$$CSD = (4u_B - u_C - 4u_D + u_E)\sqrt{\frac{r}{L}} + (-4u_B + 2u_C + 4u_D - 2u_E)\frac{r}{L} \quad (2.9)$$

Analytical solutions for COD and CSD can be obtained by evaluating the displacement components u and v given by Eqs. 2.1 and 2.2 for $\theta = +\pi$ and $\theta = -\pi$ and subtracting the values at $\theta = -\pi$ from the values at $\theta = +\pi$. Equating the like terms in the finite element and the analytical COD and CSD profiles, the stress intensity factors are given by:

$$K_I = \sqrt{\frac{2\pi}{L}} \frac{E}{2(1+\nu)(3-4\nu)} [4(v_B - v_D) + v_E - v_C] \quad (2.10)$$

$$K_{II} = \sqrt{\frac{2\pi}{L}} \frac{E}{2(1+\nu)(3-4\nu)} [4(u_B - u_D) + u_E - u_C] \quad (2.11)$$

Thus by meshing the crack tip region with quarter-point quadratic triangular elements and solving for the displacements, the stress intensity factors can be computed by using Eqs. 2.10 and 2.11. This technique does not require any special subroutines to develop the stiffness matrix for the singular elements. A single subroutine can be written to calculate the length L of the sides AC and AE, retrieve the displacement components at the nodes A, B, C, D, and E and thereby compute the stress-intensity factors using Eqs. 2.10 and 2.11.

Ingraffea and Manu (1980) have developed similar equations for the computation of stress-intensity factors in three dimensions with quarterpoint quadratic elements. In three dimensions, the crack tip is replaced by the crack front, the crack edge by the crack face.

Energy approaches for extracting stress-intensity factors make use of the fact that $K_I = [EG_I]^{1/2}$, $K_{II} = [EG_{II}]^{1/2}$, $K_{III} = [EG_{III}/(1+\nu)]^{1/2}$ for plane stress and $K_I = [EG_I/(1-\nu^2)]^{1/2}$, $K_{II} = [EG_{II}/(1-\nu^2)]^{1/2}$, $K_{III} = [EG_{III}/(1+\nu)]^{1/2}$ for plane strain. Here, G_I , G_{II} , and G_{III} are the potential energy release rates created by collinear crack extension due to Mode I, Mode II,

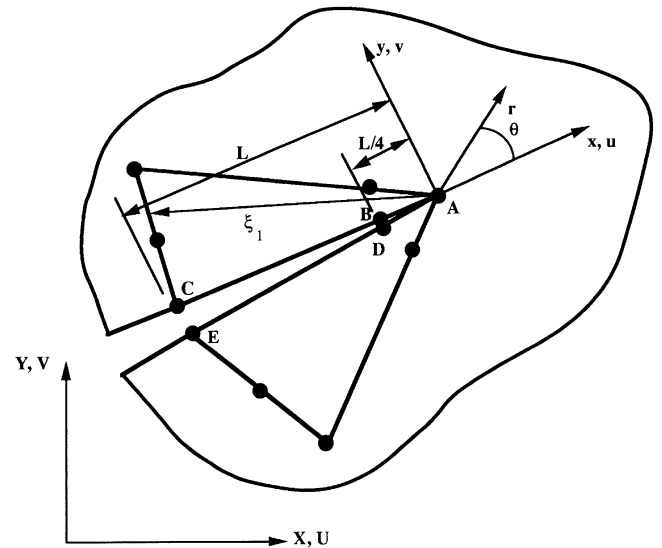


Fig. 2.2—Nomenclature for 2D quarter point singular isoparametric elements.

and Mode III deformations, respectively. In the simplest approach, the total energy release rate, $G = G_I + G_{II} + G_{III}$ can be calculated by performing an analysis, calculating the total potential energy, π_A , collinearly extending the crack by a small amount ∂a , reperforming the analysis to obtain π_B , and then using a finite difference to approximate G as $G = (\pi_A - \pi_B)/\partial a$. If G_I , G_{II} , and G_{III} are required separately, they can be calculated by decomposing the crack tip displacement and the stress fields into Mode I, Mode II, and Mode III components (Rahulkumar 1992).

The stiffness derivative method for determination of the stress-intensity factor for Mode I (2D and 3D) crack problems was introduced by Parks (1974). The method is equivalent to the J-integral approach (described later).

With reference to Fig. 2.3, any set of finite elements that forms a closed path around the crack tip may be chosen. The simplest set to choose is the set of elements around the crack tip.

The stiffness derivative method involves determination of the stress-intensity factor from a calculation of the potential energy decrease per unit crack advance, G . For plane strain and unit thickness, the relation between K_I and G is

$$G = -\left[\frac{\partial \pi}{\partial a}\right]_{load} = \frac{(1-\nu^2)}{E} K_I^2 \quad (2.12)$$

in which P is the potential energy, a is the crack length, E is Young's modulus, and ν is Poisson's ratio.

Parks (1974) shows that the potential energy, π , in the problem is given by:

$$\pi = \frac{1}{2} \{u\}^T [K] \{u\} - \{u\}^T \{f\} \quad (2.13)$$

in which $[K]$ is the global stiffness matrix, and $\{f\}$ is the vector of prescribed nodal loads. Eq. 2.13 is differentiated with respect to crack length, a , to obtain the energy release rate as

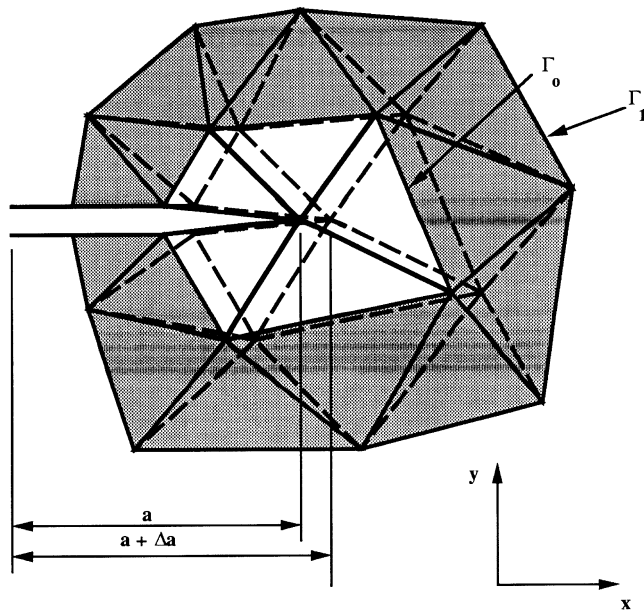


Fig. 2.3—Stiffness derivative approach for advancing nodes (Parks 1978).

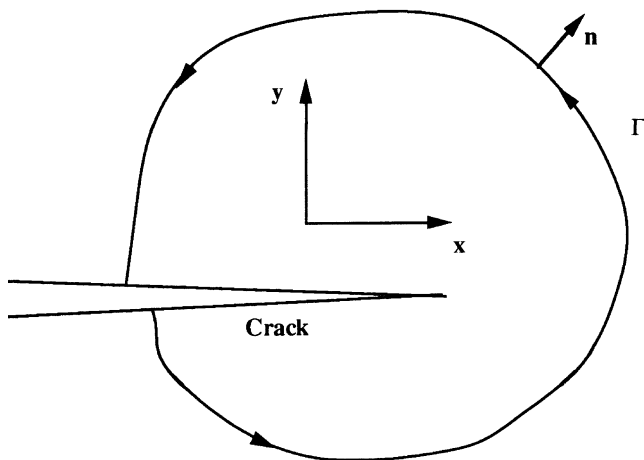


Fig. 2.4—J-Integral nomenclature (Rice 1968).

$$\left[\frac{-\partial \pi}{\partial a} \right]_{load} = -\frac{1}{2} \{u\}^T \frac{\partial [K]}{\partial a} \{u\} - \{u\}^T \frac{\partial \{f\}}{\partial a} = K_1^2 \frac{(1-\nu^2)}{E} \quad (2.14)$$

The matrix $\frac{\partial [K]}{\partial a}$ represents the change in the structure stiffness matrix per unit of crack length advance. The term $\frac{\partial \{f\}}{\partial a}$ is nil if the crack tip area is unloaded. The key to understanding the stiffness derivative method is to imagine representing an increment of crack advance with the mesh shown in Fig. 2.3 by rigidly translating all nodes on and within a contour Γ_0 (see Fig. 2.3) about the crack tip by an infinitesimal amount Δa in the x-direction. All nodes on and outside of contour Γ_1 remain in their initial position. Thus the global stiffness matrix $[K]$, which depends on only individual element geometries, displacement functions, and elastic material properties, remains unchanged in the regions interior to Γ_0 and exterior to Γ_1 , and the only contributions to the first term of Eq. 2.14 come from the band of elements between the contours Γ_0 and

Γ_1 . The structure stiffness matrix $[K]$ is the sum over all elements of the element stiffness matrices $[K_i]$. Therefore,

$$\frac{1}{2} \{u\}^T \frac{\partial [K]}{\partial a} \{u\} = \frac{1}{2} \{u\}^T \sum_{i=1}^{N_c} \frac{\partial [K_i^c]}{\partial a} \{u\} \quad (2.15)$$

in which $[K_i^c]$ is the element stiffness matrix of an element between the contours Γ_0 and Γ_1 , and N_c is the number of such elements. The derivatives of the element stiffness matrices can be calculated numerically by taking a finite difference:

$$\frac{\partial [K_i^c]}{\partial a} = \frac{\Delta [K_i^c]}{\Delta a} = \frac{1}{\Delta a} [[K_i^c]_{a+\Delta a} - [K_i^c]_a] \quad (2.16)$$

The method may be extended to mixed-mode cracks.

The *J*-Integral method (Rice 1968) for determining the energy release rate of a Mode I crack is useful for determining energy release rates, not only for LEFM crack propagation, but also for nonlinear fracture problems. For a two-dimensional problem, a path Γ is traversed in a counter-clockwise sense between the two crack surfaces, as shown in Fig. 2.4. The *J*-integral is defined as:

$$J \equiv \int_{\Gamma} \left(w dx_2 - p_i \frac{\partial u_i}{\partial x_1} ds \right) \quad (2.17)$$

where summation over the range of repeated indices is understood.

Here, $w \equiv \int_0^s \sigma_{ij} d\epsilon_{ij}$, $i, j = 1, 2, 3$ is the strain energy density, s is the arc length, and p_i is the traction exerted on the body bounded by Γ and the crack surface. The *J*-integral is equal to the energy release rate G of the crack (Rice 1968).

The *J*-integral method can be relatively easily applied to a crack problem whose stress and displacement solution is known, and is not limited to linear materials. However, elasticity or pseudoelasticity along the contour, Γ , is a requirement (Rice 1968).

Alternate energy approaches for extraction of stress-intensity factors from three-dimensional problems have been developed (Shivakumar et al. 1988). Bittencourt et al. (1992) provide a single reference that compares the displacement correlation, the *J*-integral, and the modified crack closure integral techniques for obtaining stress-intensity factors.

When using triangular quarter-point elements to model the singularity at a crack tip, meshing guidelines have been suggested by a number of researchers (Ingraffea 1983, Saouma and Schwemmer 1984, Gerstle and Abdalla 1990). When using the displacement correlation technique to extract stress-intensity factors, the guidelines are summarized as follows:

1. Use a 2 x 2 (reduced) integration scheme (Saouma and Schwemmer 1984).

2. To achieve 5 percent maximum expected error in any stress component due to any mixed-mode problem, use at least eight approximately equiangular singular elements adjacent to the crack tip node. For 1 percent error, 16 singular elements should be used (Gerstle and Abdalla 1990).

3. There is an optimal size for the crack tip elements. If they are too small, they do not encompass the near-field region of the solution, and surrounding regular elements will be “wasted” modeling the near field. If they are too big, they do not model the far-field solution accurately. The singular elements should be related to the size of the region within which near-field solution is valid. For 5 percent accuracy in stress-intensity factors, the singular elements should be about 1/5 of the size of the least dimension associated with the crack tip. For one percent accuracy, the singular elements should be about 1/20 of the size of the least dimension associated with the crack tip (Gerstle and Abdalla 1990).

4. Regular quadratic elements should be limited in size, s , by their clear distance, b , from a crack tip. The ratio of s/b should not exceed unity to achieve 30 percent error, and should not exceed 0.2 to achieve 1 percent error in the near field solution (Gerstle and Abdalla 1990).

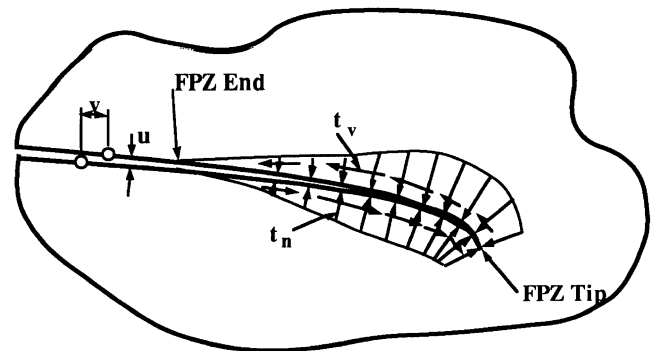
The meshing criteria given above show that a large number of elements are required at a crack tip to obtain accurate near-field stresses. Experience shows that 300 degrees of freedom are required per crack tip to reliably obtain 5 percent accuracy in the near field stresses (Gerstle and Abdalla 1990). This becomes prohibitive from a computational standpoint for problems with more than one crack tip.

Fortunately, it is not necessary to accurately model near-field stresses to calculate accurate stress intensity factors. In fact, using no singular elements, energy methods can be used, as described above, to obtain accurate stress intensity factors with far fewer than 300 degrees of freedom per crack tip.

2.3—Fictitious crack model (FCM)

Since 1961, there has been a growing realization that LEFM is not applicable to concrete structures of normal size and material properties (Kaplan 1961, Kesler et al. 1972, Bažant 1976). The FPZ ranges from a few hundred millimeters to meters in length, depending upon how the FPZ is defined and upon the properties of the particular concrete being considered (Hillerborg et al. 1976; Ingraffea and Gerstle 1985; Jenq and Shah 1985). The width of the FPZ is small compared to its length (Petersson 1981). LEFM, although not applicable to small structures, may still be applicable to large structures such as dams (Elfgren 1989). However, even for very large structures, when mixed-mode cracking is present the FPZ may extend over many meters; this is due to shear and compressive normal forces (tractions) caused by friction, interference, and dilatation (expansion) between the sides of the crack, far behind the tip of the FPZ. To clarify this notion, Gerstle and Xie (1992) have used an “interface process zone (IPZ)” to model the FPZ.

The fictitious crack model (FCM) has become popular for modeling fracture in concrete (Hillerborg et al. 1976, Petersson 1981, Ingraffea and Saouma 1984, Ingraffea and Gerstle 1985, Gustafsson 1985, Gerstle and Xie 1992, Feenstra et al. 1991a, 1991b, Bocca et al., 1991, Yamaguchi and Chen 1991, Klisinski et al. 1991, Planas and Elices 1992, 1993a, 1993b). Fig. 2.5 shows the terminology and concepts associated with the FCM. This model assumes that the FPZ is long and infinitesimally narrow. The FPZ is characterized by



Shear and normal tractions are functions of crack opening and shearing displacements and other state variables

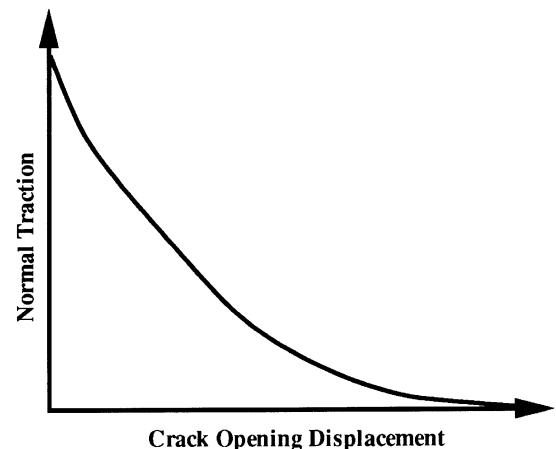


Fig. 2.5—Terminology and concepts associated with the fictitious crack model (FCM) (Hillerborg et al. 1976).

a “normal stress versus crack opening displacement curve,” which is considered a material property, as shown in Fig. 2.5.

The FCM assumes that the FPZ is collapsed into a line in 2D or a surface in 3D. A natural way to incorporate the model into the finite element analysis is by employing interface elements. The first interface element was formulated by Goodman et al. (1968) and was used in the modeling of rock joints. Since then, many types of interface and thin layer elements have been developed and are widely used in geotechnical engineering (Heuze and Barbour 1982, Desai et al. 1984). Zero-thickness elements are the most widely used type of interface, with normal and shear stresses and relative displacements across the interface as constitutive variables. Unrealistic jumps in the results of adjacent integration points of contiguous interfaces have been reported by some authors depending on initial stiffness and load conditions, although most of these problems seem to disappear with the appropriate selection of integration points and integration rule (Gens et al. 1988, Hohberg 1990, Rots and Schellenkens 1990, Schellenkens and De Borst 1993). Other investigators have implemented a semi-discrete FCM by including strain discontinuities (Ortiz et al. 1987, Fish and Belytschko 1988, Belytschko et al. 1988, Dahlblom and Ottosen 1990, Klisinski et al. 1991) or displacement discontinuities (Dvorkin et al. 1990, Lotfi 1992, Lotfi and Ingraffea 1994, 1995) within continuum elements.

The FCM has been incorporated into finite element codes with the use of interface elements. Ingraffea and coworkers (Ingraffea et al. 1984, Ingraffea and Saouma 1984, Ingraffea and Gerstle 1985, Bittencourt, Ingraffea and Llorca 1992) extended the FCM to simulate mixed-mode crack propagation analysis employing six-noded interface elements. Swenson and Ingraffea (1988) used six-noded interface elements to model mixed-mode dynamic crack propagation. Bocca, Carpinteri, and Valente (1991) have published similar work. Gerstle and Xie (1992) used a simple four-noded linear displacement interface element that was modified to allow an arbitrary distribution of tractions along its length.

Other references that implement the FCM include Rots (1988), Stankowski (1990), Stankowski et al. (1992), Hohberg (1992a), Lotfi (1992), Vonk (1992), Lotfi and Shing (1994), Garcia-Alvarez et al. (1994), Lopez and Carol (1995), and Bažant and Li (1995).

In the FCM, the stiffness of the interface element is a nonlinear function of the crack opening displacement, so that a nonlinear solution procedure is required. As with any other kind of nonlinear constitutive relation, FEM calculations with interface elements behaving in accordance the FCM require a nonlinear solution strategy. The various existing techniques such as classic Newton iteration, dynamic relaxation, and arc-length procedures have been used, with satisfactory results reported in the literature (Swenson and Ingraffea 1988, Gerstle and Xie 1992, Papadrakis 1981, Underwood 1983, Bathe 1982).

When using interface elements to model the FPZ, the elements must be very stiff prior to crack initiation to represent an uncracked material (i.e., to keep the two sides of the potential crack together). However, care must be taken not to use a stiffness so high as to cause nonconvergent numerical behavior in the finite element solution. Brown et al. (1993) successfully used interface elements with an axial stiffness equal to 50 times the stiffness of the adjacent concrete elements without numerical difficulties. Gerstle and Xie (1992) suggested using a precrack stiffness equal to the secant stiffness to a point on the descending normal traction-*COD* curve (Fig. 2.5) equal to a *COD* of 1/20 to 1/30 of the *COD* at which the normal traction drops to zero.

Some investigators have dispensed with interface elements and have instead simulated the FPZ using an influence function approach (Li and Liang 1986, Planas and Elices 1991) in which cohesive forces are applied to the crack faces (Gopalaratnam and Ye 1991). Weighted multipliers are used in the superposition of FEM solutions to satisfy overall equilibrium, compatibility and stress-crack width relations within the FPZ. This approach results in the solution of a set of nonlinear algebraic equations to determine the multipliers. In some cases, it may be appropriate to linearize the relationship between the *COD* and the tractions on the FPZ. Then linear equations can be solved to obtain the solution efficiently (Gopalaratnam and Ye 1991, Li and Bažant 1994).

Extension of the FCM with interface elements to mixed mode cracking requires a constitutive relation for the interface, in which normal and shear stresses and relative displacements are fully coupled. Crack opening and closing

conditions are expressed with a biaxial failure surface in the normal-shear stress space. Crack surface displacements, thus, have two components: opening and sliding. Several models of this kind have been proposed recently, all based on the framework of non-associated work hardening plasticity, to obtain a formulation that is fully consistent and contains fracture energy parameters (Stankowski 1990, Stankowski et al. 1993, Lotfi 1992, Lotfi and Shing 1994, Hohberg 1992a, 1992b, Vonk 1992, Garcia-Alvarez et al. 1994). After the crack is completely open, the models prevent interpenetration and provide Coulomb-type friction between crack surfaces. None of the models, however, provides secant unloading in pure tension, as is usual in the classic FCM, because this would complicate the model considerably, as discussed by Carol and Willam (1994).

An approach to nonlinear mixed-mode discrete crack propagation analysis was proposed by Ingraffea and Gerstle (1985). In this approach, the FPZ is modeled by interface elements, with the singular elements used in LEFM placed around the fictitious crack tip to predict the direction of the crack propagation. However, singular elements are not necessary for this purpose if energy release rates are calculated directly (Rice 1968, Parks 1974, Sha and Yang 1990, Rahul Kumar 1992).

Another potential approach to determining the direction of a crack is based on the very reasonable assumption that the crack will propagate when the maximum tensile principal stress at the crack tip reaches the strength of the material (Petersson 1981, Gustafsson 1985, Hillerborg and Rots 1989, Bocca et al. 1991, Gerstle and Xie 1992). The direction of crack propagation is assumed to be perpendicular to the maximum tensile principal stress. The problem with this approach is that when the FPZ becomes small compared to the crack tip element size, objectivity of the results is lost. Therefore it makes more sense to use an energy-based approach to determine crack propagation. A basis for such an approach has been developed by Li and Liang (1992). Promising results have been obtained by using energy release rate approaches to determine both the direction and the load level at which a fictitious crack will propagate (Xie et al. 1995).

2.4—Automatic remeshing algorithms

In 1981, work was completed on a two-dimensional fracture propagation code that used simple interactive computer graphics to interactively model crack propagation (Saouma 1981). Many of the tasks that Ingraffea (1977) had performed by editing files manually were now performed interactively. These tasks included semi-automatic remeshing to allow the crack to advance, limited post-processing to view stresses, deformed mesh, and stress-intensity factors, and predictions based upon the mixed-mode crack propagation theories of the crack trajectory.

Subsequently, Wawrzynek and Ingraffea (1987) developed a second generation two-dimensional interactive graphical finite element fracture simulation code based upon a winged-edge topological data structure. This program demonstrated the value of using a topological data structure in fracture simulation codes. More recently, Gerstle and Xie (1992), in collaboration with others, developed an interac-