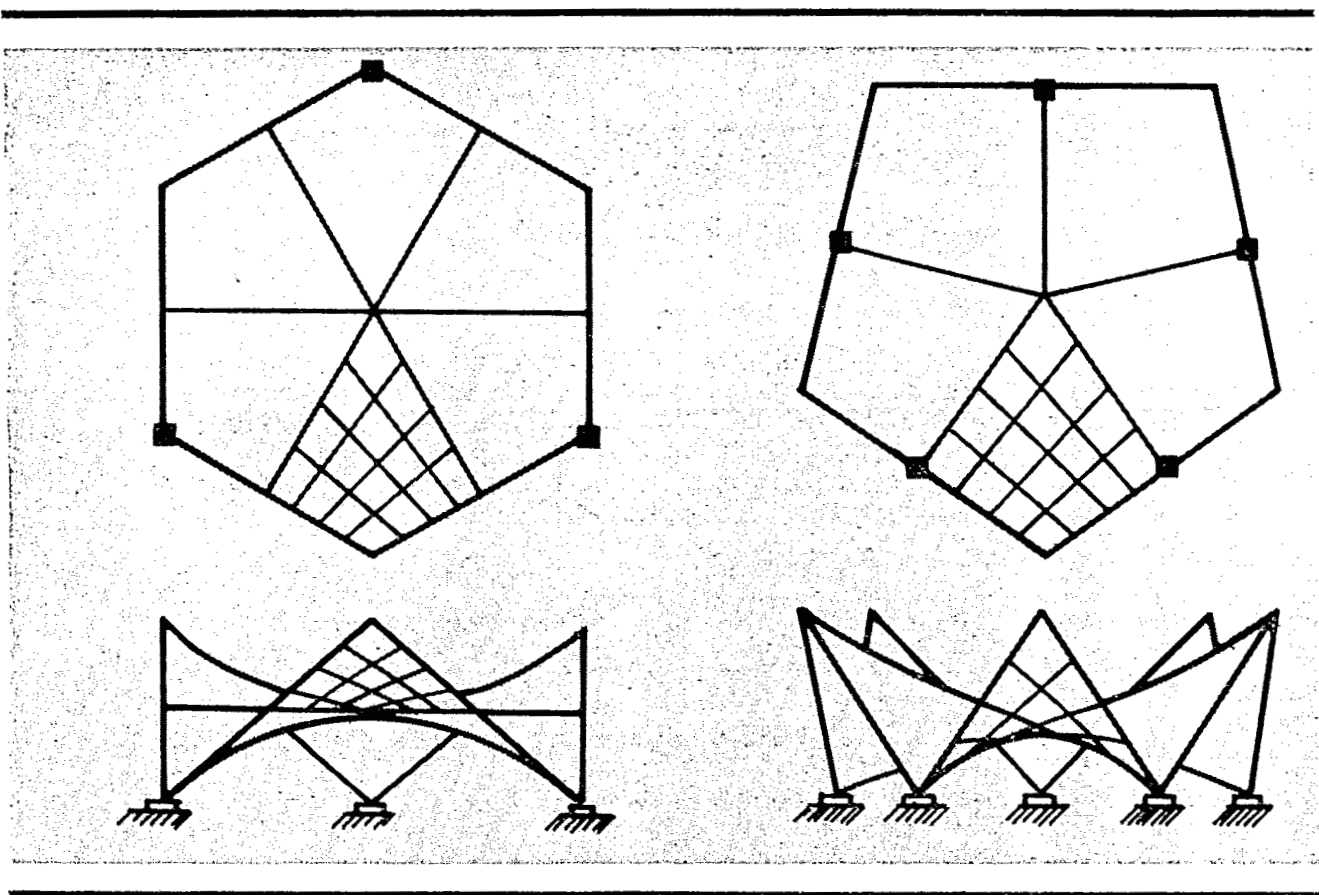


Hyperbolic Paraboloid Shells State of the Art



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PREFACE

The Hyperbolic Paraboloid (HP) concrete shell came to the attention of American designers through the work of the Spanish/Mexican engineer Felix Candela and his illustrated articles in the ACI Journal which were published in the early 1950's. Since that time many HP structures have been designed and built both in the United States and throughout the world. This publication is a result of over ten years effort by certain members of joint ACI-ASCE Committee 334 on Concrete Shell Design and Construction. This effort has been devoted to developing analysis and design data and techniques for hyperbolic paraboloid concrete shells, especially for umbrella, gable, and saddle shells. The material in this publication was presented by the authors at the ACI Fall Convention in Seattle, Washington, November 13, 1987. It is hoped the information presented in this publication will encourage the design and application of this efficient and economical type of construction.

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Chapter 1

Hyperbolic Paraboloid Shell Structures

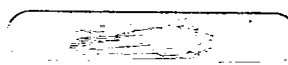
by W.C. Schnobrich

1.1 INTRODUCTION

Shell structures mobilize geometry to activate both the membrane and the flexural internal force systems to efficiently support any distributed loads applied to those structures. Based primarily on their efficiency, these geometric structural forms are employed in a number of industrial applications such as pressure vessels, containment structures, etc. where the principal function of the structure is to contain or sustain a particular loading environment. This selection is especially true when substantial loads are involved such as high internal pressures. The shell structures selected for these applications tend to be simple forms, most frequently combinations of various shells of revolution such as cylinders, cones, spheres and (for the pressure vessels) the torospherical or ellipsoidal shapes because of their ability to respond to most loads by a membrane state over the major area of the shell. Edge effects are confined to a narrow zone near the edge or around a zone of discontinuity that may be present as a result geometry changes.

In addition shell forms contain a number of functional, economic and esthetic virtues which make them logical choices for a number of applications to building structures. For repetitive structures and for those needing long, column free spans reinforced concrete shell roofs have often been chosen. Furthermore, these structures provide a clean inner surface often in a pleasant geometric shape. They have good fire resistant qualities. By proper orientation or shape selection glass areas can be placed so that natural lighting can be directed onto all or nearly all the covered ground plan.

Keywords: hyperbolic parabolic shells; reinforced concrete; roofs; shells (structural forms); structural analysis



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The hyperbolic paraboloid (HP) is one configuration of reinforced concrete shell that sustained a most dramatic growth in numbers of applications not only as roof structures but also for foundations, walls, etc. Of equal importance are its advantageous shape, being generated as it is by a system of intersecting straight lines thus making fabrication of the forms for the casting of concrete shells relatively economic and its esthetic characteristics. This single class of shell can take on an almost infinite variety of visual and functional forms depending on the amount of twist present in the individual panels, along what lines those panels are bounded and then joined to fabricate the entire surface, the location and types of supporting structure, etc. Just how versatile this class of shell is can be seen in the works of Felix Candela (2) and in the publications of Joedicke (3). Figures (1.1) and (1.2) taken from the excellent recent book by Schueller (5) show a number of possible basic shapes and sketches of a number of HP shells that have been built. These figures do show the variety of form and function that the HP can achieve.

The most popular form of HP used is the inverted umbrella. One feature that is prominent for this configuration is ability to construct the individual shells as stand alone units. This produces a system which can be constructed in an expedient manner.

1.2 EXAMPLES

One of the first roofs built with the hyperbolic paraboloid geometry was for a cosmic ray laboratory on the University of Mexico campus, Fig. 1.3. The shell was designed by Felix Candela with a thickness of $5/8$ inches. Although of relatively small span, 33 feet in the transverse direction by two 18 foot spans in the longitudinal direction, the structure ushered in an era of much activity centered around this geometric form. Candela utilized the hyperbolic shapes for many of his structures constructed not only in Mexico but throughout the world. He employed the shape not only in the umbrella, saddle, and gable configurations but conceived such combinations as the groined vault to achieve stress free edges and a light flowing appearance.

W.C. Schnobrich, member ,ACI, is a professor of Civil Engineering at the University of Illinois, Urbana. His primary area of specialization is shell structures. He is a member of ACI-ASCE Committee 334, Concrete Shell Design and Construction and is chairman of the ASCE Committee on Finite Element Analysis of Reinforced Concrete.

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The classic example of the gable roof shape was that designed by Anton Tedesko for a department store in Denver Colorado (6). This shell shows the clean graceful lines of this configuration Fig. 1.4. The shell spans 131 feet by 112 feet. It has a basic thickness of 3 inches which is increased through taper to mate with the ridge and edge beams to minimize edge effects. The shell has a rise of 28 feet so that, although shallow, it has sufficient curvature to avoid serious local bending problems. The shell has been in service since completion without any indication of distress or need of unusual maintenance.

One interesting application of the hyperbolic paraboloid shell in the inverted umbrella form was for the roof of an underground garage for the Lincoln Library in Springfield Illinois, designed by Hanson Engineers. The inverted umbrella shape was chosen so that the depressions over the columns would provide additional depth of soil to allow planting of trees and other landscaping so the building site could retain the appearance of the court house as it was when Lincoln was practicing law. The shape thus meant that less excavation was necessary to provide the necessary soil depths. An added benefit is observed in the clean cathedrel like appearance of the roof, Fig. 1.5. This shell supports a loading of saturated soil making it one of the more heavily loaded shell roofs. Thus the shell is expected to have more bending present in the surface than in a normal application of an HP shell. Some 90 shell units with spans of 30 feet were built. These shell spans were controlled by the column spacing needed for the waffle slab comprising the roof of the lower level parking. Shell thickness of 6 inches was selected based on the anticipated heavy saturated soil load. The depth of the inverted umbrella was 4 ft-4 in. Reuse of forms was achieved with the forms mounted on rollers for easy movement thereby providing an additional economic factor in this application.

Roof systems are often required to cover plants which are used for water supply or waste treatment when those plants are constructed in populated areas. These roof systems must cover large column free spaces. The HP shell has been found to be both an economic and esthetic solution for this application. Several examples include E.L. Smith Water Treatment Plant, Edmonton designed by Associated Engineering Services and the Oakwood Beach Water Pollution Project designed by Kline and Hoffman in association with Greeley and Hansen. These projects were acknowledged by engineering and architectural awards as to the excellence of the solution.

The largest saddle shaped shells were designed by Milo Ketchum to serve as maintenance hangers at TWA's overhaul base at the Kansas City airport. The shell was formed by the intersection of two saddle shapes. These shells have a span between supports of about 320 feet with the thickness being 4 inches. The edge beams are hollow triangular shapes which increase in depth from the tips to the supports. Again the versatility of shape present in the HP shell allows the roof to be adapted to the shape of the volume to be enclosed; the saddle being an ideal

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shape for a hanger for jumbo jets. By tilting the shell can be made to provide additional height for the tail assembly without raising the entire roof line.

A particularly efficient modification to the inverted umbrella shaped shell can be seen in the roof for the gymnasium at Brown University (7). The roof of the structure was to serve as additional usable area. The individual shell units were 90 feet by 60 feet. With two units joined together to form one transverse shell unit, 180 foot column free span was provided. A flat slab cast between perimeter beams has vertical members attaching this slab to the shell. This composite shell now functions as a giant, tapered cantilever beam. The top slab serves as the tension flange for the structure while the shell surface becomes the compression component. Fig. 1.6 shows an interior view of this structure. Again there are added benefits derived from the chosen shape, the air volume trapped between the shell and slab performs as good insulation for the structure. This concept was developed by Daniel Tully during his design of a number of roof structures for gymnasium applications. The top slab provides level area that can be used, in this case for tennis courts.

A particularly graceful form of HP shell is the groined vault geometry. The first application to demonstrate the beauty of this form was a restaurant constructed in Xochimilco Mexico and designed by Felix Candela. The shell covers a pentagonal ground plan with mensa basic diameter of 140 feet using a thickness of only 1.5 inches. The normal heavy perimeter beams required by many HP shells are not required in this geometry. Felix Candela was the first to develop this aspect of the design. Form costs tend to be high for this shape. A second factor that must be considered is the limited head room clearance associated with this shell. A number of replications of this shell have been built by various designers. One of special note was built in Stuttgart. It has a thickness of but one centimeter while spanning some 50 meters. The shell could be cast so thin because it utilized glass fiber for reinforcement.

Another significant shell is the sports coliseum in Ponce Puerto Rico. This HP shell was designed by the T Y Lin group (8). The roof covers a ground plan of 276 feet by 232 feet. Again a 4 inch thickness is adequate for the shell surface. Thirty inch wide edge beams vary in depth from 18 inches at their tips to a maximum of 94 inches at the abutments for the long spans. Pier supports are placed at the midpoint of each span thus making these edge beams 138 foot cantilevers. The space covered by this shell is enclosed by walls. Thus the thinness of the shell and the use of only four support piers dramatically capture the grace of this roof, Fig (1.7). Because of the size of this structure the design firm did have a comprehensive finite element analysis performed on the structure, particularly including the supporting edge members and their eccentric placement relative to the shell.

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The largest concrete dome in the world, the King Dome in Seattle, Wa., is constructed of HP saddle segments. The dome has a clear span of 661'-4". The shell slabs are 5" thick and are bounded by arched rib stiffeners. See Fig.1.8.

Other HP shells that must be mentioned include the inverted umbrella shells that makeup the roof for the terminal buildings of the Newark International Airport. These shells represent rather large spans for the inverted umbrella being 115 feet by 120 feet with a rise of about 20 feet. Another Candela shell of special note is the Church of Our Miraculous Lady in the Navarte district of Mexico City. This shell of gable units has a very steeply inclined roof. Because of the steepness of the slope Candela provided special counter weights along the ridge lines of the shells to balance anticipated upward thrusts expected to be present in the shell because of the steep slopes involved.

One can not list some of the more interesting HP shells without making special note of St Marys Cathedral located in the heart of San Francisco. The roof consists of eight hypar shells which are approximately 50 feet wide by 130 feet high placed on a very steep slope. The shell was shotcreted on top of precast concrete ribbed forms. The basic structure covers a 205 foot square plan while rising some 190 feet high. The ribs forming the base of the roof system enhance the interior view of the roof. This can be seen in an interior view Fig. (1.9). Because of the complexity of the structure extensive model tests and computer analyses were performed.

1.3 ANALYSIS

Over the years since the HP form of shell construction first became popular many papers have been written concerning their analysis and behavior. The initial papers dealt with either a simplified procedure such as a membrane analysis (1) or a more elaborate bending analysis method applied to a simplified structure. The results produced by a membrane analysis do indicate the basic mechanism of how the shell globally responds to the load. Both approaches, the membrane and the more complex bending theory, however left many questions unanswered. The membrane theory, although adequate for preliminary design does not provide complete enough information necessary for the final design of the supporting members. Questions of shallowness (how shallow can a shell be before slab bending starts to dominate), how the weight of any stiffener or edge beam is carried locally (either by the shell or by the beam itself), etc. remained unanswered by a membrane theory. The elaborate bending analysis when performed required such unrealistic support conditions as to often render the results of little practical interest. Most of the shells built during this era were sufficiently small that the deficiencies of the theory could be overcome and were overcome by good design procedures.

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With the development of the capability to preform large systems of numerical calculations, finite element and finite difference procedures have evolved that are capable of properly treating this structure in its many forms. These numerical schemes provide a suitable analysis procedure for these structures considering their properties to be orthotropic, edge or support beams in any placement relative to the shell, any shape covering any ground plan, partial loadings covering limited areas of the shell etc. Currently procedures are being developed which will allow other influences to be considered such as creep, large deflections, concrete cracking, etc. Analysis procedures have been developed to a sufficient level of capability as to handle most cases that the designer will have to consider.

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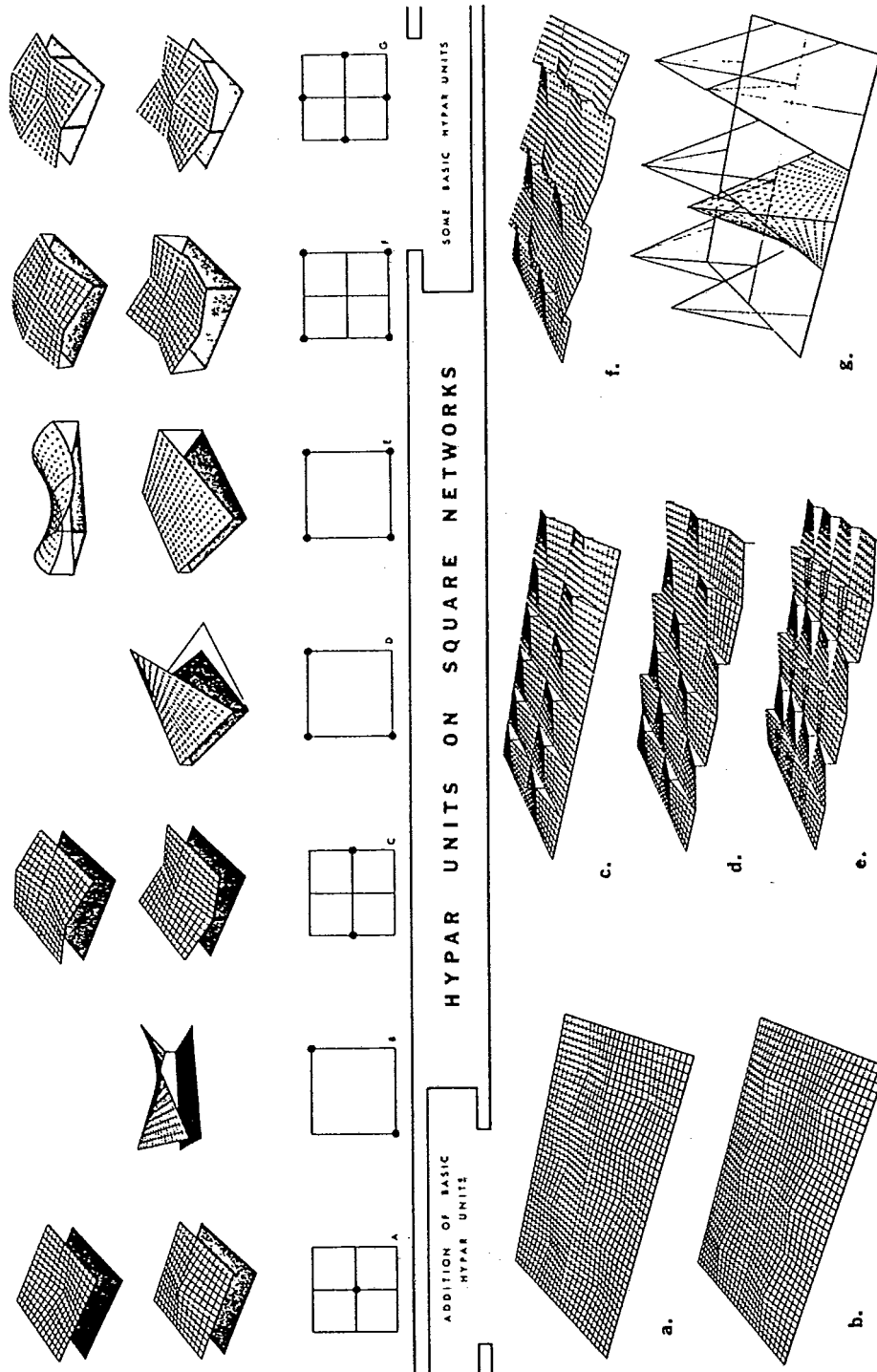


Fig. 1.1--Basic HP shapes

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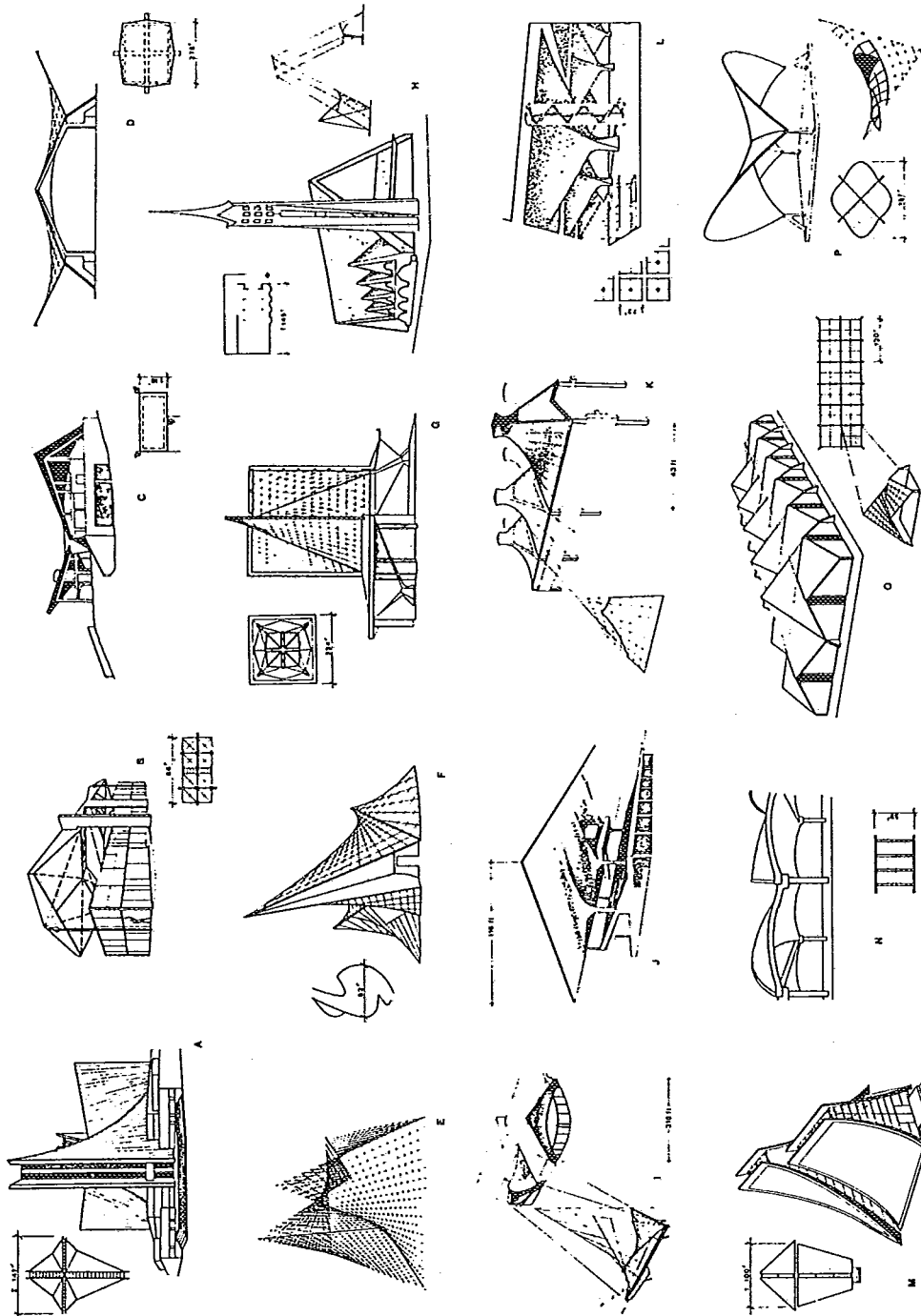


Fig. 1.2--Examples of constructed HP's

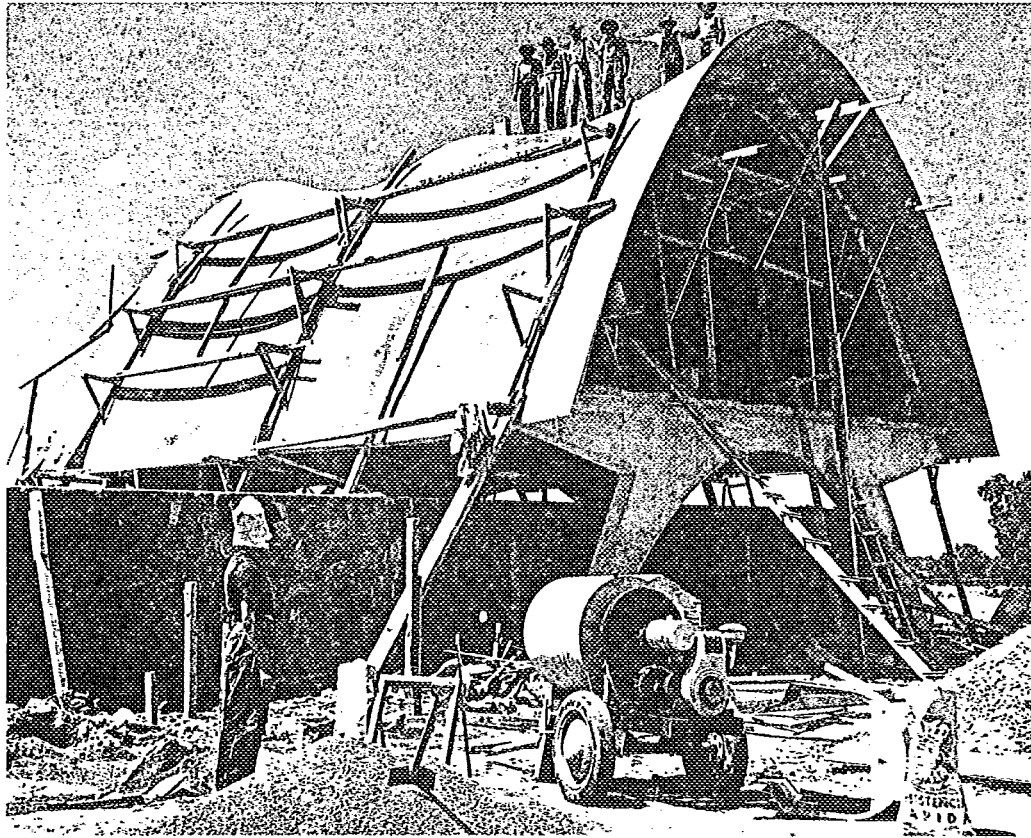


Fig. 1.3--Candela's Cosmic Ray Laboratory

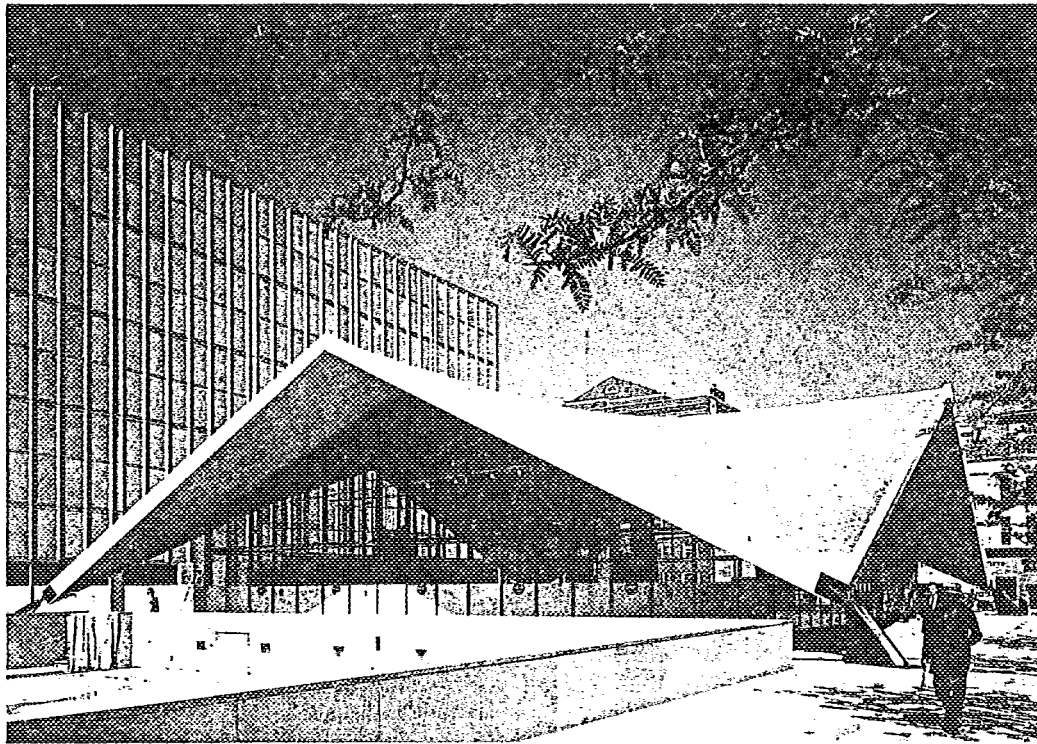


Fig. 1.4--Gable HP roof, Denver, Colorado
(Courtesy Anton Tedesko)

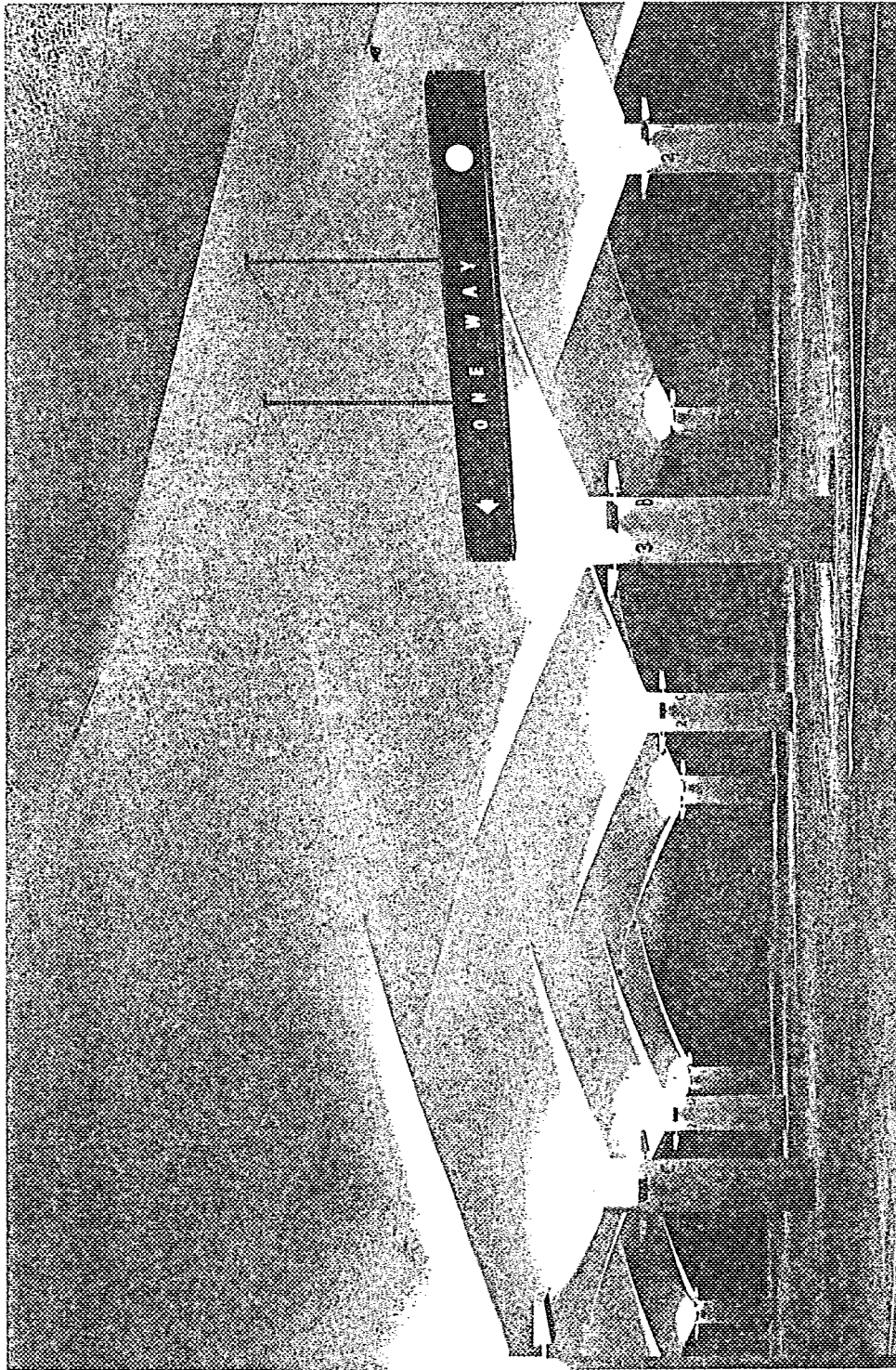


Fig. 1.5---Underground parking garage at Lincoln Library (Courtesy Hanson Engineers)

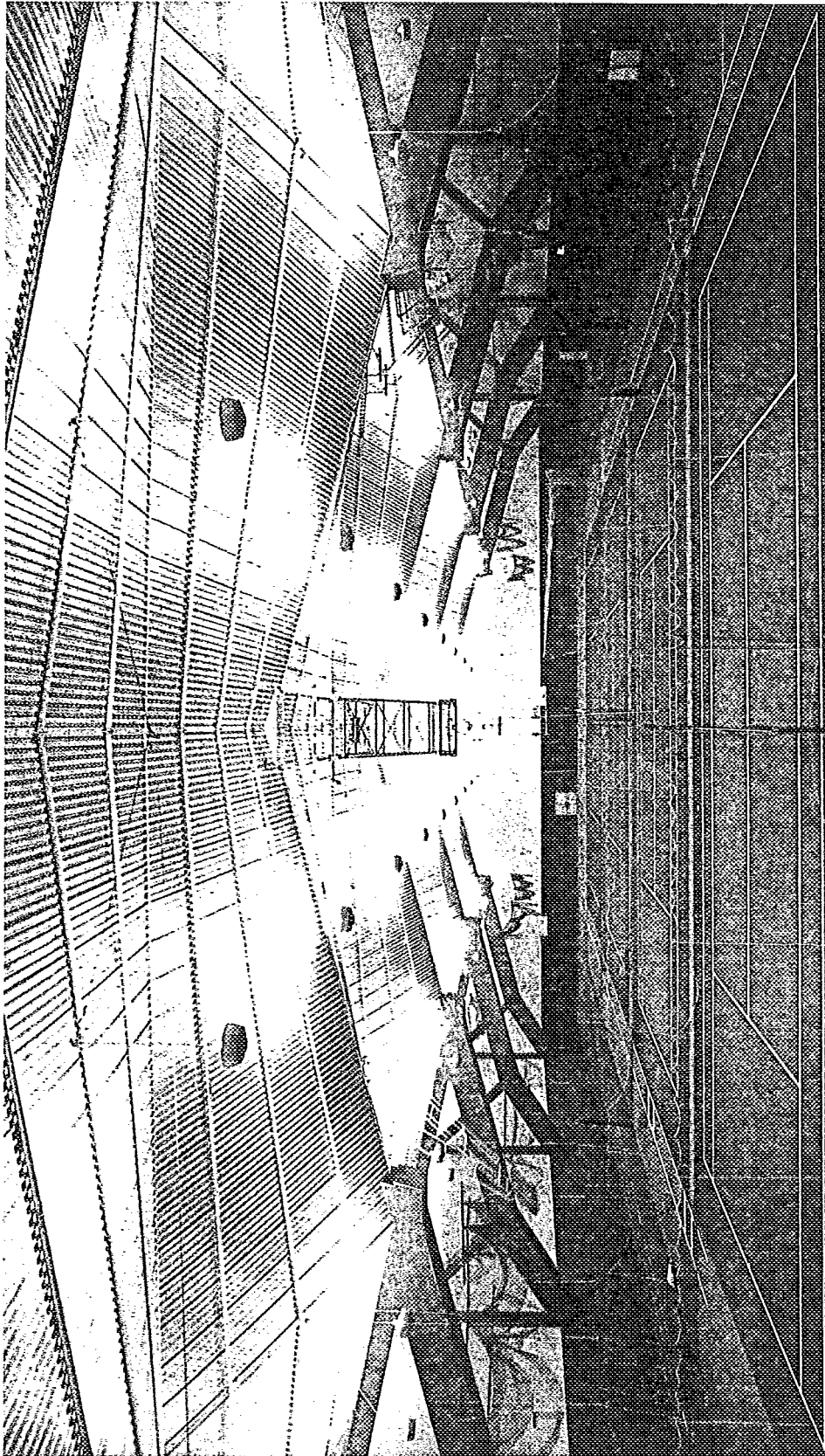


Fig. 1.6--Brown University gymnasium (Courtesy Tully Associates)

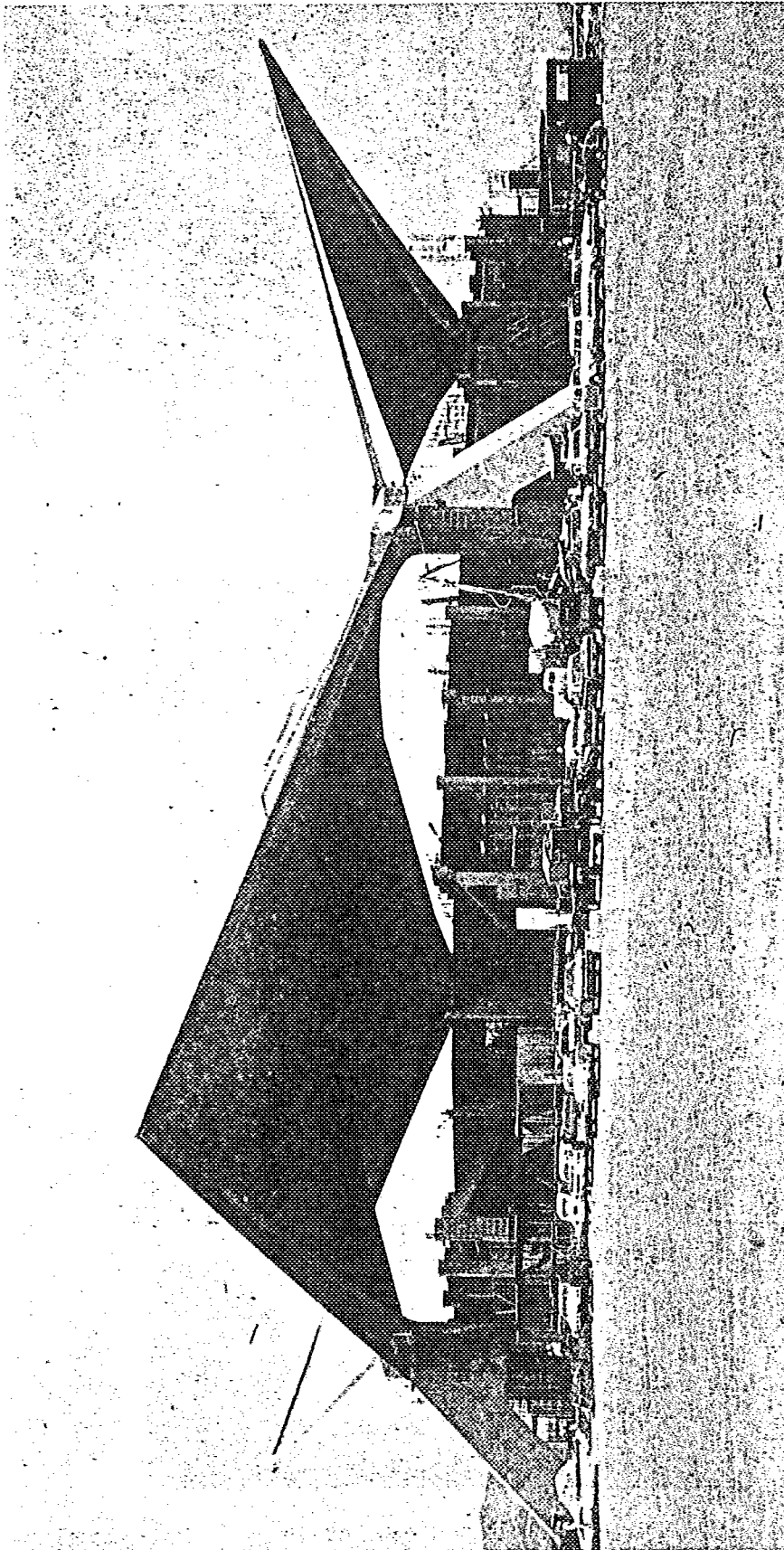


Fig. 1.7--Ponce Coliseum Puerto Rico (Courtesy A.C. Scordelis)

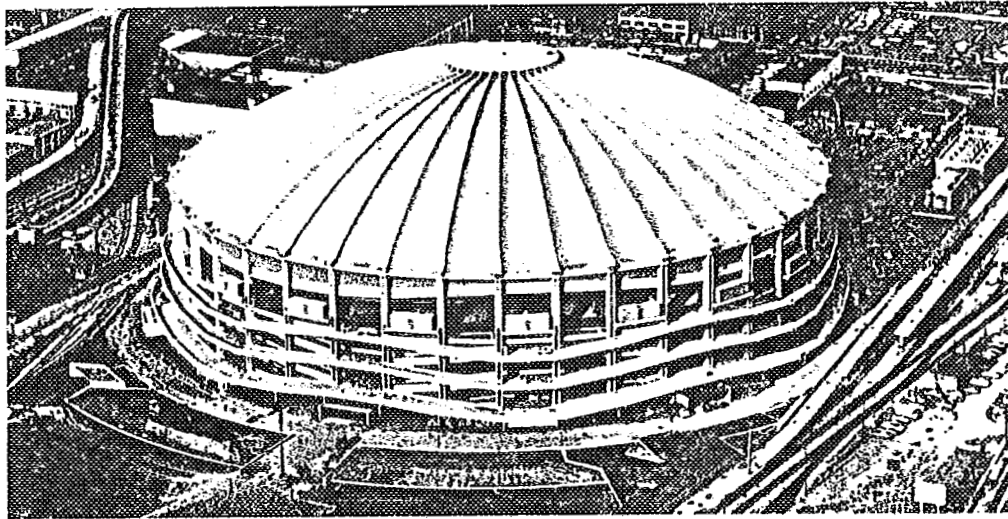


Fig. 1.8--King Dome, Seattle, Washington

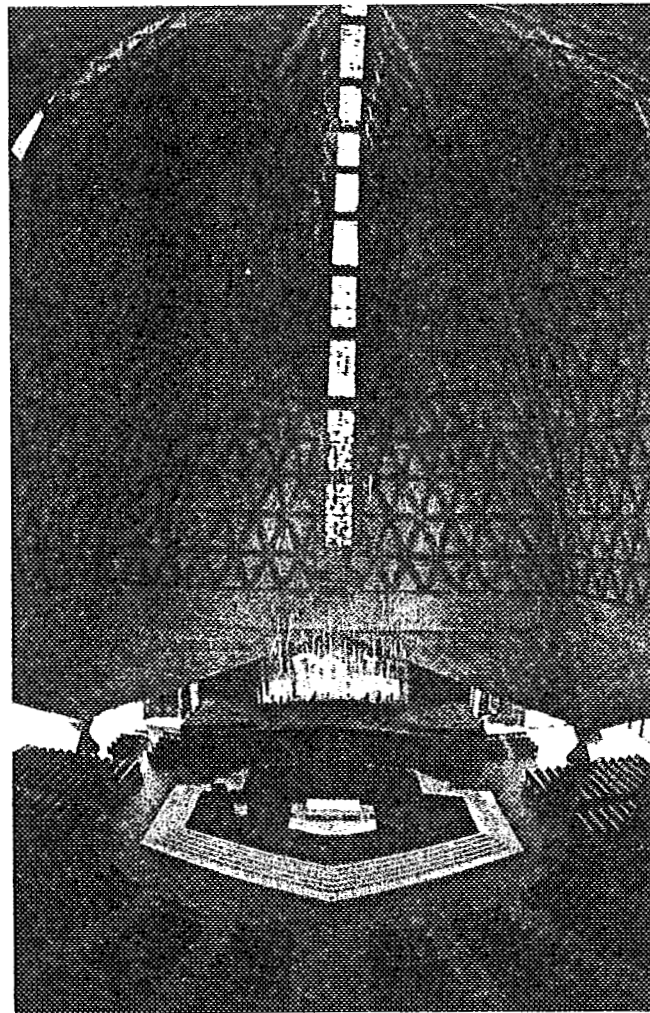


Fig. 1.9--Interior view of Saint Mary's Cathedral

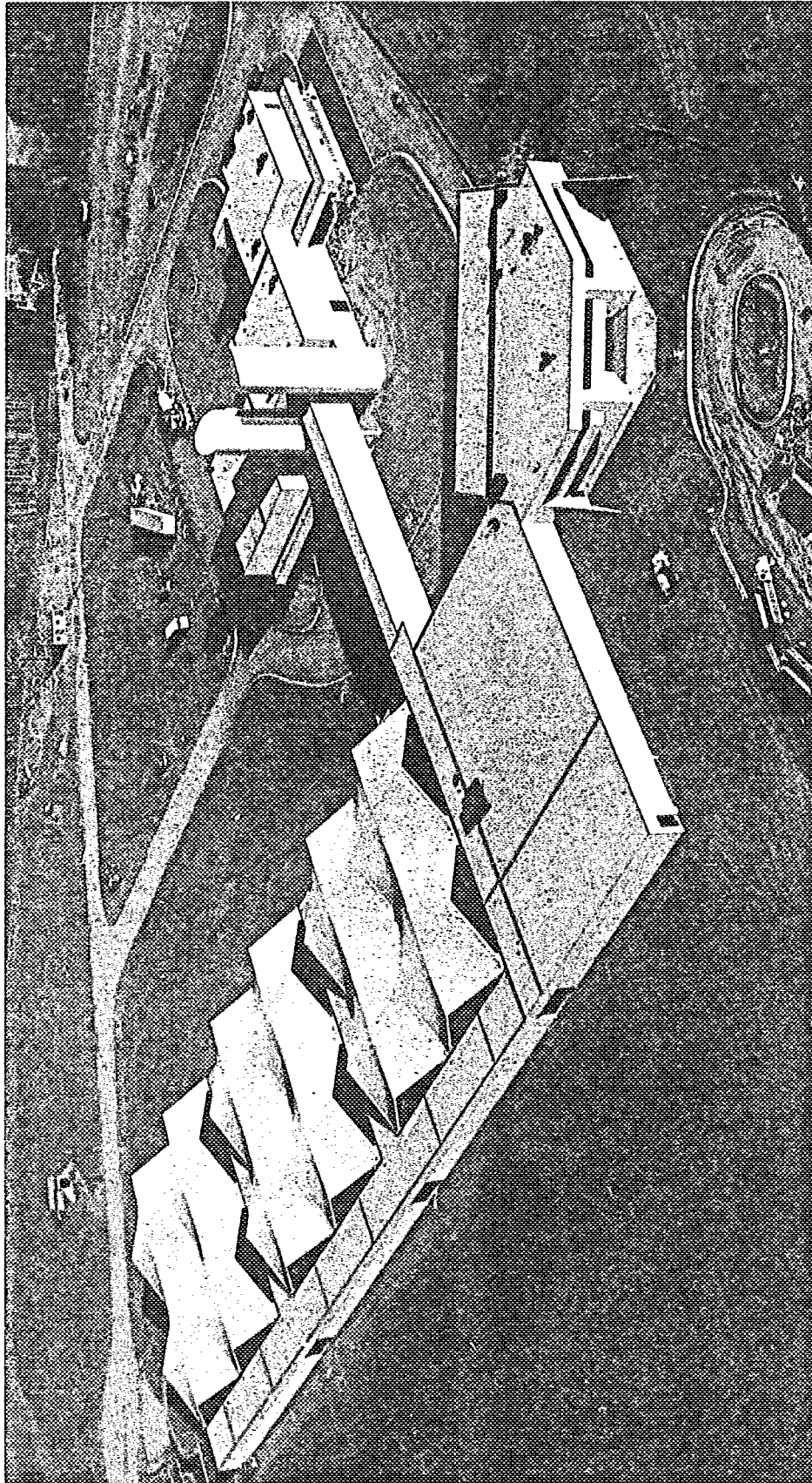


Fig. 1.10--Edmonton water treatment plant (Courtesy S. Simmonds)

Chapter 2

Membrane Analysis

by A.C. Scordelis

Synopsis: A review of the state of the art for the membrane analysis of HP shells is presented. Membrane solutions using simple statics are given for HP shells having square, rectangular, or parallelogram shapes in plan and subjected only to uniform vertical loading over the horizontal projection of the shell. Statics of shell-edge beam systems are discussed for saddle shells, inverted umbrellas, gable shells, and shells on two supports.

Membrane stresses for general loadings on HP shells having a parallelogram shape in plan obtained using a differential equation approach are described. Stress transformation formulas are given which can be used to find principal membrane stresses in the shell and boundary stresses to be transmitted to the edge members.

Membrane stresses in HP shells having an arbitrary quadrilateral shape in plan are discussed and procedures for determining them are described.

Keywords: hyperbolic parabolic shells; membrane theory (shells); reinforced concrete; roofs; shells (structural forms); structural analysis

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INTRODUCTION

The majority of the designs of hyperbolic paraboloid (hereafter termed HP) shell roofs during the 1950's and 1960's were achieved by making use of the membrane theory of translational shells and considering approximate [2.1, 2.6] or intuitive expressions [2.3] for flexural stresses at points, where it was known that the membrane state of stress could not exist and flexural stresses of significant magnitude might be present. With the advent of the computer in the 1960's, methods of analysis using finite elements or finite differences, which incorporated both membrane and bending action in the shell, became possible. Today, many general computer programs are available to obtain these more accurate solutions, which should be used in final designs.

While the limitations of the membrane theory are now recognized for HP shells, it still is a useful approach to understanding the overall statics of these structures. Membrane theory assumes that only in-plane membrane forces N_x , N_y , and N_{xy} can exist in the shell. Shell bending and torsional moments M_x , M_y , and M_{xy} , transverse shears Q_x and Q_y , shell deformations, and compatibility of strains between the shell and its edge members are ignored. The membrane analysis satisfies statics, but not compatibility.

In what follows, first, a membrane solution for the special case of a uniform vertical load over the horizontal projection will be described using simple statics. Then, a more general membrane solution for arbitrary loadings will be outlined using a differential equation approach. Appropriate references will be given for those wishing to study these in greater detail.

SOLUTION USING SIMPLE STATICS

The doubly curved surface of an HP shell may be defined in two ways, either as a surface of translation, or as a warped parallelogram, Fig. 2.1. In the first case, the surface is defined by translating a vertical arching parabola having upward curvature, cod, over another sagging parabola with downward curvature, jok, Fig. 2.1. The HP surface may also be

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generated by translating a straight line ij along two straight lines ik and $j\ell$, Fig. 2.1, to form a warped parallelogram.

Consider, first, the case where $L_1 = L_2 = L$ and $c_1 = c_2 = c$. Thus, the arching and sagging parabolas would be identical in Fig. 2.1. A plan view of the warped parallelogram portion of the shell, $ijkl$, to be analyzed, is shown in Fig. 2.2. Note that for this case the HP is square in plan and the straight line generators are orthogonal to each other in the plan view. Now assume only a uniform vertical loading of w psf of horizontal projection acting on the shell. It will be shown that if it is assumed that at all points the arching parabolas carry $w/2$ and the sagging parabolas carry $w/2$ of the total load w , the boundary forces required at the edges of the shell for equilibrium will consist only of a uniform membrane shear N_{xy} (force/length) along the edges given by:

$$N_{xy} = \frac{wa^2}{2c} \quad (2.1)$$

This value is the same at every point along the edges, as well as within the shell. No membrane forces, N_x or N_y , normal to the edges are required for equilibrium. This remarkable fact indicates that the shear from Eq. (2.1) may be picked up by edge members under pure axial force, with no bending, and transmitted to the supports at selected corners of the HP shell.

Equation (2.1) may be verified using simple statics as described in the following paragraphs.

Consider the one-foot wide arching parabola, $i\ell$, and sagging parabola, jk , strips spanning between the corners, Figs. 2.2 and 2.3. Assuming that each takes a distributed load of $w/2$ and noting that $L = \sqrt{2}(2a)$, then in both cases the end reactions are

$$V = \frac{(w/2)L}{2} = \frac{wL}{4} = \frac{\sqrt{2}wa}{2} \quad (2.2)$$

$$H = \frac{(w/2)L^2}{8c} = \frac{wL^2}{16c} = \frac{wa^2}{2c} \quad (2.3)$$

Next, consider one-foot parabolic strips L_A and L_S meeting at a common point p on the shell edge, Figs. 2.2 and 2.3. Since these strips also carry uniformly distributed loads of $w/2$ and have shapes identical to the parabolas spanning a distance L between the corners, then the horizontal reactions H are the same as given by Eq. (2.3). As a matter of fact, this

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is true for all points in the shell. Note also, for all points p on the edge of the shell, $L_A + L_S = L$, and, thus

$$V_A + V_S = \frac{wL_A}{4} + \frac{wL_S}{4} = \frac{wL}{4} = \frac{\sqrt{2}wa}{2} = V \quad (2.4)$$

which is the same as Eq. (2.2).

A small wedge-shaped differential element at edge point p is taken out as a free body and shown in Fig. 2.4. H and V_A , from the arching parabolic strip, and H and V_S from the sagging parabolic strip, calculated from Eqs. (2.3) and (2.4), are balanced by the boundary forces F_x , F_y , and F_z . Summing forces in three orthogonal directions x, y, z , with z being a vertical direction, the boundary forces F_x , F_y , and F_z are

$$F_x = H \left(\frac{dx}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} \right) + H \left(\frac{dx}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} \right)$$

$$F_x = \frac{wa^2}{2c} dx \quad (2.5)$$

$$F_y = H \left(\frac{dx}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} \right) - H \left(\frac{dx}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} \right) = 0 \quad (2.6)$$

$$F_z = V_A \left(\frac{dx}{\sqrt{2}} \right) + V_S \left(\frac{dx}{\sqrt{2}} \right)$$

$$= (V_A + V_S) \left(\frac{dx}{\sqrt{2}} \right) = V \left(\frac{dx}{\sqrt{2}} \right)$$

$$= \frac{\sqrt{2}wa}{2} \left(\frac{dx}{\sqrt{2}} \right) = \frac{wa}{2} dx \quad (2.7)$$

and noting that $dz/dx = c/a = \text{slope of the shell edge}$, $dx = (a/c)dz$, then Eq. (2.7) becomes

$$F_z = \frac{wa^2}{2c} dz \quad (2.8)$$

Thus, at all points along the edges the force F_y normal to the edge is zero and the resultant of F_x and F_z is a pure shear force R parallel to the sloping edge, Fig. 2.4.

$$R = \frac{wa^2}{2c} ds \quad (2.9)$$

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This verifies Eq. (2.1) and indicates that to find the total force along an edge member, or its components in the x or z directions, one simply needs to multiply N_{xy} (force/length) by the slope length or its projections in the x or z directions. It should be noted that only if it is assumed that the arching and sagging parabolic strips each take one-half of the load w will the normal force F_y on the boundary, Eq. (2.6), have the desired zero value. The above solution would represent only one of the many membrane solutions which could satisfy statics, if normal boundary forces were permitted.

To summarize for this special case of a uniform load w psf of horizontal projection, there exists at all points in an HP shell a state of uniform pure shear N_{xy} parallel to the straight line generators and uniform horizontal compressive or tensile components of thrust H in the direction of arching or sagging parabolas, respectively. The value of these forces per unit length is given by the simple formula

$$N_{xy} = H = \frac{wa^2}{2c} \quad (2.10)$$

Figure 2.5 illustrates these forces per unit length on typical elements taken from the shell.

Using a procedure similar to that described, HP shells having the geometries shown in Fig. 2.6 may be analyzed for a uniform vertical load w psf of horizontal projection [2.5]. Formulas for N_{xy} , the uniform membrane shear, at all points in the shell are given in Fig. 2.6 for each case. Note that for all of the cases shown the arching (A) and sagging (S) vertical parabolas are orthogonal to each other. When these parabolas have different rise/span ratios, the straight line generators will form a parallelogram in plan with a plan angle ω between the edges, Figs. 2.6c,d,e. For these latter cases the arching and sagging parabolas are still assumed to each take $w/2$. However, the horizontal compressive or tensile components of the thrust are no longer equal to each other because of the difference in rise/span ratios. Nevertheless, when a boundary wedge element similar to that shown in Fig. 2.4 is investigated, it is once again found that only a pure shear boundary force N_{xy} parallel to the edge is required for equilibrium.

In Figs. 2.6a,b,c,d it can be seen that three of the corner points have the same z coordinate, while only one has a z coordinate equal to c . For the general case, Fig. 2.6e, of all four corners having different z coordinates, c , cannot be easily found, and it is better to use the general expression for N_{xy} given in Fig. 2.6e [2.5].

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$$N_{xy} = \frac{w}{2K} \sin \omega \quad (2.11)$$

where K is called the twist or warp of the shell and can be found easily from

$$K = \frac{z_l - z_j - z_k + z_i}{ab} \quad (2.12)$$

EDGE MEMBERS AND SUPPORTING SYSTEMS

Figure 2.7 illustrates the statics of a saddle shell-edge beam support system, by membrane theory, under a uniform vertical load w psf of horizontal projection. Figure 2.7a shows free body diagrams of the shell, the edge beams, and the joints at the corners. The directions of the uniform boundary shears N_{xy} acting on the shell edge can easily be established since they must go from the low points to the high points in order to balance the downward load w . Equal and opposite shears act on the edge beams which produce a linearly increasing axial force in the edge beam from the high unsupported points, j and k , to the low supported points, i and l . The joints at the low points, i and l , are held in equilibrium by vertical and horizontal reactions, R_V and R_H , at the supports, Figs. 2.7a,b.

The value of N_{xy} (force/length) can be calculated using Eq. (2.11). From this the maximum axial force P in the edge beam and the reactions R_V and R_H can be found by multiplying N_{xy} by the appropriate projected length

$$N_{xy} = \frac{wa^2}{2c} \quad (2.13)$$

$$P = N_{xy} \sqrt{(2a)^2 + (2c)^2} \quad (2.14)$$

$$R_V = 2 N_{xy} (2c) = 2wa^2 \quad (2.15)$$

$$R_H = \sqrt{2} N_{xy} (2a) = \frac{\sqrt{2}wa^3}{c} \quad (2.16)$$

As a check, note that R_V as calculated above equals the total load $W = w(2a)^2$ divided by 2 as it should.

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Figures 2.7c,d indicate an alternate way of looking at the membrane solution. For the cantilever of Fig. 2.7c, the external moment $(W/2)(x_1)$ is balanced by an internal couple $Ty_1 = Cy_1$ where T equals the sum of the uniform shell membrane tensile forces H , and C equals the horizontal compressive thrust in the edge beams. For the arch of Fig. 2.7d the external moment $(W/2)(x_2)$ is balanced by the couple $Cy_2 = R_{HY2}$, where C equals the sum of the uniform shell membrane compressive forces H , and R_H equals the external horizontal reaction at the support.

Figure 2.7c clearly indicates that while the membrane theory satisfies statics, it violates compatibility of strains at the shell-edge beam juncture since the shell is in tension and the edge beam is in compression.

Figures 2.8 and 2.9 illustrate how the same two basic HP shell panels may be assembled on different edge beam supporting systems to form an inverted umbrella, Fig. 2.9a, a gable shell, Fig. 2.9b or a shell on two supports, Fig. 2.9c. The directions of the boundary shear forces, N_{xy} , acting on the edge beam systems, Fig. 2.9, can easily be established by recognizing that they must flow towards the low point in each panel. The sign and magnitude of the axial force in each supporting beam and of the reactions that must be supplied at the supports can be determined as described previously for the saddle shell. Note that it is essential that the supports be capable of carrying all of the necessary horizontal, as well as vertical, reactions delivered to them by the edge beam system.

Figure 2.10 indicates for each of the three shells an alternate way of looking at the statics of the shell-edge beam systems. In the free body diagrams shown on the left, it is evident that the external statical moment in each case is being taken by a T-C couple provided by horizontal support reactions and/or edge beam forces. Note that for the sections shown the shell edges have only pure shear N_{xy} acting on them and, thus, do not contribute to the T-C couple.

MEMBRANE STRESSES FOR GENERAL LOADINGS ON HP SHELLS HAVING A PARALLELOGRAM SHAPE IN PLAN

For loadings other than a uniform load w psf of horizontal projection, such as dead load, wind load, and equivalent static earthquake load, the simple statics approach used above cannot be used easily to find the membrane stresses in the shell N_x , N_y , and N_{xy} . Using a differential equation approach, Candela [2.2], Tester [2.8], and Ramirez, Scordelis, and Ngo [2.5, 2.7] have developed general formulas for a variety of loadings and geometries. These equations become algebraically involved;

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however, computer programs [2.5, 2.7] are available* which provide solutions for the membrane stresses in the shell and boundary forces. A brief description of the differential equation approach is given in the following paragraphs.

Using a righthand skew coordinate system xyz , the geometry of any HP having a parallelogram shape in plan is defined in Fig. 2.11 by the plan dimensions a and b , the skew angle ω in the horizontal xy plane, and the z coordinates of the four corners i , j , k , and ℓ . The equation of the surface is:

$$z = A + Bx + Cy + Kxy \quad (2.17)$$

where

$$A = z_i; \quad B = \frac{z_k - z_i}{a}; \quad C = \frac{z_j - z_i}{b}; \quad K = \frac{z_\ell - z_j - z_k + z_i}{ab} \quad (2.18)$$

Differentiating Eq. (2.17), let:

$$\xi = \frac{\partial z}{\partial x} = B + Ky; \quad \eta = \frac{\partial z}{\partial y} = C + Kx \quad (2.19a)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2} = 0; \quad \frac{\partial^2 z}{\partial x \partial y} = K \quad (2.19b)$$

The positive directions of the membrane stresses and the external loading are shown in Fig. 2.12. The problem to be solved may be stated simply as: Given: an HP shell with known geometry and boundary conditions (Fig. 2.1) and known external loading, X , Y and Z (Fig. 2.12); Find: the membrane stresses N_x , N_y , and N_{xy} in the true surface of the shell (Fig. 2.12). The general method of solution was originally presented by Pucher [2.4] and basically consists of projecting the membrane stresses on the horizontal plan xy and first solving for these projected stresses from which in turn the true stresses may be found. The "projected" stresses $\bar{N}_x$, $\bar{N}_y$, and $\bar{N}_{xy}$ are in the direction of the x or y axes and are forces per horizontal unit length in the x or y directions. The "true" stresses N_x , N_y , and N_{xy} are in the direction of the x or y generators on the true surface of the shell and are forces per true unit length of generator. The external loads X , Y , or Z are forces per unit of horizontal projected area. The slope of the generators

*Detailed information on the input-output data as well as the availability of these computer programs is available from the author.

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in the x and y directions are defined by the $\tan \phi$ and the $\tan \psi$, respectively, and the angle between the generators in the true surface is γ .

Using equations of equilibrium with the projected stresses, the following expressions are obtained:

$$\frac{\partial \bar{N}_x}{\partial x} + \frac{\partial \bar{N}_{xy}}{\partial y} = -X \sin \omega; \quad \frac{\partial \bar{N}_y}{\partial y} + \frac{\partial \bar{N}_{xy}}{\partial x} = -Y \sin \omega \quad (2.20)$$

$$\bar{N}_x \frac{\partial^2 z}{\partial x^2} + \bar{N}_y \frac{\partial^2 z}{\partial y^2} + 2\bar{N}_{xy} \frac{\partial^2 z}{\partial x \partial y} = \left[X \frac{\partial z}{\partial x} + Y \frac{\partial z}{\partial y} - Z \right] \sin \omega \quad (2.21)$$

Substituting the geometric properties of the HP shell from Eqs. (2.19) into Eq. (2.21), the following results is obtained:

$$\bar{N}_{xy} = \frac{1}{2K} [\xi X + \eta Y - Z] \sin \omega \quad (2.22)$$

Note that by Eq. (2.22), $\bar{N}_{xy}$ is uniquely determined at any point in the shell and is independent of the boundary conditions. Once $\bar{N}_{xy}$ is known, $\bar{N}_x$ and $\bar{N}_y$ are obtained by suitable integration of Eq. (2.20):

$$\bar{N}_x = - \int \left[\frac{\partial \bar{N}_{xy}}{\partial y} + X \sin \omega \right] dx + f_1(y) \quad (2.23a)$$

$$\bar{N}_y = - \int \left[\frac{\partial \bar{N}_{xy}}{\partial x} + Y \sin \omega \right] dy + f_2(x) \quad (2.23b)$$

The arbitrary functions $f_1(y)$ and $f_2(x)$ can be determined from the specified boundary values of $\bar{N}_x$ and $\bar{N}_y$ at any two adjacent edges of the shell.

Finally, the true stresses (Fig. 2.12) are obtained from the projected stresses by the following relationships:

$$N_{xy} = \bar{N}_{xy}; \quad N_x = \bar{N}_x \mu; \quad N_y = \frac{\bar{N}_y}{\mu} \quad (2.24)$$

where

$$\mu = \frac{\cos \psi}{\cos \phi} = \sqrt{\frac{1 + \xi^2}{1 + \eta^2}} \quad (2.25)$$

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The solution of Eqs.(2.22) and (2.23), except for the well known case of uniform vertical load Z equal to w psf of horizontal projection, is extremely tedious and algebraically involved.

For the special loading $Z = w$, $X = 0$, $Y = 0$, substituting into Eqs. (2.22) and (2.24), N_{xy} is found to be constant throughout the shell and to be equal to the same value as found by the simple statics approach, Eq. (2.11). Also, it can be seen from Eq. (2.23) that if the boundary forces $\bar{N}_x$ and $\bar{N}_y$ are specified to be zero at boundary edges $x = 0$ and $y = 0$, respectively, then they will also be zero at the boundaries $x = a$ and $y = b$, respectively. This again verifies the solution found by statics for the special loading of $Z = w$.

For other general loadings, $\bar{N}_{xy}$ will not be constant throughout the shell, but can be found using Eq. (2.22). Formulas for $\bar{N}_{xy}$, $\bar{N}_x$, and $\bar{N}_y$ for various loadings are available [2.5, 2.7]. In order to satisfy the requirements of the membrane analysis, the boundary stresses $\bar{N}_x$ and $\bar{N}_y$ must be specified at two intersecting edges of the shell. It is always possible to choose the coordinate axes so that the z -axis passes through the point of intersection of these edges (Fig. 2.13). In this way, the stresses $(\bar{N}_x)_{x=0}$ and $(\bar{N}_y)_{y=0}$ acting on the boundaries at the y and x axes, respectively, can be chosen as desired while the boundary stresses at the opposite edges $(\bar{N}_x)_{x=a}$ and $(\bar{N}_y)_{y=b}$ are fixed. Thus, only two of the four edges of the HP can be freed of normal boundary forces for a general loading.

The stresses N_x , N_y , and N_{xy} are oblique components of the stresses in the true surface of the shell (Fig. 2.12). The true normal and shearing stresses N_β and T_β on any plane making an angle β in the true surface with the positive part of the x -generator may be found by considering the equilibrium of an element such as that shown in Fig. 2.14.

$$N_\beta = N_x \frac{\sin^2 \beta}{\sin \gamma} + 2N_{xy} \frac{\sin \beta \sin(\beta-\gamma)}{\sin \gamma} + N_y \frac{\sin^2(\beta-\gamma)}{\sin \gamma} \quad (2.26)$$

$$T_\beta = -N_x \frac{\sin \beta \cos \beta}{\sin \gamma} - N_{xy} \frac{\sin(2\beta-\gamma)}{\sin \gamma} - N_y \frac{\sin(\beta-\gamma) \cos(\beta-\gamma)}{\sin \gamma} \quad (2.27)$$

The true angle γ between the generators at any point in the shell can be found by the following formula:

$$\begin{aligned} \cos \gamma &= \cos \phi \cos \psi \cos \omega + \sin \phi \sin \psi \\ &= \frac{\cos \omega + \xi \eta}{\sqrt{(1+\xi^2)(1+\eta^2)}} \end{aligned} \quad (2.28)$$

The planes on which the principal stresses N_1 and N_2 act can be found by setting $T_\beta = 0$ in Eq. (2.27).

$$\tan 2\beta = \frac{2N_{xy} \sin \gamma + N_y \sin 2\gamma}{N_x + 2N_{xy} \cos \gamma + N_y \cos 2\gamma} \quad (2.29)$$

The two angles given by Eq. (2.29) are designated as $\theta_1 = 2\beta/2$ and $\theta_2 = \theta_1 + 90^\circ$, which define the planes on which N_1 and N_2 act, respectively. Substituting each of these angles individually into Eq. (2.26) for β , the values of N_1 and N_2 can be calculated.

The true angles between the positive part of the x-generator and the directions of N_1 and N_2 themselves (Fig. 2.14) are simply

$$\rho_1 = \theta_1 - 90^\circ; \quad \rho_2 = \theta_2 - 90^\circ \quad (2.30)$$

The projection of each of these angles ρ_1 and ρ_2 on the horizontal xy plane may be found by the following formula:

$$\tan \bar{\rho} = \frac{\mu \sin \rho \sin \omega}{\cos \rho \sin \gamma + \sin \rho (\mu \cos \omega - \cos \gamma)} \quad (2.31)$$

The computation of membrane stresses in HP shells under general loadings, if done manually, is extremely tedious and time-consuming. However, the procedure has been programmed for a digital computer so that the results can be obtained simply and rapidly using an available computer program [2.5] which requires a minimum of input data.

MEMBRANE STRESSES IN HP SHELLS HAVING AN ARBITRARY QUADRILATERAL SHAPE IN PLAN

A method for deriving general formulas for membrane stresses in HP shells having a parallelogram shape in plan was described above. Now, it will be shown how those formulas, with proper modifications, can be used to determine membrane stresses in HP shells having a quadrilateral shape in plan in which the four corners may have arbitrary and independent elevations in space. The solution described enables the designer to analyze the membrane stresses in a variety of interesting architectural and structural combinations of HP shells, a few examples of which are shown in Fig. 2.15.

A general HP in space is shown in Fig. 2.16. $x_1x_2x_3$ is a right hand orthogonal coordinate system with the x_1x_2 plane parallel to a horizontal plane. xyz is a right-hand skew

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coordinate system with the z axis normal to the xy plane and the skew angle between the x and y axes equal to ω . The HP shown is formed by two series of straight line generators represented by the unit vectors $\hat{I}$ and $\hat{II}$, which are parallel to planes I and II, respectively. Planes I and II in turn are respectively parallel to the xz and yz planes of the skew coordinate system. A projection of the HP on the xy plane will always appear as a parallelogram (Fig. 2.17a) while a projection of the HP on any other plane, for example x_1x_2 , will appear as an arbitrary quadrilateral (Fig. 2.17b).

General formulas for membrane stresses in HP shells may be derived as described above for the case of the geometry and loading given in the skew coordinate system xyz . The present problem is: given an HP with geometry and loading in the orthogonal coordinate system $x_1x_2x_3$, find the true membrane stresses in the shell. A complete and detailed solution to this problem is described in Ref. [2.7].

The location of a point $P(\alpha, \beta)$ on the HP (Figs. 2.16 and 2.17) may be conveniently defined by the parameters α and β . α represents the fraction of the total distance between $\alpha = 0$ and 1 that the generator $\hat{II}$ has traveled. β is found similarly using the generator $\hat{I}$ between $\beta = 0$ and 1. In this manner, the equation of the HP in the $x_1x_2x_3$ coordinate system may be written as

$$x_3 = x_{i3} + (x_{k3} - x_{i3})\alpha + (x_{j3} - x_{i3})\beta + (x_{l3} - x_{j3} - x_{k3} + x_{i3})\alpha\beta \quad (2.32)$$

and in the xyz skew coordinate system as

$$z = z_i + (z_k - z_i)\alpha + (z_j - z_i)\beta + (z_l - z_j - z_k + z_i)\alpha\beta \quad (2.33)$$

or, if desired in the xyz system, as given in Eq. (2.17) above for an HP with a parallelogram shape in the xy plane.

$$z = A + Bx + Cy + Kxy \quad (2.17)$$

For an HP with an arbitrary quadrilateral shape, the given data consist of the x_1, x_2, x_3 coordinates of points i, j, k , and l , Figs. 2.16 and 2.17b, and the loading vectors in the orthogonal $x_1x_2x_3$ coordinate system. Using vector algebra and the given data, the geometry of the shell $a, b, \omega, z_i, z_j, z_k$, and z_l , Figs. 2.16 and 2.17a, and the loading vectors in the skew xyz coordinate system can be found. Once this is done, the general formulas derived for an HP with a parallelogram

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shape in the xy plane may be used to determine the membrane stresses N_x, N_y, N_{xy} , the principal membrane stresses and their directions, and the boundary stresses at the edges. A computer program [2.7] is available for performing this analysis with a minimum of input data.

CONCLUSIONS

Methods for determining membrane stresses in HP shells using simple statics for simple loadings and geometries and using a differential equation approach for more general loadings and complex geometries have been presented.

The importance of understanding the statics of the combined shell-edge beam system has been emphasized.

Computer programs are available [2.5, 2.7] for performing the membrane analyses described with a minimum of input data.

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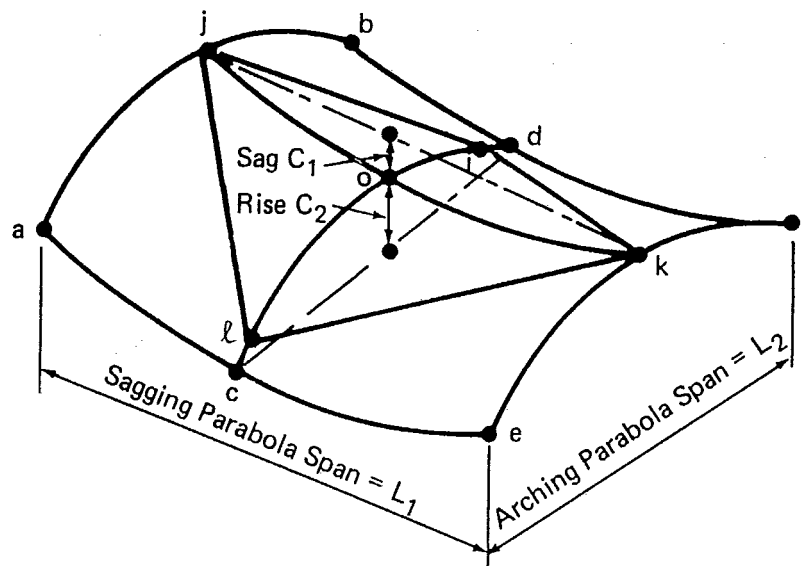
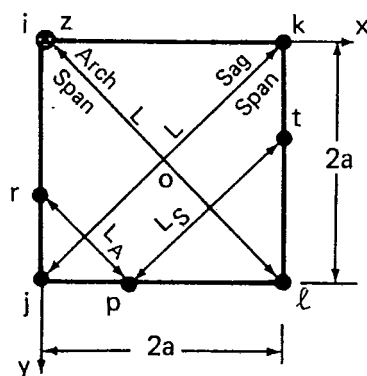
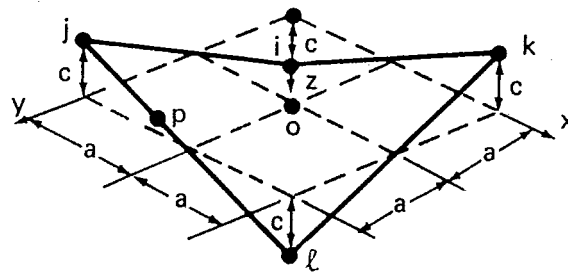


Fig. 2.1--Hyperbolic paraboloid (HP) surface



a) Plan View



b) Three Dimensional View

Fig. 2.2--Square HP shell with $L_1 = L_2 = L$ and $c_1 = c_2 = c$

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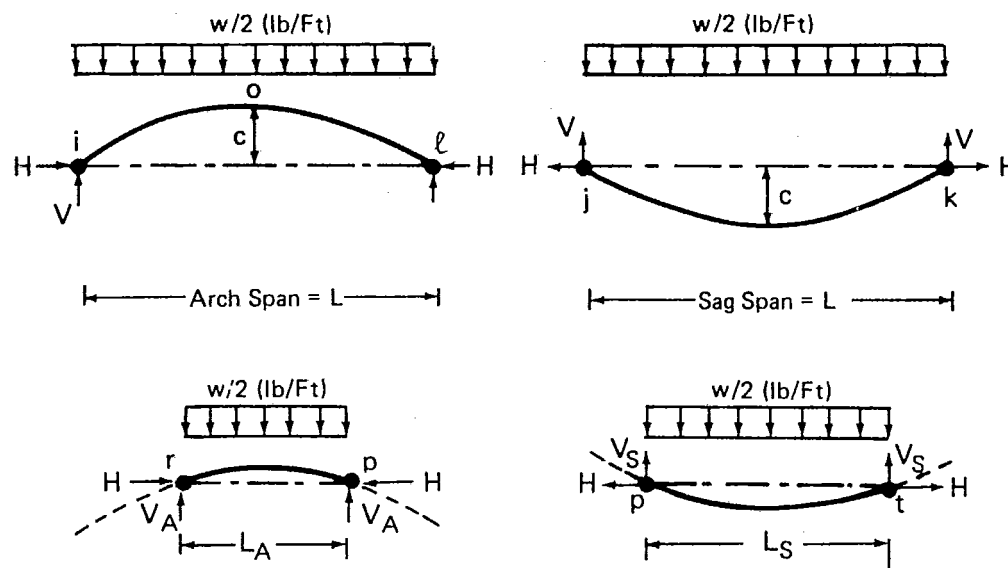
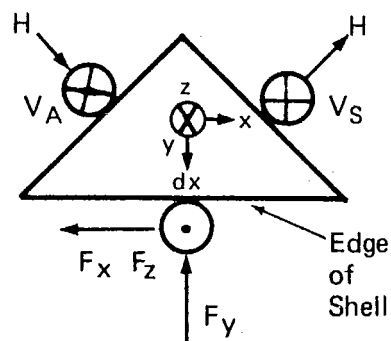
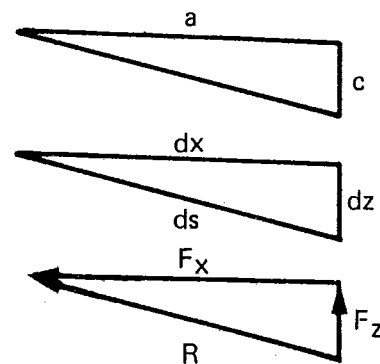


Fig. 2.3--Loads and reactions on one-foot-wide arching and sagging parabolic strips



a) Plan View



b) Elevation Views

Fig. 2.4--Differential wedge element at point p on shell edge boundary

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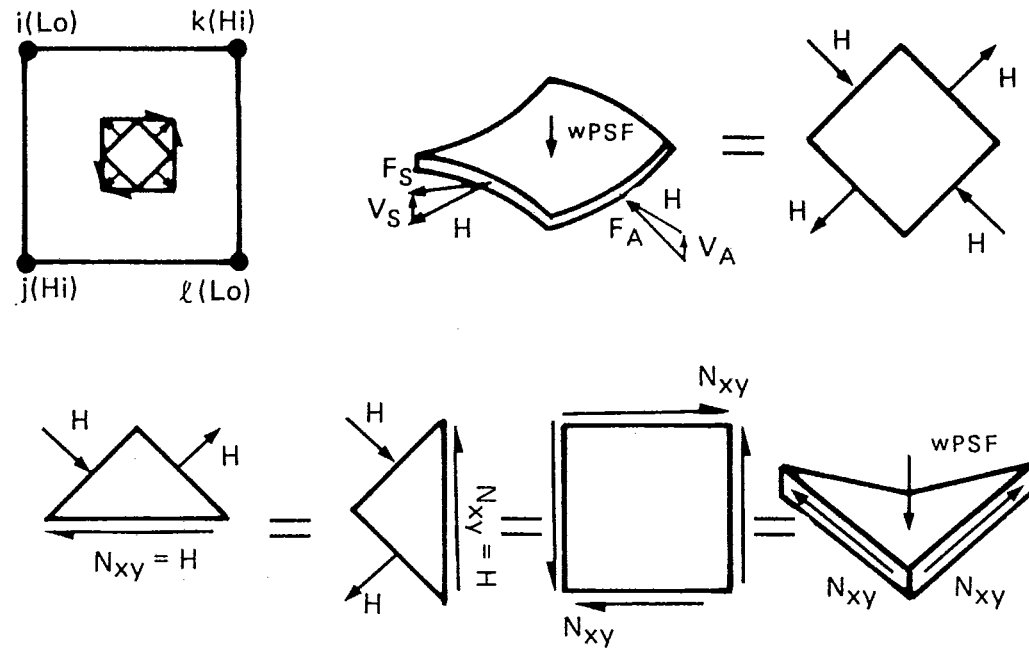


Fig. 2.5--Stress (force/length) states on a typical differential element

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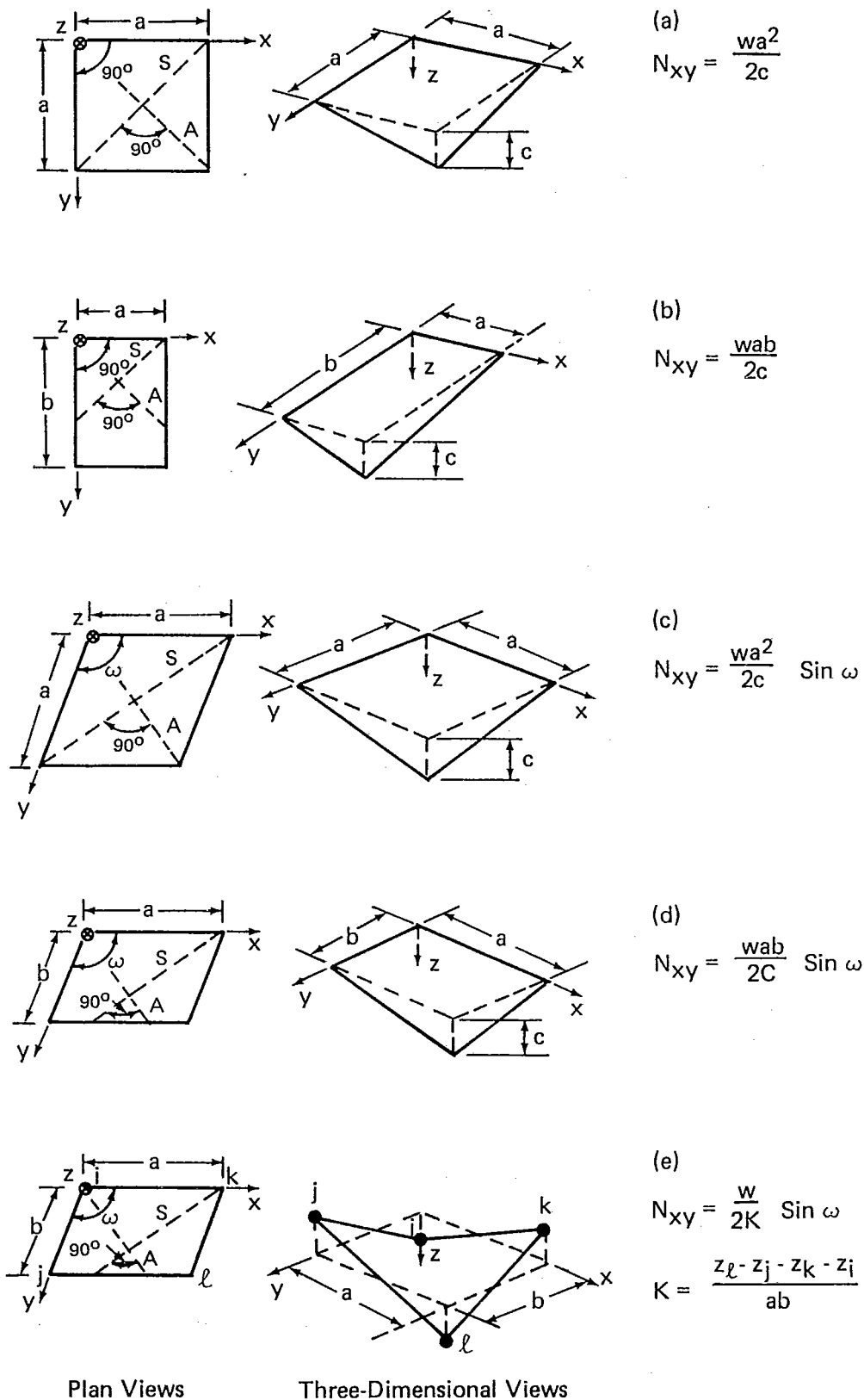


Fig. 2.6--Membrane shear N_{xy} for case of uniform load w psf of horizontal projection

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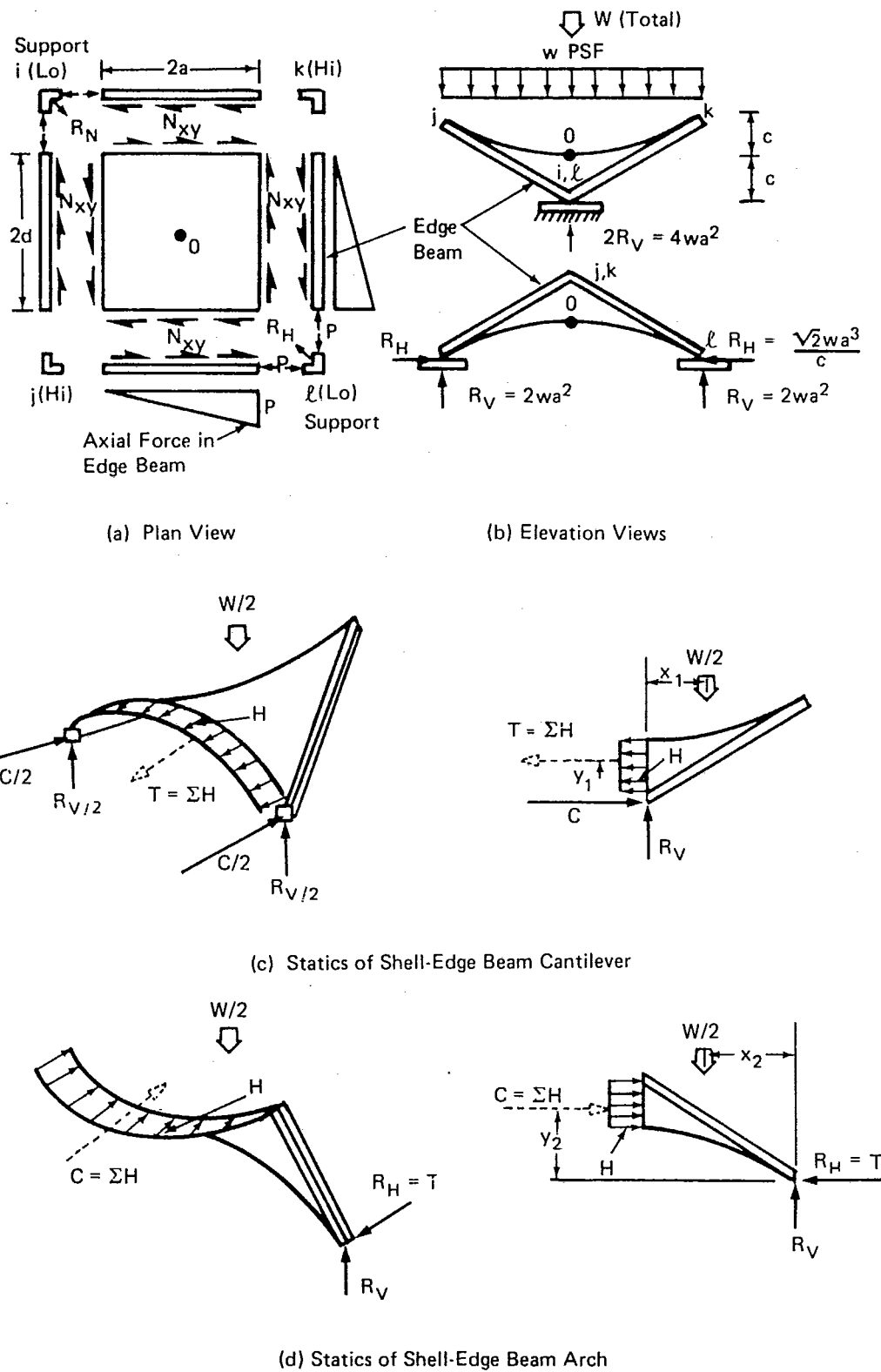


Fig. 2.7--Statics of saddle shell-edge beam support system

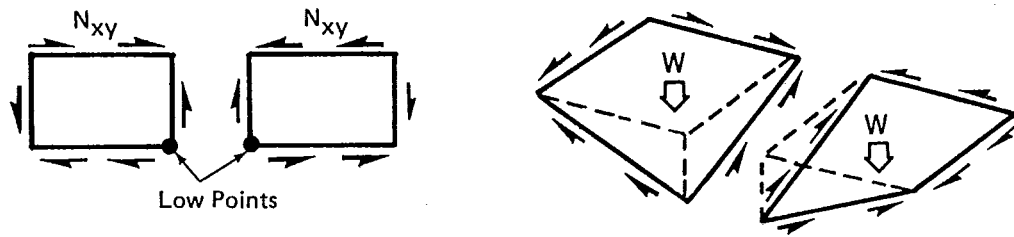


Fig. 2.8--Boundary shear forces on HP shell panels

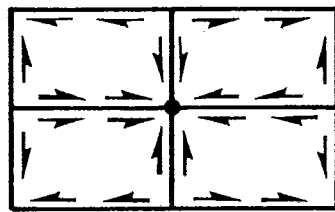
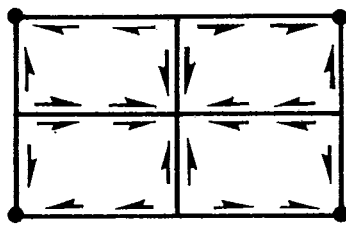
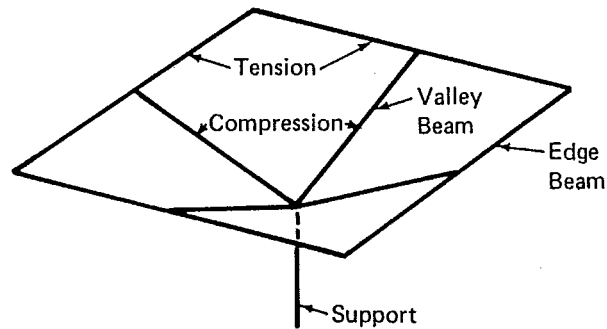
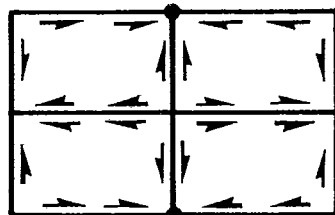
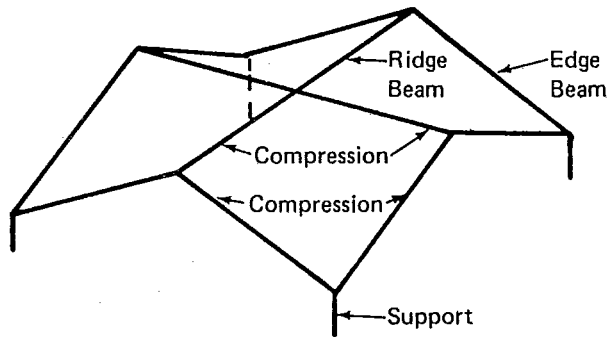
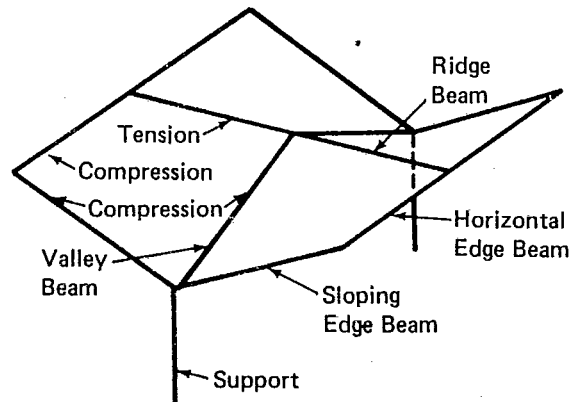
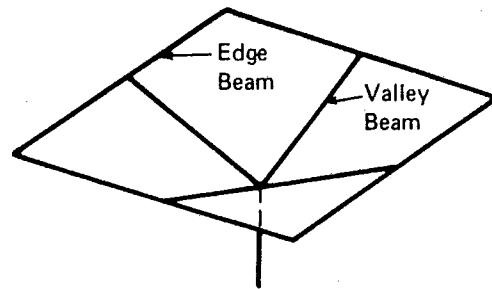
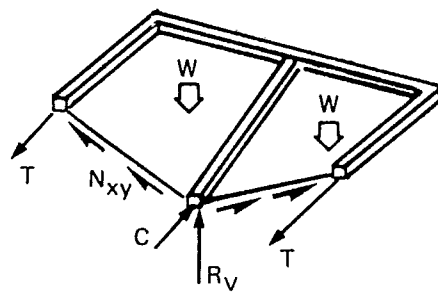
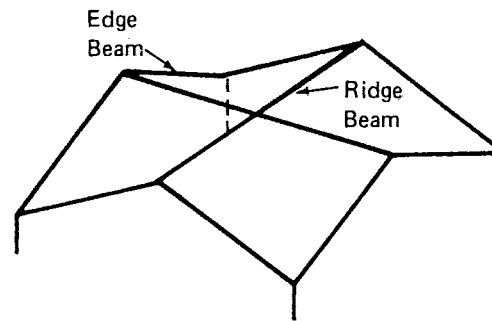
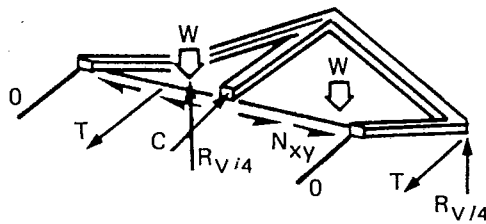
a) Inverted Umbrella
(Single Column Support)b) Gable Shell
(Four Supports)c) Shell on Two
(Column Supports)

Fig. 2.9--Shear forces acting on edge beam systems

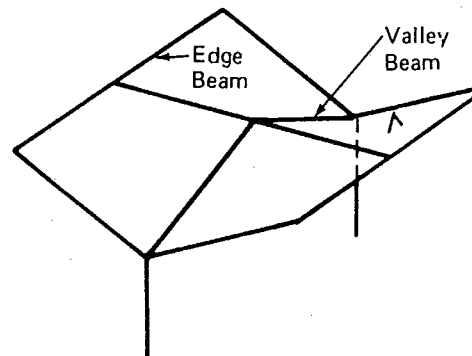
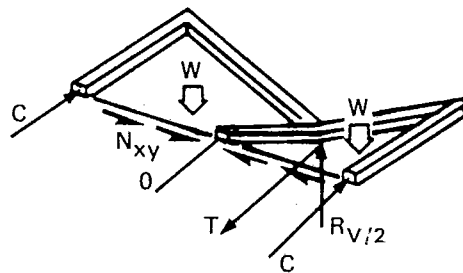
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a) Inverted Umbrella on Single Support



b) Gable Shell on Four Supports



c) Shell on Two Supports

Fig. 2.10--Statics of shell-edge beam systems

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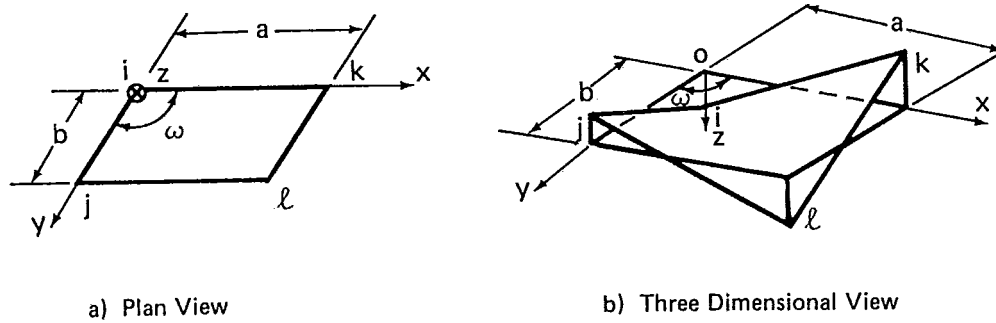


Fig. 2.11--Geometry of the HP shell

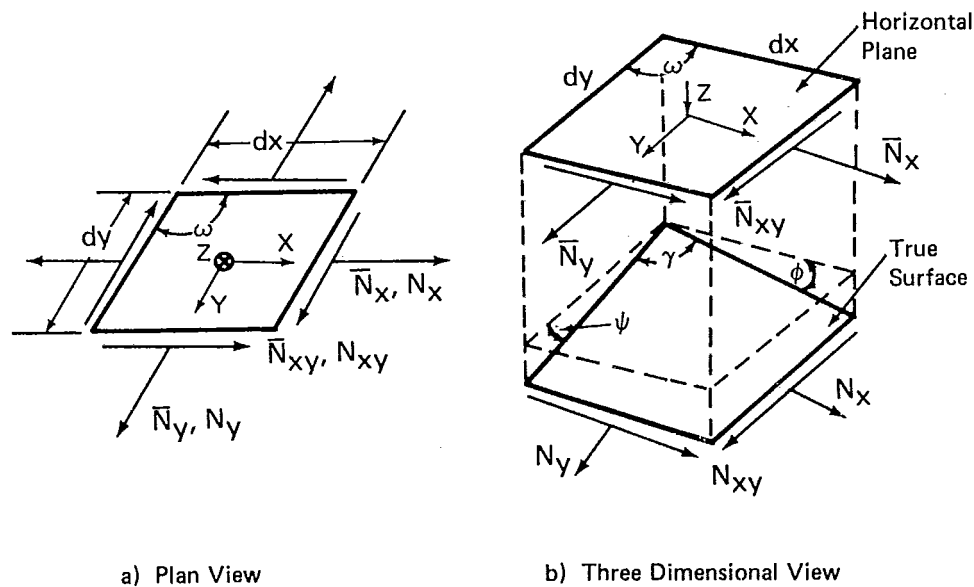
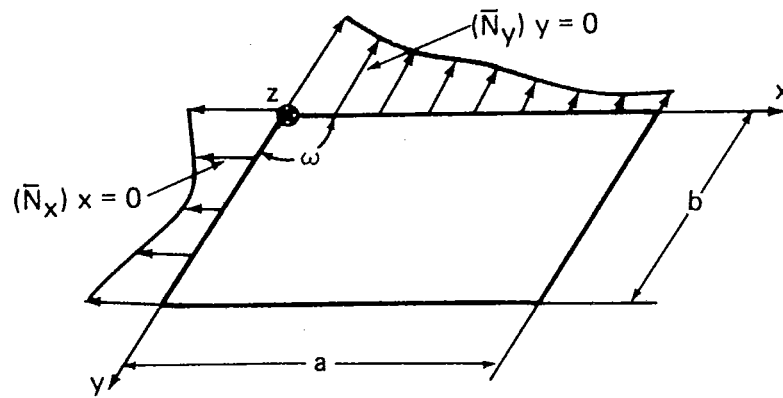
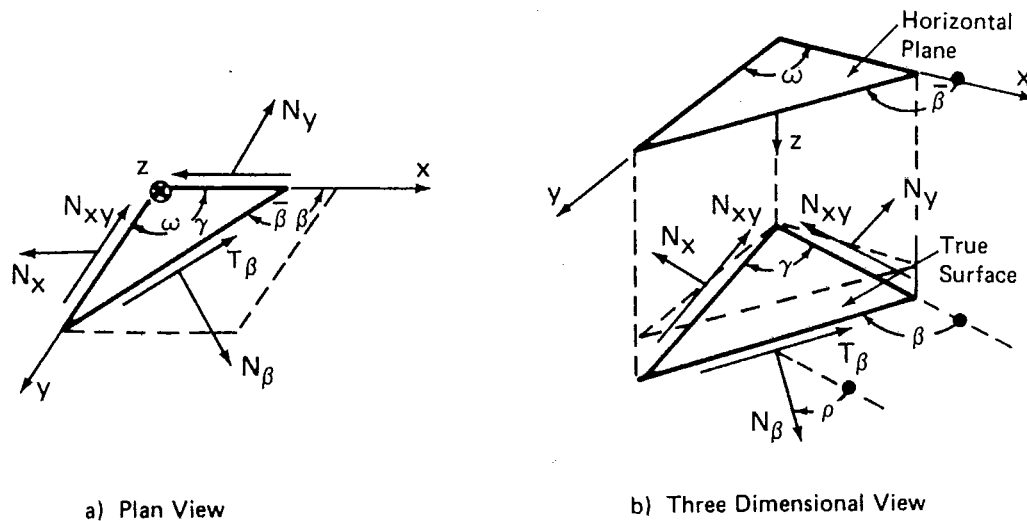
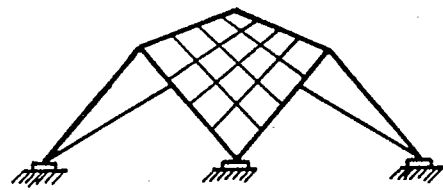
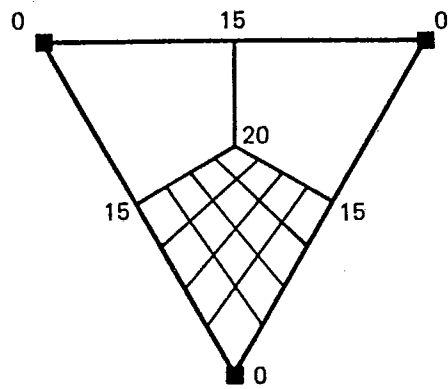


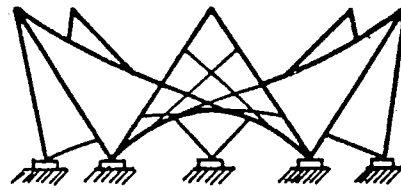
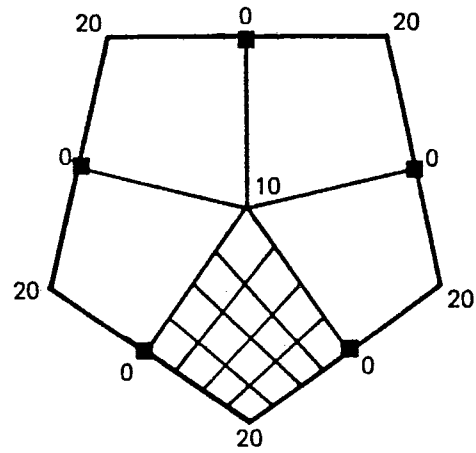
Fig. 2.12--Membrane stresses on true surface and in projected xy plane

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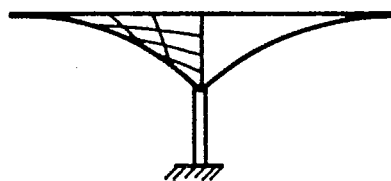
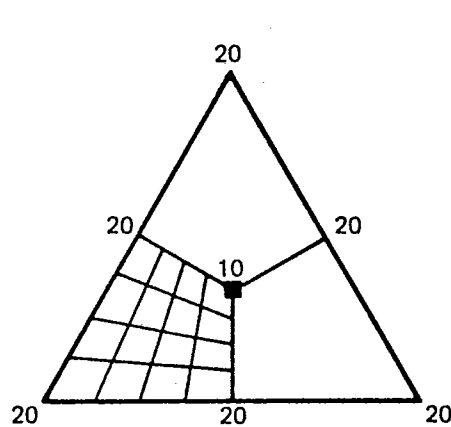
Fig. 2.13--Boundary stresses at $x = 0$ and $y = 0$ Fig. 2.14--Normal and shearing stresses, N_β and T_β , on an arbitrary plane on the true surface of the shell



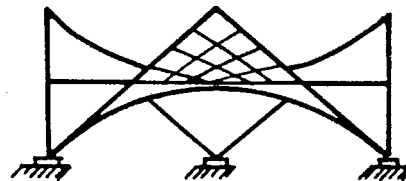
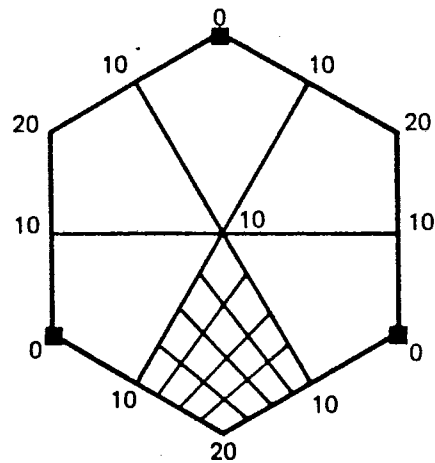
(a) Example 1



(b) Example 2



(c) Example 3



(d) Example 4

Fig. 2.15--Structural systems using HP shell units having arbitrary quadrilateral shapes

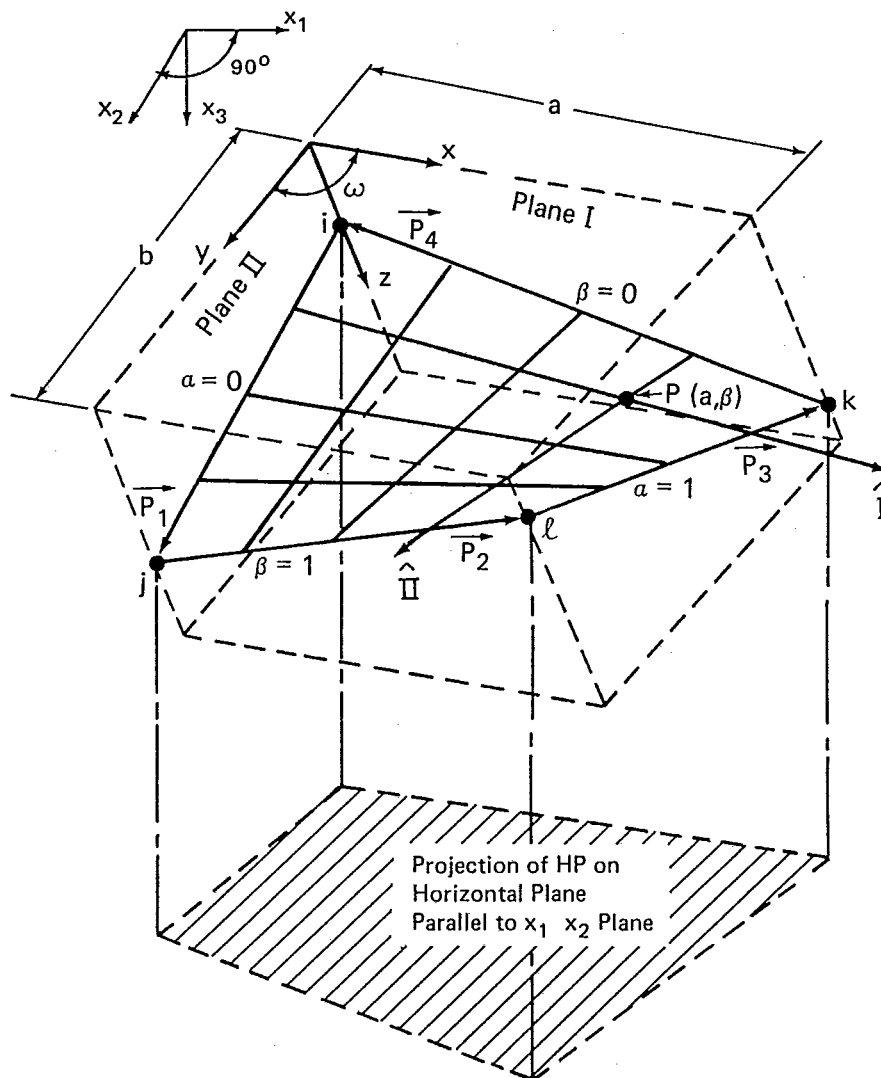
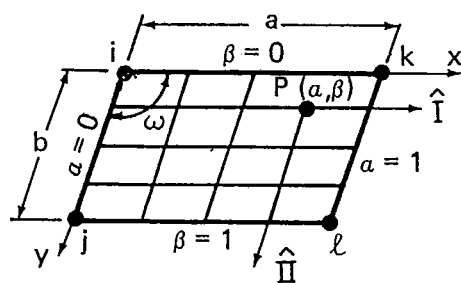
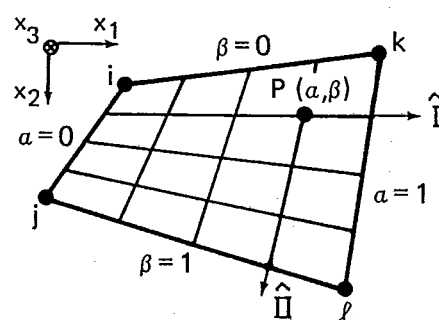


Fig. 2.16--Geometry of a general HP shell



a) xy Plane



b) $x_1 x_2$ Plane

Fig. 2.17--Projections of HP on xy plane and horizontal x_1x_2 plane

Chapter 3

Bending Theory

by W.C. Schnobrich

Synopsis:

The equations for a bending theory of shallow shells are specialized to the case of the hyperbolic paraboloid shell. The nature of the analytic solutions to these equations that have been achieved for this class of shells is described. Such analytical solutions involve boundary conditions which depart significantly from those present in real shell applications. Types of numerical approximate solutions present as alternatives are noted and the details of one of these, the Finite Elements Method, are described. The importance of the dead weight of perimeter or support beams is noted, demanding a bending solution with real support conditions. Some representative solutions are included as examples.

Keywords: bending; edge beams; finite element method; hyperbolic parabolic shells; reinforced concrete; shells (structural forms); structural analysis

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3.1 INTRODUCTION

The severe deficiencies of the membrane analysis, i.e., the inability to account for the dead weight of supporting and/or edge beams and the failure to achieve compatibility between the shell and those beams, can result in a significant departure of computed membrane results from reality. For a number of HP configurations a bending analysis is essential to the proper definition of the behavior of the shell-beam system even at service loads.

Many discussions of general bending theories adequate to the HP problem exist in the literature [3.2,3.6,3.13,3.17,3.25]. Basic assumptions common to all these shell theories are made during the development of these theories

- a) The thickness dimension is small relative to the other characteristic dimensions of the shell.
- b) The displacements are small relative to the dimensions of the shell. Equilibrium equations being written relative to the undeformed configuration.
- c) Transverse shearing strains are neglected. This assumption translates into normals remaining normal. This assumption may easily be relaxed, however, for those cases where shear deformations may be significant.
- d) Normal stresses in the thickness direction are neglected.

The HP as used in a roof shell configuration does not require a particularly rigorous theory to be sufficient for the problem. The equivalent of a Donnell type theory will suffice. However, the actual support conditions present in the real structure must be closely simulated if an acceptable solution is to be attained. Classic approaches using boundary conditions for which bending solutions can be achieved through analytic means, may differ as far from the real case as would a pure membrane solution. Fortunately, use of one of the readily available numerical procedures circumvents the problem. They provide the

designer with an analysis method adequate to the problem. Thus, these procedures will be emphasized in this report.

3.2 GOVERNING EQUATIONS

The governing equations expressing the bending theory for hyperbolic paraboloidal shells can be cast in several forms, depending on the coordinate system used when defining the shell surface. The first equations presented below are specialized to the particular case of the hyperbolic paraboloid described by characteristic lines, i.e., the surface is related to a coordinate system which coincides with its asymptotic or straight lines, Fig. 3.1. The equation of the surface in this form is

$$Z = kxy; \quad k = \frac{f}{ab}.$$

The terms are defined in the figure. Poisson ratio is assumed zero and higher order terms are neglected.

The equations expressing the equilibrium of a small element cut from the shell are:

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + P_x &= 0 \\ \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} + P_y &= 0 \\ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + 2kN_{xy} + P_z &= 0 \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x &= 0 \\ \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y &= 0 \end{aligned} \quad (3.2)$$

The geometric relations defining the strains in terms of displacements are expressible by

$$\epsilon_x = \frac{\partial u}{\partial x} = u_{,x}; \quad \epsilon_y = \frac{\partial v}{\partial y} = v_{,y}; \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2kw \quad (3.3)$$

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where $(\)_{,x} = \partial(\)/\partial x$.

With a linear elastic theory, the stress resultants and displacements are related by the following equations

$$\begin{aligned} N_x &= K u_{,x} \\ N_y &= K v_{,y} \\ N_{xy} &= \frac{k}{2} (u_{,y} + v_{,x} - 2kw) \\ M_x &= -D w_{,xx} \\ M_y &= -D w_{,yy} \\ M_{xy} &= -D w_{,xy} \end{aligned} \quad (3.4)$$

where

$$K = Et \text{ and } D = \frac{1}{12} Et^3$$

On substituting the following nondimensional coordinates

$$\begin{aligned} \bar{u} &= u/a & \bar{v} &= v/b & \bar{w} &= kw & \lambda &= t^2/12rf^2 \\ \xi &= x/a & \eta &= y/b & r &= a/b & \bar{Q} &= P_z a^2/Etf \end{aligned} \quad (3.5)$$

into equations (3.2) and (3.4) then those results into equilibrium equations (3.1), the result can be written in the following form:

$$\bar{v}_{,\xi\eta} + 2\bar{u}_{,\xi\xi} + r^2\bar{u}_{,\eta\eta} - 2r\bar{w}_{,\eta} = P_x a^2/K \quad (3.6)$$

$$r^2\bar{u}_{,\xi\eta} + 2r^2\bar{v}_{,\eta\eta} + \bar{v}_{,\xi\xi} - 2r^2\bar{w}_{,\xi} = P_y a^2/K \quad (3.7)$$

$$(2r\bar{w} - r^2\bar{u}_{,\eta} - \bar{v}_{,\xi}) + (\bar{w}_{,\xi\xi\xi\xi} - 2r^2\bar{w}_{,\xi\xi\eta\eta} + r^4\bar{w}_{,\eta\eta\eta\eta}) = Q \quad (3.8)$$

If the displacements are not nondimensionalized, the equations are

$$2u_{,\xi\xi} + r^2u_{,\eta\eta} + r v_{,\xi\eta} - 2r a k w_{,\eta} = P_x a^2/K \quad (3.6a)$$

$$r u_{,\xi\eta} + 2r^2 v_{,\eta\eta} + v_{,\xi\xi} - 2a k w_{,\xi} = P_y a^2/K \quad (3.7a)$$

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$$ru_{,\eta} + v_{,\xi} - 2akw + \bar{\lambda}^2 \bar{\Delta}^4 (kaw) = P_z a^2 / kK \quad (3.8a)$$

with

$$\bar{\lambda}^2 = \frac{1}{12} \left(\frac{h}{a}\right)^2 \left(\frac{b}{f}\right)^2 \bar{\Delta}^4 = \frac{\partial^4}{\partial \xi^4} + 2r^2 \frac{\partial^4}{\partial \xi^2 \partial \eta^2} + r^4 \frac{\partial^4}{\partial \eta^4}$$

It is possible to reduce the governing equilibrium equations expressed in the displacement unknowns, as the above three equations represent, into a system of two equations in two unknowns by selecting a stress function which is chosen so as to automatically satisfy the first two of Eq. (3.1). This is achieved by defining the in-plane stress resultants as

$$\begin{aligned} N_x &= \Phi_{,yy} + \int P_x dx \\ N_y &= \Phi_{,xx} + \int P_y dy \\ N_{xy} &= -\Phi_{,xy} \end{aligned} \quad (3.8)$$

The equilibrium equations, (3.1), now reduce to

$$D\Delta^4 w = Z - 2k\Phi_{,xy} \quad (3.9)$$

where

$$\Delta^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2$$

With the introduction of the stress function as an unknown, it becomes necessary to enforce the compatibility of strains by the following equation

$$\Delta^4 \Phi = 2Etkw_{,xy} \quad (3.10)$$

Unless this equation is satisfied, there is no guarantee that the deformed structure will be a smooth continuous structure. In order to determine the complete deflected shape of the shell, the tangential displacements must be determined. These displacements are computed from the stress resultant-displacement equations, (3.4). Thus, the tangential displacements are related to the stress function by the equations

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$$\begin{aligned}
 u &= \frac{1}{Eh} \int (\Phi_{,yy} - \nu \Phi_{,xx}) dx \\
 v &= \frac{1}{Eh} \int (\Phi_{,xx} - \nu \Phi_{,yy}) dy
 \end{aligned}
 \tag{3.1}$$

If in-plane loads exist they should be added to these equations. The fact that the tangential displacements are integral equations of the stress function is a severe handicap when the boundary conditions pertinent to the structure involve these displacements. Several solutions are published in the literature combining the Φ and w equations into a single equation. Apeland and Popov [3.3] present the solution form for two opposite edges being moment free knife edge supports for example.

The classical thin shell solution thus involves the solution of either the three partial differential equations with displacements unknowns (u - v - w formulation Eqs. (3.6-3.8) or two equations with the normal displacement and the stress function as unknowns (Φ - w formulation, Eqs. 3.9 & 3.10). Which of these forms or variations therefrom is used as a matter of personal preference, the method to be used to achieve a solution to whatever equations are used, and/or the boundary conditions present for the real shell.

If the coordinate axes are aligned to the principal curvatures, the equation of the surface can be expressed as, Fig. 3.2,

$$Z = -C_x \left(\frac{x}{a}\right)^2 + C_y \left(\frac{y}{b}\right)^2 \quad \text{or} \quad Z = \frac{x^2}{2r_x} + \frac{y^2}{2r_y}$$

where r_x and r_y are the radii of curvature in the x and y directions. This equation can also be derived from the previous equation by a simple coordinate transformation. Note for the HP, r_x is negative while r_y is positive. The equilibrium equations are now

$$\begin{aligned}
 \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + P_x &= 0 \\
 \frac{\partial N_y}{\partial x} + \frac{\partial N_{xy}}{\partial y} + P_y &= 0 \\
 \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{N_x}{r_x} + \frac{N_y}{r_y} + P_z &= 0
 \end{aligned}
 \tag{3.12}$$

The bending equations remain the same as before, (Eq. 3.2). The strain-displacement equations now take the form