

STATUS AND POTENTIALITIES OF NONLINEAR DESIGN OF CONCRETE FRAMES

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SYNOPSIS

Because structural failure generally occurs in successively more severe stages at successively less probable loads, design should ideally account for all stages and be based on comprehensive analysis utilizing a comprehensive, non-linear, force-strain relationship. The criterion for optimum design, using the failure-stage-versus-load profile, is derived. For frames, a method of comprehensive analysis based on a multilinear moment-curvature relationship, using critical moments and "plasticity factors," is presented. Procedures and the relative economics of comprehensive design and its special cases, elastic, plastic, and ultimate strength designs, are compared. A bilinear design procedure for concrete frames, based on two failure stages, is presented.

THE STAGES OF FAILURE AND COMPREHENSIVE DESIGN

Notation.—The symbols adopted for use in this paper are defined where they first appear and are listed alphabetically in Appendix II.

Although asked by the program committee to present a keynote survey paper to the Symposium, it is hardly necessary to state that the views to be presented may not coincide with those of committee members. Rather, what follows will be an effort to establish some basic concepts (flavored by personal biases) by which some of our differences may be clarified, if not reconciled.

As the loads increase on a structure, structural failure may be considered to occur in stages, each more severe than the preceding stage, and each representing a higher relative loss, L , to the owner and users of the structure. However, the probability, p , that the load corresponding to any failure stage will occur in the life of the structure decreases with increase in load. Thus,

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a low probability tends to compensate for the high loss of a severe failure stage, and any failure stage may be the most critical in the design of the structure (1).²

Therefore, in the ideal, design is a comprehensive procedure by which the resistances of a structure to the various failure stages are correlated to the probabilities of the corresponding loads so that the total cost, including the first cost and the expected losses from all failure stages, is minimized. This comprehensive design procedure necessarily requires a knowledge or determination of the failure-stage-versus-load relationship for any structure assumed by the designer. This relationship must be obtained by some sort of comprehensive analysis—based on a model by which the designer may predict the behavior of the structure with respect to all possible critical failure stages.

Thus, the ultimate concern of the designer is comprehensive design. Elastic design, limit design, or ultimate strength design are but limited aspects of this concern, and so these terms have been purposely eliminated from the title of this symposium.

THE FAILURE-STAGE VERSUS LOAD RELATIONSHIP

Each load on a structure has its own characteristic failure-stage-versus-load relationship. Also, these relationships vary from structure to structure; such relationships for reinforced concrete structures generally differ from those for steel structures, and characteristic differences exist for statically determinate and indeterminate structures.

Fig. 1 shows the writer's concept of this relationship for a possibly typical load on a typical statically indeterminate reinforced concrete building, including an attempted approximation of the losses corresponding to the failure stages, and a plot of these losses versus the probability of the load in the life of the structure on logarithmic scales.

For this case, the first failure stage shown is that caused by the small creep deflections and minor tensile cracking which always occurs and which causes very small loss (i.e., slight loss in corrosion or weathering resistance). With higher load, the yield point of the reinforcement is exceeded at more and progressively longer regions, leading to wide cracking, objectionable deflections, loss of user-confidence, and need for repairs. Then, with further increase in load, the concrete strains begin to exceed the value, approximately 0.004, at which spalling will occur, and deflections become excessive, soon leading to abandonment of the building. The loss at this point will include not only the cost of the building, but the cost of wrecking and the losses from interruption of the affected businesses.

The final stages are the actual collapse of portions of the frame, followed by the limit stage of collapse of the entire frame. Both of these stages usually occur at appreciably higher loads than spalling because of: The redistribution of forces that continues after spalling, the increase of resistance of some sections after spalling, and the resistance to collapse of those elements of the building usually neglected in the frame design such as walls, floors and roofs. The loss at total collapse has been estimated so as to include

² Numerals in parenthesis refer to corresponding items in the Appendix.

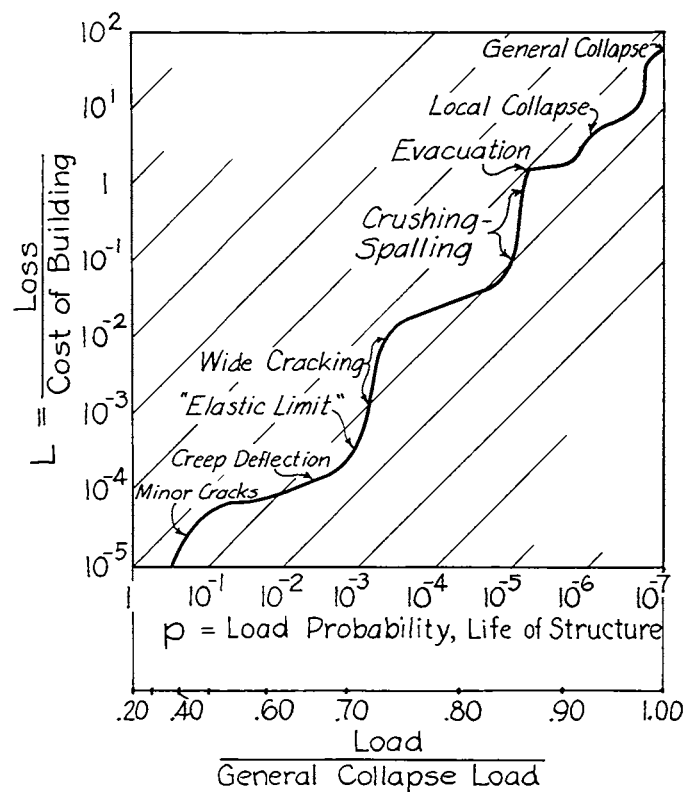


FIG. 1

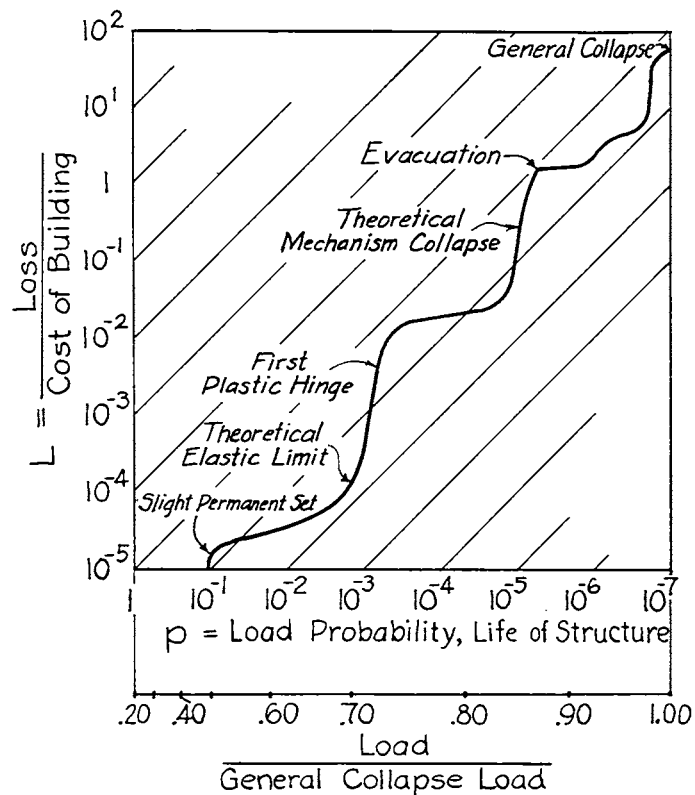


FIG. 2

NONLINEAR DESIGN

the losses from destruction of the tools and records of the businesses and the loss from a 0.5 probability of loss of life of the occupants of the building. Evaluating the loss of a life is difficult and even controversial, and the loss has here been assumed at twice the total future earnings of the occupant, with one occupant assumed for each 250 sq ft of an average-cost building.

A comparable and somewhat typical relationship for a building with continuous steel framing is shown in Fig. 2. Here, the successive failure stages for successively higher loads involve very small, and then larger, amounts of permanent set and deflection, with abandonment of the building following soon after the formation of sufficient hinges for the theoretical collapse mechanism, and with actual collapse occurring at an appreciably higher load because of the strength reserves from strain hardening, the moment gradients at the hinges, and the elements of the building neglected in the frame design.

A significant feature of relationships similar to those of Figs. 1 and 2 is that the relative total expected loss in the life of the building from all failure stages for the particular loading is closely given by the area under the curve:

$$\frac{\text{Total Expected Loss}}{\text{Cost of Building}} = \int_{p=0}^{p=1} L dp = \int_{L=0}^{L=L_c} p dL \dots (1)$$

in which L_c is the value of L at collapse. Mathematically, the error in this relationship becomes negligible for low probabilities, and this error may be neglected for probabilities of magnitude corresponding to the failure stages shown.

Because both scales of Figs. 1 and 2 are logarithmic, the areas of these plots are of variable density. The parallel lines inclined at 45° shown on these figures are contours of constant density, with each successive line indicating a change in density by a factor of ten. Thus, the expected loss is very sensitive to horizontal displacement of any "hump" on the curve representing a failure stage on the curve, and the horizontal placement of these humps is the prerogative of the structural designer.

Two goals of comprehensive design, as distinguished from design for one failure stage, may now be stated in terms of the loss-failure-load relationship. The major objective is the optimum horizontal placement of the failure-stage profile. A minor objective is the adjustment of this profile within the rather severe limitations imposed by statics and other design requirements, toward the optimum shape, for which the failure stages are located close to a common 45°-inclined line.

OPTIMUM PLACEMENT OF FAILURE-STAGE PROFILE

The optimum horizontal placement of the failure-stage profile will be considered first. For an infinitesimal increase in strength or load factor of the structure all failure stages will occur at an infinitesimally higher load with corresponding decreases in probability of occurrence in the life of the structure, dp . That is, each point of the curves of Figs. 1 or 2 will move to the right a distance dp . (The scale of load relative to true collapse load will move likewise.) The corresponding change in expected loss will be the change in area under the curve of the figure plus the changes in areas of all other

similar curves representing all other modes of failure and loading:

$$\frac{d(\text{expected loss})}{C_T} = \sum_{\text{All Modes}} \int_{L_1}^{L_c} (dp) dL$$

in which C_T is total cost of the building, and L_1 is the L-axis intercept (a more significant value of L_1 will be established later). Because, mathematically

$$dp = p d(\log_e p)$$

the change in expected loss becomes:

$$\frac{d(\text{expected loss})}{C_T} = \sum_{\text{All Modes}} \int_{L_1}^{L_c} d(\log_e p) p dL \dots (2)$$

The optimum or most economical horizontal position will occur when the sum of the change in expected loss and the change in the cost of the building is zero. The change in the cost of the building may be obtained from the curve of frame cost versus collapse load of which Fig. 3 is an example. Obviously,

$$\begin{aligned} \frac{d(\text{cost frame})}{C_T} &= \frac{\partial \left(\frac{C_F}{C_T} \right)}{\partial (\log_{10} p_c)} d(\log_{10} p_c) \\ &= 0.434 \frac{\partial \left(\frac{C_F}{C_T} \right)}{\partial (\log_{10} p_c)} d(\log_e p_e) \dots \dots \dots (3) \end{aligned}$$

Adding Eqs. 2 and 3 and equating to zero

$$\begin{aligned} &0.434 \frac{\partial \left(\frac{C_F}{C_T} \right)}{\partial (\log_{10} p_c)} d(\log_e p_c) \\ &+ \sum_{\text{All Modes}} \int_{L_1}^{L_c} d(\log_e p) p dL = 0 \dots \dots \dots (4) \end{aligned}$$

In general, the horizontal movements dp , actually $d(\ln p)$, of the points on a curve such as Figs. 1 or 2 will not be equal. Instead, the increase in load for any given failure condition will tend to be proportional to the load. Furthermore, the load scale corresponding to the logarithmic probability scale will likely be somewhat irregular and nonlinear, as shown in Figs. 1 and 2. The value of load corresponding to a probability of unity will usually be the dead

load; the value of load corresponding to a probability of 10^{-1} will usually be greater than the service load; and the collapse load probability is usually not less than 10^{-6} . Therefore, the load-scale tends to expand from left to right.

Both of these factors cause the values of $d(\ln p)$ for $1 > p > 10^{-1}$ or even $1 > p > 10^{-2}$ to be much smaller than the values for the remainder of the scale. Therefore, a good approximation of the integral of Eq. 4 may be obtained by ignoring values of L equal to or less than those for the lowest plateaus of Figs. 1 and 2, and the limit L_1 may then be redefined as the value of L at the top of this lowest plateau, roughly corresponding to the theoretical elastic limit for either steel or reinforced concrete.

A second approximation, involving more error than the one considered above, but nevertheless potentially useful, now becomes apparent. Between L_c and the redefined L_1 , corresponding to values of p on Figs. 1 and 2 between approximately 10^{-7} and 10^{-3} , the load scale is almost linear. If it is assumed that for this range the value of $d(\log_{10} p)$ is constant, Eq. 4 becomes

$$-0.434 \frac{\partial \left(\frac{C_F}{C_T} \right)}{\partial (\log_{10} p_c)} = \sum_{\text{All Modes}} \int_{L_1}^{L_c} p \, dL \dots \dots (5)$$

This basic relationship may be written thus: For optimum placement of the failure-stage profile, Fig. 1, the slope of the frame-cost versus collapse-load-probability curve, Fig. 3, is numerically equal to the expected loss from all modes of failure for L exceeding the redefined L_1 .

Eq. 5 has more potential significance in the quantitative, rational determination of load factors than is indicated by the rigor of some of its assumptions because of the extremely large changes in probability corresponding to small changes in load.

VARIATION IN SHAPE OF PROFILE

The shape of the failure stage profile is subject to great variation from causes that may or may not be controllable by the designer. At one extreme there is the practically vertical line profile corresponding to an elastic-range instability, fortunately rare, and at the other extreme, there is the structure with broad steps for successive failure stages representing great reserves of strength beyond the elastic limit.

Factors in this variation over which the designer has some control are: (a) The configuration of the cross section of members—for reinforced concrete the choice of tee-beam, flat slab, or folded plate is significant; (b) the ductility of the materials in the members, even considering only reinforced concrete, the choice is first whether or not to prestress; the designer then knows that ductility varies inversely with the net reinforcing index, $q - q'$, over a wide range; and (c) finally, the interaction with flexure of "secondary" effects, often critical, including shear, bond, and the participation of floors, walls, and roofs in flexural action, over which the designer may exert limited control.

However, the designer has very little control over other important causes of variation, most important of which are the configuration of the loads and

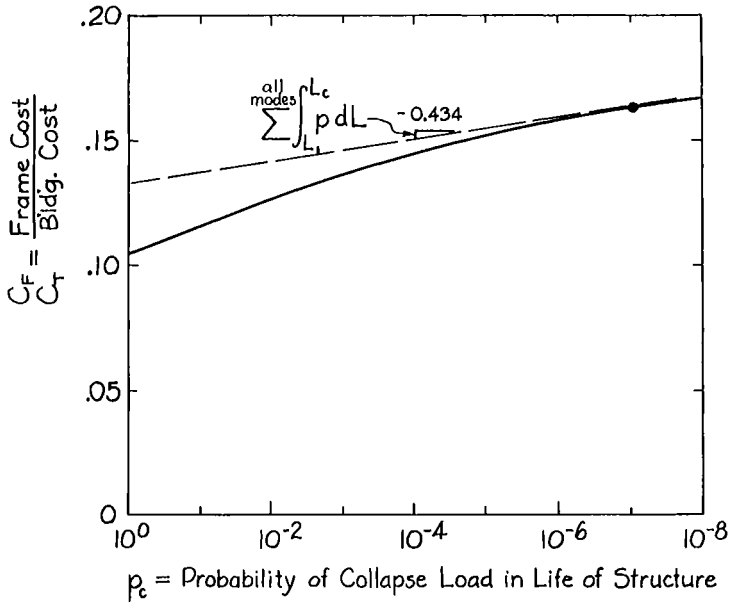


FIG. 3.—FRAME COST VS. COLLAPSE LOAD

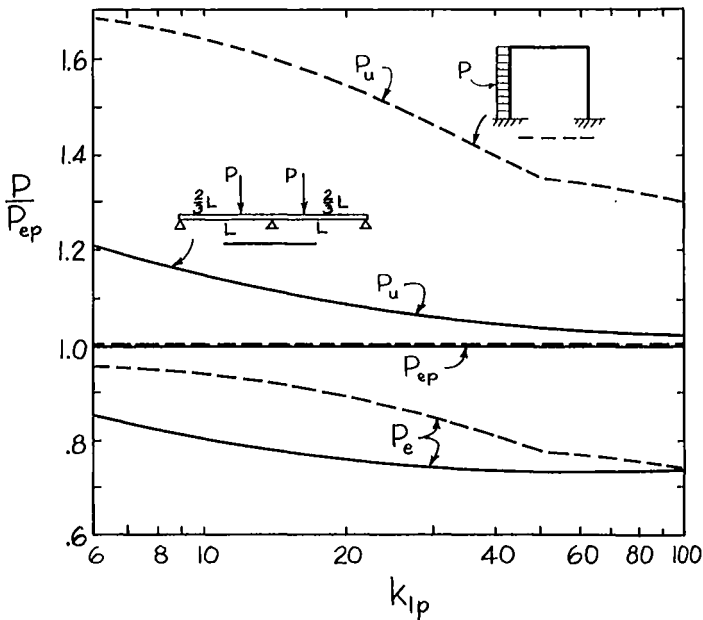


FIG. 4.—VARIATION IN FAILURE-STAGE LOADS WITH DUCTILITY AND STRUCTURE-LOAD CONFIGURATION

related configuration of the axes of members placed to support these loads. An example of variation from these causes is shown in Fig. 4, which illustrates the theoretical relative load levels for the two statically indeterminate structure-load configurations shown over a range of ductility indicated by the "plasticity factors," k_{1p} , characteristic of bilinear moment-curvature relationships for reinforced concrete (defined in Fig. 8). The load levels are those given by an elastic distribution of moments, P_e , an inelastic distribution of moments limited by an ultimate compressive concrete strain of 0.0038, P_{ep} , and a complete rigid-plastic redistribution distribution of moments, P_u . These levels represent approximately a wide-cracking, a crushing-spalling, and a collapse failure stage, respectively (the last neglecting the restrictions of limited ductility on a full redistribution of moments). Variations in maximum moment at a section for these failure stages have been neglected; therefore, the variations in strength shown are only those caused by differing redistributions of moment. Obviously, the three curves for any statically determinate structure would all coincide on the straight line, $P/P_{ep} = 1$.

Thus, Fig. 4 shows, for only three parameters (configuration of structure and load and ductility) of the several mentioned: First, a wide variation in horizontal spread in failure stage profile (vertical spread on Fig. 4), and second, a wide variation in relative position of the failure stages with respect to each other.

TRADITIONAL VERSUS COMPREHENSIVE DESIGN

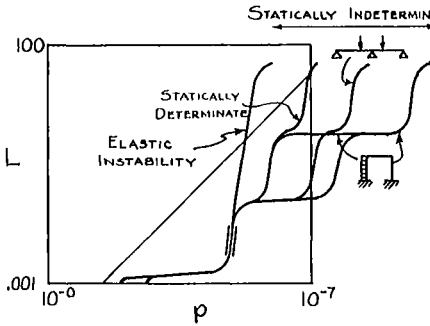
Whereas Eq. 5 determines the optimum horizontal placement of any failure-stage profile regardless of its shape or spread, the traditional design procedure of the past has been based on a single failure stage, and placement of the entire profile has depended on a fixed (constant load factor) placement of one hump of the profile. The results for elastic moment design are shown in Fig. 5. In order to assure a permissible horizontal position for the collapse hump for any possible structure (here the statically-determinate structure profile is usually critical), the design loads or load factor for the elastic analysis must be set very high, resulting in overly safe and uneconomical designs for those structures and loadings with the highest reserves of strength from redistribution of moments.

As Fig. 6 shows, a similar situation exists for the plastic design of steel, except that control of an intermediate hump probably provides slightly better control over the humps at either extreme. Here, to insure proper position of permanent-set and over-all collapse humps, the loads for mechanism collapse (probably closest to actual local collapse loads) must be set uneconomically high.

Thus, the primary advantage of comprehensive design is that it leads to the use of lower, more economical, load factors.

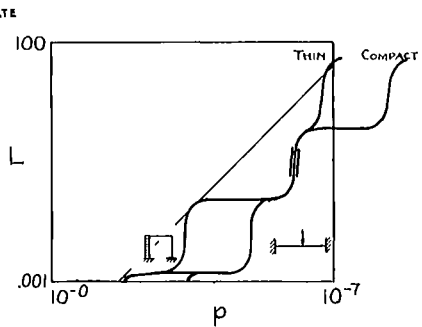
COMPREHENSIVE ANALYSIS

Of course, the prerequisite to the optimum placement of the load-failure-load profile is the determination of this profile through a comprehensive analysis, and the solution to this problem will now be considered.



ELASTIC MOMENT DESIGN
W.S.D. AND U.S.D.

FIG. 5



PLASTIC DESIGN - STEEL
STATICALLY INDETERMINATE ONLY

FIG. 6

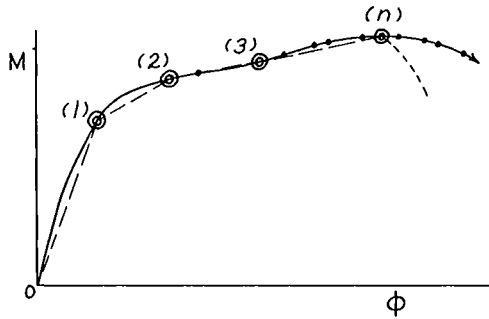


FIG. 7.—TYPICAL $M-\phi$ RELATIONSHIP,
REINFORCED CONCRETE

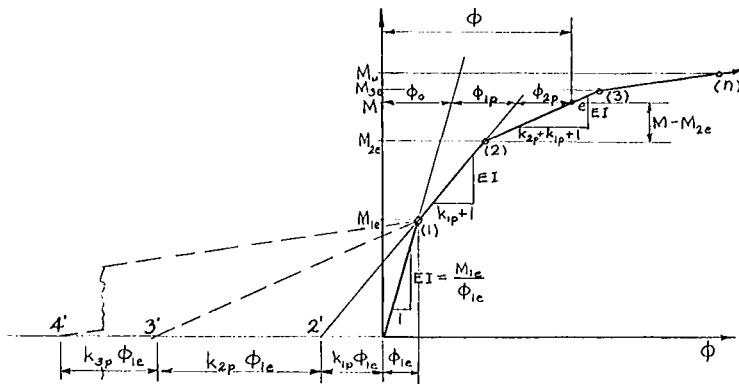


FIG. 8.—MULTI-LINEAR $M-\phi$ RELATIONSHIP

For analytical purposes, the basic interrelationship between behavior and failure-stages at a section and for the entire structure is best defined and accounted for through the moment-curvature relationship for the section and the curvature diagram for the structure.

Fig. 7 shows a typical moment-curvature ($M-\phi$) relationship for a member at one of its cross sections. All failure stages associated with a limiting value of compressive or tensile strain (cracking, permanent set, crushing-spalling) can be represented by reasonably definite points (1), (2), (3) . . . , on this relationship. However, the criterion for both local collapse and general collapse is not a limiting strain, but the condition that the derivative of load on the structure with respect to deflection is zero: $\partial W / \partial \Delta = 0$. Therefore, referring to Fig. 7, collapse does not occur at a particular moment on any $M-\phi$ (or $M-\theta$) relationship. Rather, it occurs when roughly half of the moments at the inelastic or "hinging" regions are on the ascending portions of their respective $M-\phi$ curves to the left of the point of maximum moment, and the remainder are on the descending portions to the right of the maximum point, all as shown schematically by the black dots of Fig. 7.

To the writer's knowledge, no serious attempt at accurate evaluation of the corresponding "true" collapse load has been made, perhaps more because of the inherent uncertainties and errors in determining $M-\phi$ relationships and in translating them into M -rotation relationships for the descending range (2) than because of the analytical complexities. Instead, two approximate, simpler solutions have been proposed with accuracy perhaps more compatible with the inherent uncertainties.

One solution, by far the better known, is the nondeformational, rigid-plastic solution equivalent to the assumption that the maximum moments occur at all sections simultaneously. Obviously, this assumption always theoretically results in an upper bound solution, but in practice, this solution is often conservative because of the conservative values usually assumed for ultimate moment (3) and the neglect of nonframe portions of the structure which also resist collapse. However, if, as sometimes occurs with reinforced concrete, a sudden fracture at maximum moment causes a precipitous loss in strength such as that shown by the dotted line of Fig. 7, the moments at practically all other inelastic (often called "hinging") regions will be on the ascending curve, and the upper bound solution may be grossly high. (In connection with the rigid-plastic solution it is suggested that the terms "hinge" and "lower bound," although useful for the plastic theory involving a perfectly plastic material, can be dangerously misleading and should be avoided for a material of limited ductility such as reinforced concrete. The concept of a hinge, involving indefinite rotation capacity at constant moment, does not apply with any precision to reinforced concrete. And the "lower bound" for a perfectly plastic material is not at all the lower bound for any material of limited ductility; in fact, for such a material, a statically admissible "lower bound" may far exceed the actual ultimate strength.)

The second approximate solution involves the assumption that the collapse load is adequately given by a deformational analysis, using $M-\phi$ relationships, with no deformation or curvature to exceed the maximum-moment curvature at any section. Thus, for this solution, no moments would be on the descending portion of the $M-\phi$ relationships. Obviously, this solution would always be a lower bound solution, even for the case of sudden fracture,

and therefore, in the writer's opinion, it is better suited for use on materials of limited ductility.

Another advantage of this solution is with respect to a comprehensive analysis; for this solution the collapse failure stage, as well as other failure stages, are represented by definite points on the $M-\phi$ relationship. It follows that any method for translating such $M-\phi$ relationships into the behavior of the structure under a set of loads will produce a complete failure load profile and be a method of comprehensive analysis within the limitations of the assumptions involved.

For a practical method of comprehensive analysis based on a moment-curvature relationship such as that of Fig. 7, it is first proposed that this relationship be approximated by any convenient series of straight lines, shown by the dashed lines of Fig. 7. (It is perhaps desirable, but not necessary, that these lines intersect on failure points.) With this approximation, the nonlinear problem in analysis becomes, more precisely, a multilinear problem in analysis.

Referring to the multilinear $M-\phi$ relationship of Fig. 8, first the total curvature, ϕ , corresponding to the bending moment, M , will be evaluated. It is convenient to divide ϕ (line $M-e$) into component parts defined by its intersections with extensions of all lower segments of the $M-\phi$ diagram, these components being, for the case shown, ϕ_o , ϕ_{1p} , and ϕ_{2p} . Also, as shown, lines parallel to all segments are constructed through point (1) so as to intersect the horizontal axis; for example, line (1)-3' is parallel to segment (2)-(3). It is then convenient to define a "plasticity factor" (e.g., k_{2p}) as the factor by which the "elastic-limit" curvature, ϕ_{1e} , must be multiplied to obtain a corresponding intercept (e.g., intercept 2'-3' equals $k_{2p}\phi_{1e}$). Now, the components of the desired curvature may be found from pairs of similar triangles, one of each pair with an intercept as its base and the other with the curvature component as its base. Considering, for example, the triangles defined by base $k_{2p}\phi_{1e}$ and point (1) and base ϕ_{2p} and point (2)

$$\frac{\phi_{2p}}{k_{2p}\phi_{1e}} = \frac{M - M_{2e}}{M_{1e}} \quad \text{or} \quad \phi_{2p} = \frac{\phi_{1e}}{M_{1e}} (k_{2p})(M - M_{2e})$$

Adding all curvature components found by such pairs of similar triangles, for the general case, the total curvature for a moment in the segment immediately above point "n" will be

$$\phi = \frac{\phi_{1e}}{M_{1e}} \left[M + k_{1p} (M - M_{1e}) + k_{2p} (M - M_{2e}) + \dots + k_{np} (M - M_{ne}) \right] \dots \dots \dots (6)$$

The convenience of this Eq. 6 in evaluating the multilinear curvatures in any structure is shown by the length of beam and corresponding moments shown in Fig. 9. From Eq. 6 it is clear that the curvature at the point of Fig. 9 with moment M is: The total ordinate M , plus k_{1p} times the total ordinate above line M_{1e} , plus k_{2p} times the total ordinate above line M_{2e} —all times the factor $1/EI$ (or ϕ_{1e}/M_{1e}). Therefore, for calculation of rotations and deflections from multilinear curvatures, the same moment-area principles

used for purely elastic curvatures may be used if, in addition to the total moment-area, the total area above M_{1e} expanded by the factor k_{1p} , the total area above M_{2e} expanded by k_{2p} , and the total area above M_{ne} expanded by the factor k_{np} , are also included.

With this simple evaluation of curvatures in terms of moments, each stage in the comprehensive analysis of a structure then consists of arriving at a statically admissible set of moments for which the curvatures, rotations,

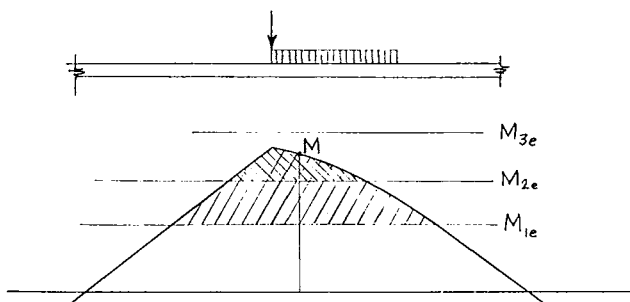


FIG. 9.—TYPICAL PORTION OF M-DIAGRAM, MULTI-LINEAR ANALYSIS

and deflections satisfy the structure's boundary conditions. Techniques for this procedure have been presented elsewhere (4, 5).

DESIGN FOR TWO FAILURE STAGES

With the previous analysis of the optimum placement of the failure-stage load profile and a method of comprehensive analysis by which the profile may be obtained, an ideal has been presented. But, obviously, this ideal is not an immediate practical possibility in design. Probably, the next step in the evolution of the design and analysis of reinforced concrete frames will be to a design and analysis based on two failure stages, or a bilinear method.

The reasons for this step are several, and, upon examination, very convincing. The present "Ultimate Strength Design" method of ACI 318-63 is a hybrid method combining elastic analysis, which is only effective for cracking failure, with a maximum, "ultimate-strength" moment, which is essentially the limit for a crushing-spalling failure. Therefore, this method is really effective for neither type of failure; it tends to give unsafe predictions of ultimate strength because the elastic moment distribution assumed, and thus designed for, does not exist at ultimate load; and it loses effect in control of cracking because the ultimate strength moment is not governed by cracking. Clearly, a two-failure-stage design, by its superior, two-point control of the load-failure profile, will result in some combination of better behavior and more economy (through reduction of load factors). Of course, the more of one benefit, the less of the other.

But additional benefits inhere from the substitution of a nonlinear design method for elastic-distribution design, because such a substitution in effect

provides the designer with a new degree of freedom. With elastic design, the relative distribution of moments for a loading is fixed, but, as is known, the ductility of nonlinearity is a reserve that a redundant structure will spend in trying to adjust its moments to the strengths provided by the designer. (Of course, the only restriction, besides statics, is that this reserve of ductility not be overestimated. Obviously, we should not change from the elastic-design assumption of no ductility to the other extreme of infinite ductility.)

For the rare structure subjected to a single, fixed arrangement of loading, this new freedom allows redistributions that will: (a) Reduce the congestion of reinforcement at critical regions and (b) optimize economy.

For a beam of certain external dimensions, simple economy of reinforcing steel is advanced by minimizing the total moment area. An easily proven geometric theorem that is a useful guide in design is that these areas are minimum for a position of the base line or, in general, base curve of a moment diagram such that the horizontal projections of the intercepts of the intersecting curves are balanced, that is:

1. The total lengths of these projections for the positive moment areas and the negative moment areas are equal.
2. The total first moments of these projections for the positive moment areas and the negative moment areas are equal.

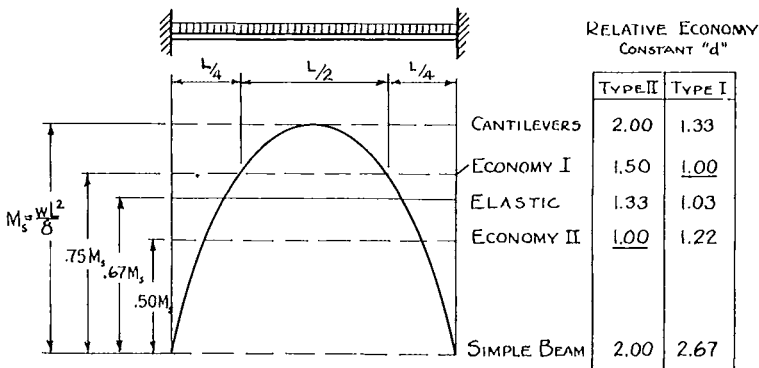


FIG. 10.—RELATIVE ECONOMY FOR DISTRIBUTIONS OF MOMENT

For example, for the simple case of Fig. 10, these conditions are satisfied by the base line marked "Economy I." (In contrast, the most economical position for design of a constant strength "type II" beam, e.g., a rolled steel section, is on the other side of the elastic base line, as shown.)

Unfortunately, for the single-loading beam, reduction of congestion and optimization of economy are usually opposing trends and, as shown by Fig. 10, where congestion would be at the fixed ends, a designer can only advance toward one objective at the expense of retreating from the other.

However, the situation differs for the more common structure subjected to several arrangements of loading. Such a structure must be designed for

TABLE 1.—SCOPE OF EXPERIMENTAL WORK, STATICALLY DETERMINATE REINFORCED CONCRETE^a

Parameter		Primary		Secondary										Tertiary										
		$q - q'$	V_d/M	$\partial(M_m/M_u)/\partial t$	P_d/M_u	q'/q	P_t	f_y	f'_c	d	n	$(M_m)_{min}/(M_m)_{max}$	M_{beam}/M_{col}	l/r	s/d	t_c/d	T	H	f_{bear}/f'_c	w/d	N	M_t/M_{tu}	D/d	t_f/d
Reference																								
6. Gaston, et al		X			X	x		X																
7. McCollister		X			X			X																
8. Burns		X			X				X	x	x					x								
9. Yamashiro		X		X												x								
10. Ernst		X	x		x											X								
11. Sawyer		X	x					x																
12. Gerstle		x									X													
13. Mattock		X	X					x	x															
14. Hognestad		X		X				X																
15. Viest			X	x				X																
16. Chan				X									x	X										
17. CEB Series 1		X		X																				x
17. CEB Series 2		X		X			X																	x
17. CEB Series 3		x		X				x																x
17. CEB Series 4		X		X		X																		x
17. CEB Series 5		X		x						x														x
17. CEB Series 6		X		X	x																			
18. Proposed CEB 7		x														X								
18. Proposed CEB 8		X																		x				
18. Proposed CEB 9																		X						
18. Proposed CEB 10																					X			
18. Proposed CEB 11															X									
18. Proposed CEB 12			X																					
18. Proposed CEB 13			x																					
18. Proposed CEB 14											X													
18. Proposed CEB 15		x					X																	
18. Proposed CEB 16		x					x															X		
18. Proposed CEB 17																							X	x
Started, U. of Fla.		X	X	x																				

x—Investigation of two values of parameter

X—Investigation of three or more values of parameter

^a Notation for table is given in Appendix II.

the outer envelope of moments from all loadings, and the ductility-adjustment of the moments of any loading toward fitting any given envelope is a major source of economy. This source of economy does not inherently conflict with reduction of congestion, rather, by its tendency toward a general reduction in reinforcement it tends to reduce congestion.

It may be concluded that the adoption of two-failure-stage design would lead, for the important case of a redundant structure with varying load configurations, not only to better behavior or a reduction in load factors, or both, but to a new degree of freedom in design allowing exploitation of natural, ductilely-caused adaptability so as to increase economy and decrease congestion.

In a move to two-stage design, the most important question which arises is: Design for which two failure stages—wide cracking, crushing-spalling, local collapse, over-all collapse, or some other? Then, a secondary question will arise only if one of the selected failure stages is a collapse stage (because a deformational analysis is mandatory for all other stages): Should a deformational analysis with one moment attaining maximum value, a rigid-plastic analysis with all moments at maximum, plus the necessary restrictions, or perhaps a more rigorous method, be used to evaluate the collapse load?

These questions do not have obvious answers, and one reason for this uncertainty is the existence of the gaps in our knowledge of the flexural behavior of reinforced concrete, made evident in Table 1 a,b. Yet, we are now at a stage from which we can probably attain good answers to these questions. The word “good” was used advisedly because a good answer is very nearly as valuable as the ideal answer, and the ideal now beyond our reach does not prevent our grasping the good. The writer believes this Symposium will help in achieving good answers.

CONCLUSIONS

1. Failure occurs in successively more severe stages at successively less probable loads. Comprehensive design ideally considers all stages, using a comprehensive analysis with elastic design, limit design, and ultimate strength design as special aspects.

2. Optimum design loads may be obtained by relating the slope of a frame-cost versus load-probability curve to the expected loss. Such loads would represent, for many structures, lower and more economical load factors than for conventional design.

3. Comprehensive analysis may be based on the effective moment-curvature relationships of a structure. A multilinear idealization of these relationships with relative slopes defined by “plasticity factors” is analytically efficient. However, there are difficulties (not insoluble) in determining the collapse load for structures of materials of limited plasticity.

4. Design based on analysis for failure from two failure stages is a step toward comprehensive design: Improved behavior or economy, or both, and a new degree of freedom in the distribution of strength within the limitations of statics.

ACKNOWLEDGMENTS

The generous support of the University of Florida, Gainesville, Fla., and the National Science Foundation, and the valuable assistance of W. C. Lin in the preparation of this paper are gratefully acknowledged.

APPENDIX I.—BIBLIOGRAPHY ON NONLINEAR DESIGN
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APPENDIX II—NOTATION

The following symbols have been adopted for use in this paper:

- C_f = cost of structural frame of building;
- C_t = total cost of building;
- E = modulus of elasticity;
- I = effective moment of inertia of area of cross-section;
- k_{1p}, k_{2p} , etc. = plasticity factor for corresponding inelastic segment of $M-\phi$ relationship, Fig. 8;
- L = loss from failure with respect to cost of building;
- L_c = value of L for collapse of building;
- L_1 = value of L at elastic limit;
- M = bending moment;
- M_{1e}, M_{2e} , etc. = maximum bending moment for corresponding segment of $M-\phi$ relationship, Fig. 8;
- M_u = ultimate bending moment;
- p = probability of load in life of structure;
- p_c = probability of collapse load in life of building;
- P, W = load on structure;
- ϕ = curvature from bending;
- ϕ_{1e} = maximum curvature for first (elastic) segment of $M-\phi$ relationship, Fig. 8;
- ϕ_o = component of curvature from extension of first segment of $M-\phi$ relationship, Fig. 8; and
- ϕ_{1p}, ϕ_{2p} , etc. = inelastic components of curvature from extension of corresponding inelastic segments of $M-\phi$ relationship, Fig. 8.

Notation used in Table 1 is as follows:

Primary	$q - q'$	Net reinforcing index
	$\frac{V_d}{M}$	Relative moment-gradient or shear
	$\frac{\partial(M_m/M_u)}{\partial t}$	Rate of Loading
	$\frac{P_d}{M_u}$	Axial Load ratio
Secondary	q'/q	Compressive to tensile reinforcement ratio
	P_t	Areawise transverse reinforcement ratio
	f_y	Yield strength of steel, tensile
	f'_c	Strength of concrete, compressive
	d	Scale
	n	Repetitions of load
	$\frac{(M_m)_{\min}}{(M_m)_{\max}}$	Ratio of minimum M_m to maximum M_m , repeated loadings
Tertiary	$M_{\text{beam}}/M_{\text{col}}$	Ratio of beam and column moments at a joint
	l/r	Slenderness ratio (for columns only)
	$\frac{s}{d}$	Relative spacing of transverse reinforcement
	t_c/d	Column-to-beam depth ratio at a joint
	T	Temperature
	H	Relative humidity
	$\frac{f_{\text{bearing}}}{f'_c}$	Ratio bearing stress at M_m to compressive strength
	w/d	Relative width of bearing plate
	N	Number of inelastic regions
	M_T/M_{Tu}	Relative torsional moment
	D/d	Ratio, bar size to beam size
	t_f/d	Ratio, flange thickness to depth, tee beams

DISCUSSION

MICÍK TICHÝ³ AND MILOŠ VORLIČEK⁴.—The author's concept of the failure-stage-versus-load relationship seems to be a good starting base for

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⁴ Bldg. Research Inst., Tech. Univ. in Prague, Prague, Czechoslovakia.

further considerations concerning economical aspects of structural design. However, a relationship such as that shown in Fig. 1 is only valid for that structure for which the mechanical properties and dimensions are precisely known in advance. In reality, when dimensioning a structure, these quantities must be assumed by certain design values (e.g., means) and, necessarily, deviations from them must be expected. Disregarding the systematical deviations caused by the human factor, those deviations are random. Therefore, the relationship under consideration is a random function and the value of load for which certain type of failure occurs must obviously be a random variable. This fact might be schematically shown by means of irregular bundle of relationships and by random distribution of load related to a chosen failure stage (Fig. 11). It should be noted that the important part of the deviations is in the domain of maximum density of the area under curve and, consequently, random deviations have an appreciable influence on the expected loss.

If a structure is designed so that the predicted failure-stage profile will lie somewhere in the bundle of curves (curve A in Fig. 11), there always is a risk that the real loss will be greater than the expected one. This fact must be taken into account when minimizing the building costs.

The latest method of structural design based on the concept of limiting states, which is now used in several countries and by the C.E.B., might perhaps be denoted in the author's terms as a three-stage design. According to this method, three limiting states are decisive, namely: Cracking state, deformation state, and ultimate strength state. Their position at the failure-stage profile may be schematically shown, as in Fig. 12.

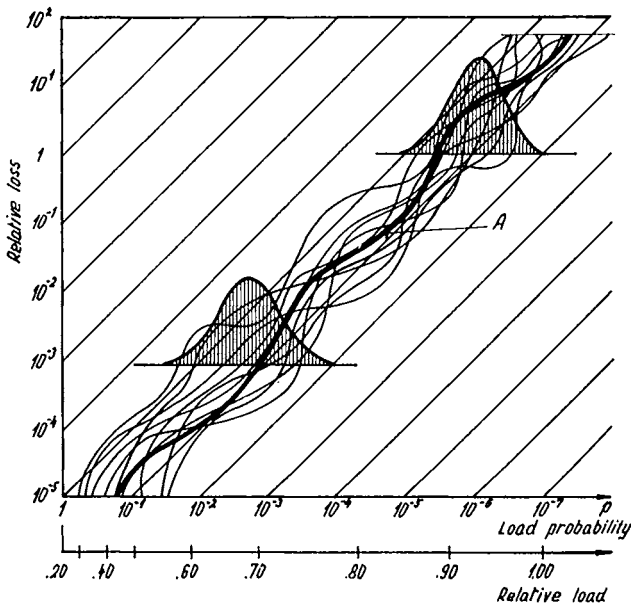


FIG.—11

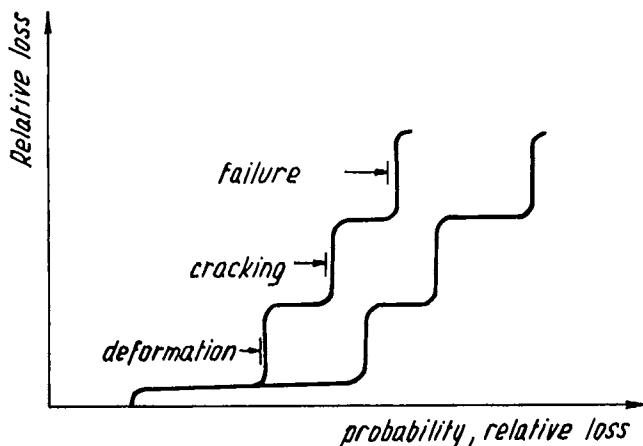


FIG.—12

The definition of the first two limiting states is sufficient from the practical point of view. But the definition of the last one is, for many structures, unclear. Different solutions are therefore possible for identical conditions. An optimum failure-stage profile attached to a fixed point (defining for example collapse or other stage) might furnish a reasonable and unequivocal solution.

Figs. 5 and 6 are rather optimistic. The present design methods (the limiting states method included) give limits only from the stand-point of serviceability. The question of costs is left to experience and there is no stress on the designer to lead the failure-stage profile through a bottleneck.

HERBERT A. SAWYER, JR.—Tichý and Vorličěk, as well as Macchi, have raised the very important point of the effect of variation in materials and in structural dimensions on the conclusions of the paper.

The most recent ACI Building Code (ACI 318-63) and, as Professor Macchi pointed out in the free discussion, the recent recommendations of the C.E.B. for an international code of practice, incorporate the desired safety or reliability into a design via two somewhat independently determined factors—one related to the loads and the other related to material and structural strengths. This paper was prepared within this frame of reference and is exclusively concerned with load factors; variations in material strengths would not affect the paper's theory and conclusions regarding the relationships between loads, safety, and costs. Incidentally, Tichý's paper (19) for this symposium complements this paper in that it is concerned with the relationships between strengths and safety.

As Tichý and Vorličěk state, the inevitable constructional variations in material strengths do cause variations in the failure-stage profile, but these variations are more limited and more systematic than those shown by the discussers' Fig. 11.

First, a variation in material strength has no effect on the magnitude of loss for any failure stage. Therefore, contrary to the discussers' Fig. 11, the elevation of each of the humps of the failure-stage profile will remain constant, and strength variations will not cause significant vertical deviations or spread in the essentially horizontal portions of the profile curves. Furthermore, if the designer can determine the effective material strengths for design and use them in conjunction with valid comprehensive analyses, the vertical portions of the profile curves for various material strengths will also tend to coincide. This tendency is evident from the fact that loads are plotted relative to the calculated general collapse load, thus eliminating all spread in the vertical portions of the curves immediately below "General Collapse" of Fig. 1, and reducing spread in all other vertical portions because of the reasonable expectation that loads corresponding to all stages of failure would be changed in somewhat the same proportion by a material-strength variation.

The adjective "effective" was used advisedly in modifying "material strengths" because if many structures were to be built in accordance with a certain structural design, the inevitable variation in materials and dimensions in this "population" of structures would cause variation of failure loads in the population for each failure stage. Therefore, it is true that vertical portions of the failure-stage profiles for this population of as-built structures will show a horizontal spread or distribution with a frequency distribution which may be similar to the two frequency distribution curves of Fig. 11. It follows that a complete consideration of structural economics and safety for a design will involve: (a) a prediction of these distribution curves for each failure stage, using a statistical extrapolation of past experience, and (b) the purely mathematical determination from these curves of the single effective profile to represent average losses of the population of structures for a determination of load factors according to the principles of this paper.

Although the problem of the prediction of these frequency distribution curves is beyond the scope of this paper, it might be said that little seems to have been done on this problem, with the outstanding exception of work reported by Tichý¹⁹ to this symposium. Probably the most solid starting point for obtaining these curves is information on the expected variation in strength of the structural materials. However, the variation in strengths of a population of structures is but indirectly related to the variation in test-specimen strengths of the materials of these structures. Such test results must be averaged to an extent which depends on the size and redundancy of the structure and somewhat on the severity of the failure stage, and the variance for the frequency distribution for structural strengths is reduced in inverse proportion to the extent of this averaging. On the other hand, this variance is increased by dimensional and other constructional variations, as well as any tendencies for constructional variations to be non-random.

The mathematical determination of a single effective profile curve to represent failure losses equal to the average failure losses represented by any given frequency distribution of curves is not difficult, although it is complicated by the variable density of area under the failure stage profile, which causes the effective curve to always be to the left of the mean of the frequency distribution curve, and its displacement on the logarithmic load-probability scale from the mean depends on the second power of the standard deviation, σ , of the frequency distribution curve. For example, assuming normal frequency distributions, if a factor-of-ten change of load probability

equals σ , (a dispersed curve), the effective curve will be 2.32σ , from the mean, whereas if a factor-of-ten change of load probability equals $4 \sigma_2$ (a narrow curve), the effective curve will be but $0.58 \sigma_2$ from the mean.

Obviously, not enough is known about either the frequency distributions of expected loads or of expected structural strengths to allow precise determination of load factors using the methods of this paper, but initially, even approximate determinations along such rational lines will probably be significant. It is believed that such determinations for some high loadings of low probability—notably blast and seismic loadings—may significantly alter present design standards. It might be noted that for an approximate determination a sufficiently accurate effective load-profile curve might be simply derived from the nominal material-strength because such a profile curve will always be to the left of the mean of all curves based on actual material-strengths. This suggestion is in accord with the author's opinion that the range of variation of expected loads on a structure is more important to the question of structural safety than the range of variation of material strengths. Some support for this opinion is offered by Freudenthal,²⁰ who stated, regarding aircraft structures, that the rare occurrences of ultimate failure are not caused by structures of extraordinarily low strength but by structures of adequate strength encountering extreme loading conditions.

With respect to the discussers' comments on the present use of three-stage design in various countries, it is true that present methods involve rule-of-thumb-type precautions and preventatives for several failure stages. However, a multi-staged design, as described in this paper, involves analyses based on force-strain or moment-rotation relationships which are appropriate to each stage investigated and which are thus usually nonlinear. To the author's limited knowledge, no nonlinear relationship has yet been codified for general use in any country. The recent C.E.B. recommendations merely state that an analysis "may be done by the limit state method if moment rotation diagrams are available . . . from tests representing conditions similar to those applying to the structure."²¹

The discussers' comments have helped to place the paper in better perspective and are greatly appreciated.

REINFORCED CONCRETE LINEAR STRUCTURES AT ULTIMATE LOAD

E. F. P. Burnett¹ and C. W. Yu²

With discussion by Edward G. Nawy, H.A. Sawyer, M.Z. Cohn, and
E.F.P. Burnett and C.W. Yu

SYNOPSIS

An attempt is made to evaluate our present knowledge with regard to the analysis and design of reinforced concrete linear structural systems at ultimate load. The fundamental difference between the moment curvature concept and moment rotation concept is emphasized and discussed in detail. The authors have attempted to outline previous significant work, to underline a few basic principles, bearing in mind the difference between these two concepts, and to indicate the present extent of our knowledge of this subject with an appreciation of the assumptions and simplifications that are entailed. Readers are assumed to have some basic knowledge of some of the better known work on the subject, such as Sawyer's or Baker's work.

INTRODUCTION

In statically determinate two-dimensional linear structural systems the bending moments and shears are dependant solely on equilibrium criteria. However, in the analysis of statically indeterminate systems, for any solution to be valid the following criteria must be satisfied i.e., (1) equilibrium, (2) compatibility of deformation, (3) the behavioral relationship at each individual cross section. In steel structures this last criterion is a question of satisfying the appropriate stress-strain relationship. However, with Reinforced Concrete the most fundamental relationship that is practically applicable is the moment-curvature relationship for any particular cross section. Therefore, provided the moment curvature relationships are known it is theoretically possible to analyze the structure for any loading up to and including the ultimate load.

¹ M. S. Yolles and Assoc., Ltd., Toronto, Ontario, Canada.

² Dept. of Civ. Engrg., Imperial Coll. of Science and Tech., London, England.

The following approach which has been previously used by Y. Guyon,^{3,4} is similar to that used in problems in Elasticity. Being a trial and error technique, its application is limited by the complexity of the problem. The following assumptions which have been shown to be substantially true, being made:

(1) The contribution to the flexural strength of the member from the tensile strength of the concrete is negligible.

(2) Strains are proportional to the distance from the neutral axis.

For e_c = the strain in the extreme fibre of the concrete in flexural compression,

for R = the radius of curvature of the member at a particular cross-section,

and nd = the distance from the neutral axis to the extreme fibre of the concrete in flexural compression.

It can readily be shown that

$$\frac{e_c}{nd} = \frac{1}{R} = K \dots \dots \dots (1)$$

in which K = the radius of curvature of the member at that particular cross-section. It is also well known that provided deformations are small (i.e., the slope of the member at any stage of loading is small compared to unity) and there are no deformational discontinuities, then the rotation, θ , that occurs between any two points, A and B, in the structure is given by:

$$\theta_{AB} = \int_A^B K ds \dots \dots \dots (2)$$

This is a completely general expression provided that for inelastic behavior the load is monotonically (always increasing) varying.

For the purposes of illustration consider a continuous beam with a uniformly distributed loading. Any bending moment distribution will be fully defined by a knowledge of the moment values at the supports because between supports, the bending moment distribution must be in equilibrium with the loading i.e., the free span bending moment distribution. For any assumed bending moment distribution a set of compatibility equations involving deformation and incorporating the necessary behavioral relationships may be decided. Any deformational criterion may be used but it is usually convenient to consider that there can be no discontinuous rotation at the supports. At the section D in Fig. 1 for instance the compatibility criterion is that:

$$\theta' + \theta'' = 0 \dots \dots \dots (3)$$

$$\text{It can readily be shown that } \theta' = \int_{\text{span CD}} \frac{x}{l} k dx \dots \dots \dots (4)$$

span CD

³ Guyon, Y., "The Strength of Statically Indeterminate Prestressed Concrete Structures," Proceedings, Symposium on the Strength of Concrete Structures, London, England, May, 1956.

⁴ Guyon, Y., "Note on the Ultimate Load Design of Statically Indeterminate Structures," C. E. B. Bulletin No. 21, January, 1960.

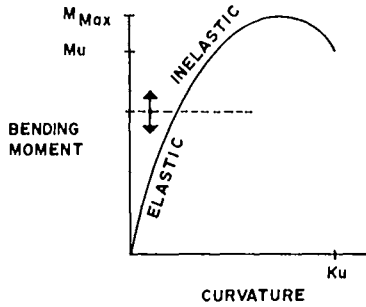


Fig. 1 (i)
MvK RELATIONSHIP

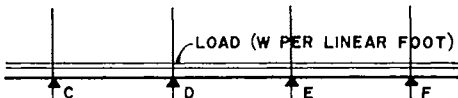


Fig. 1 (ii)
LOAD DIAGRAM

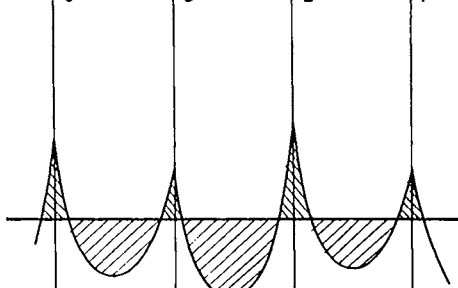


Fig. 1 (iii)
B.M. DIAGRAM

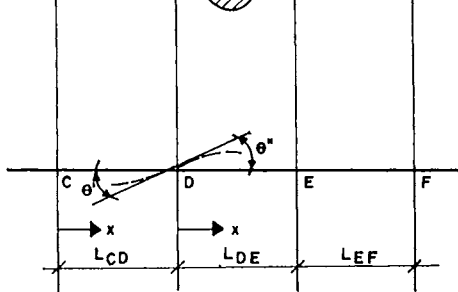


Fig. 1 (iv)
DEFORMATION
AT D

FIG. 1

$$\text{and } \theta'' = \int_{\text{span DE}} \left(1 - \frac{x}{l}\right) k \, dx \dots\dots\dots (5)$$

$$\text{Hence, } \int_{\text{span CD}} \frac{x}{l} k \, dx + \int_{\text{span DE}} \left(1 - \frac{x}{l}\right) k \, dx = 0 \dots\dots\dots (6)$$

An equation similar to Fig. 6 may be obtained for each internal support section. In general, the number of compatibility equations necessary will be equal to the number of statical indeterminacies involved.

For some assumed bending moment distribution to be correct these equations must be satisfied. As has been demonstrated by Guyon,^{3,4} the application of this basic method is both simpler and quicker to apply than would appear at first sight. It may be applied for any load stage up to failure of the structure. Depending on the moment curvature relationship, failure may be said to occur either when one or more cross-sections rupture, or when the structure or part of it becomes kinematically unstable.

Along similar lines but still based on a knowledge of the moment curvature relationship, L.P. Johnson,⁵ and in particular H.A. Sawyer,^{5,6} have developed techniques which may be applied to various structures at ultimate load. It would thus appear that provided the moment curvature relationship is known (for the appropriate sections) then in theory many linear reinforced concrete structural problems may be solved. Apart from the practical difficulties involved in a general application of this method, there are a number of reasons why this approach is undesirable as seen from the following.

THE MOMENT CURVATURE RELATIONSHIP

Because this relationship is so vital to the calculations it is of interest to consider a typical relation obtained from tests. The following results are based on beams tested at Imperial College by Brenner.⁷ A number of simply supported beams, with a point load at midspan, were tested with continuously increasing load. The member was said to have ruptured when spalling became excessive. By means of clinometers, strain gages and deflection gages it was possible to obtain a fairly accurate picture of the behavior of the member for each stage of loading. Typical results are shown in Fig. 2 to 4. The following load stages are to be considered:

1. Well before yield or apparent yield of the steel.

⁵ Johnson, L. P., and Sawyer, H. A., "Elasti-Plastic Analysis of Continuous Beams and Frames," *Journal of the Structural Division*, ASCE, Vol. 84, No. ST8, Proc. Paper 1879, December, 1958.

⁶ Sawyer, H. A., "Elastic-Plastic Design of Single Span Beams and Frames," *Proc. Paper No. 851*, ASCE, Vol. 81, December, 1955.

⁷ Brenner, T., "The Influence of the Reinforcement on the Elastic and Inelastic Properties of Reinforced Concrete Frames," thesis presented to the University of London, at London, England, in 1962, in partial fulfilment of the requirements for the degree of Master of Science.

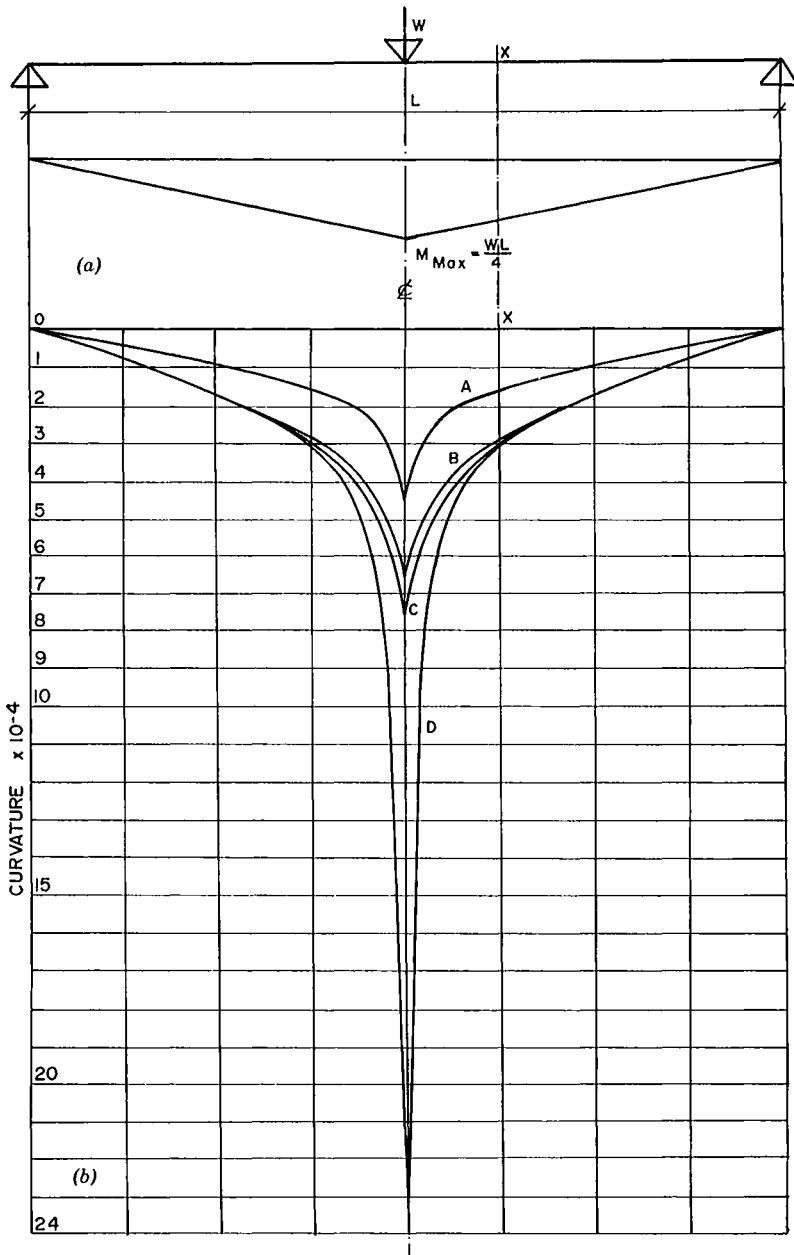


FIG. 2.—BENDING MOMENT AND CURVATURE DISTRIBUTION

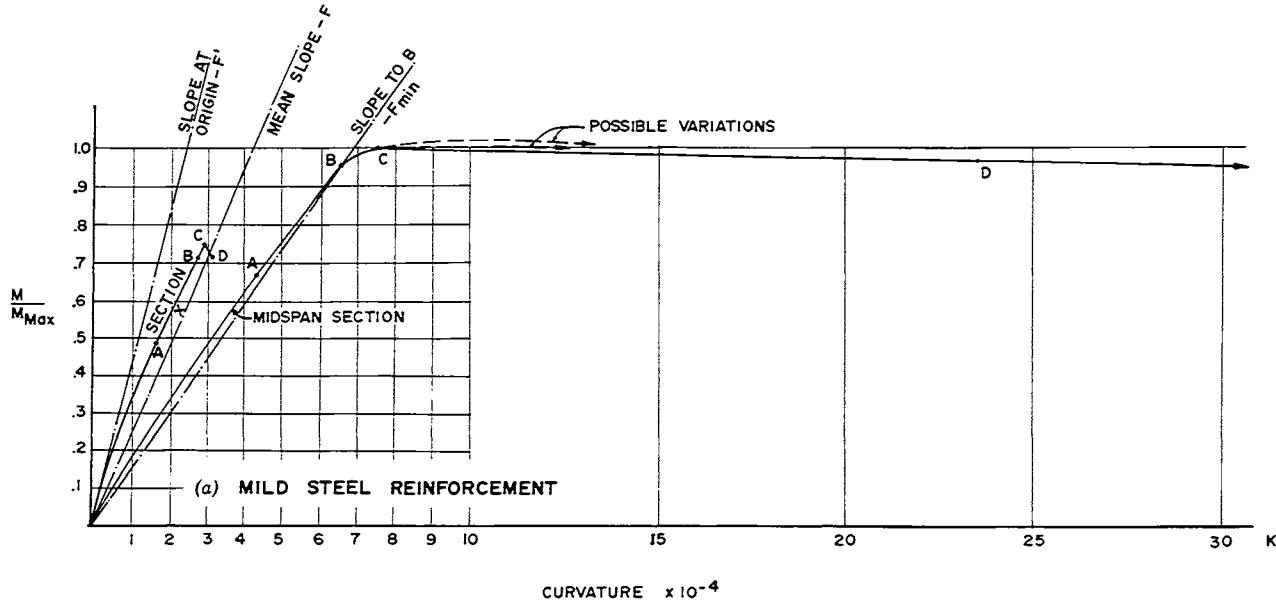


FIG. 3.—TYPICAL MOMENT-CURVATURE RELATIONSHIPS FOR THE MIDSPAN SECTION OF AN UNDER-REINFORCED CONCRETE BEAM

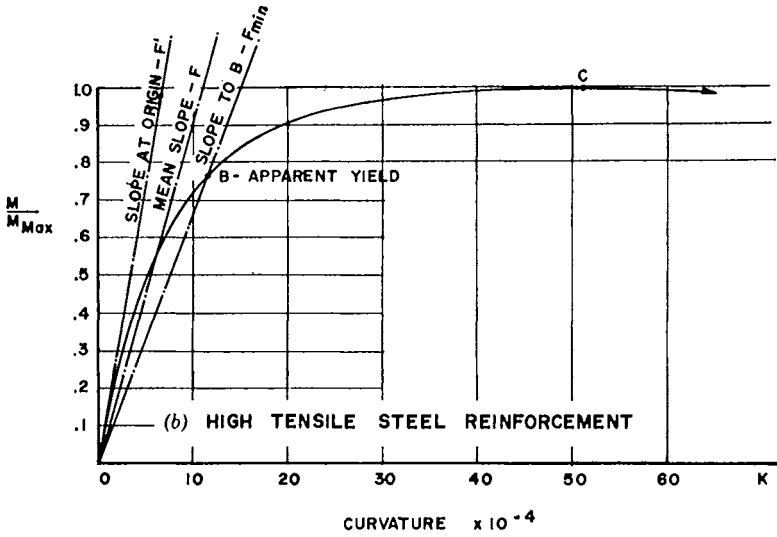


FIG. 3b

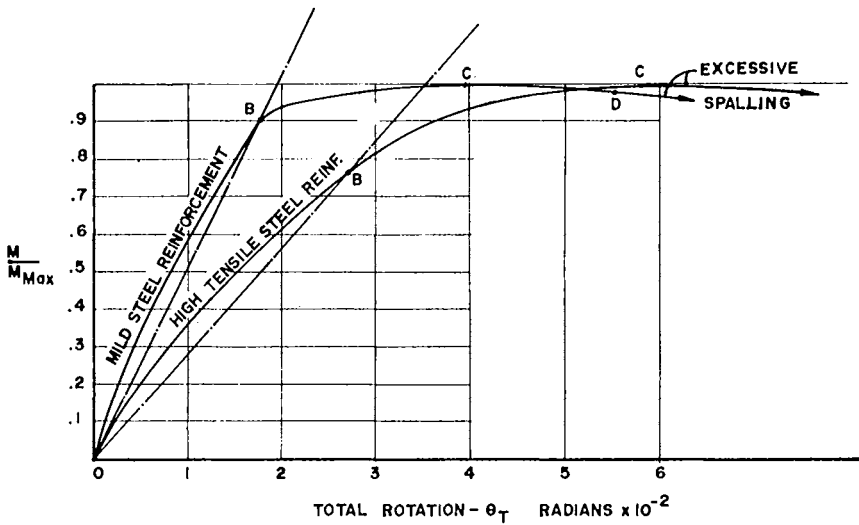


FIG. 4.—TYPICAL MOMENT-ROTATION RELATIONSHIP FOR MEMBER WITH POINT LOAD AT MIDSPAN

2. At first yield of the mild steel reinforcement (or if High Tensile steel, its apparent yield point).
3. At maximum strength of the member.
4. At a stage when spalling has occurred and the critical section may be considered to be near failure. With high tensile steel reinforcement this may occur before the member attains its maximum strength.

Although the bending moment distribution was linear, it is seen that with the exception of possibly a brief period when the member was initially uncracked, the curvature distribution across the member is nonlinear. Results obtained from these and other experiments indicate that the moment curvature relationship will vary from section to section within the member. It is unlikely, apart from symmetry, that any two cross sections will have the same moment curvature diagram. Therefore, to apply this concept to any design method, it would require a knowledge of the moment curvature relationship at every section. Also at no stage, except possibly in the initial, short lived, uncracked range, will the moment curvature relationship for any particular cross section have a truly linear range.

These are points that are not generally appreciated, it usually being implied that the behavior of a critical section is representative of the behavior of the member as a whole. This is not the case, however, and in calculation it will result in deformations that are grossly overestimated. This will give a conservative result in the design of a statically determinate system, but in statically indeterminate systems the solution may be seriously affected, the over-estimation of deformation not necessarily giving a "safe" answer. Furthermore, if it is to be assumed that prior to yield or apparent yield of a member, its behavior is to be linearly elastic then to enable realistic or correct deformations to be evaluated, a mean value for the flexural rigidity of this member must be used. Not only will this mean value assure that deformations prior to yield are correct, but also that the elastic and inelastic deformations are realistically proportioned. The choice of a mean flexural rigidity has been discussed in greater detail by Burnett^{8,9} and as a summary the following points are now considered.

The major advantage of including inelastic behavior in a design method is that the moment redistribution that occurs in the structure can appreciably increase the strength of the structure over and above that indicated by the Elastic Theory. This additional strength is limited by the deformational capacity of the structure. To make optimum use of these inelastic properties it would seem that it would be advantageous to use the maximum elastic flexural rigidity i.e., the mean, initial, uncracked value for the slope of the moment curvature relationships. This would give the minimum "elastic" deformation and consequently a much greater proportion of the available deformation would be inelastic. However, this is shown to be not the case. In theory, because cracking occurs with the first application of load, it would mean that for all loads there would be at each section an elastic and an inelastic contribution to the behavior of the structure. Now the extension of any basic theory to the

⁸ Burnett, E. F. P., "The Ultimate Load Design of Statically Indeterminate Concrete Structures," thesis presented to London University, at London, England, in 1963, in partial fulfilment of the requirements for the degree of Master of Science.

⁹ Burnett, E. F. P., "Flexural Rigidity, Curvature and Rotation and Their Significance in Reinforced Concrete Design," Magazine of Concrete Research, June, 1964.

practical solution of the structures is dependant on the assumption that the inelastic behavior, in the vicinity of any critical section, be considered to be concentrated at that section. Using a maximum flexural rigidity would give a maximum spread of inelasticity and it can be shown^{8,9} that the error inherent in making this assumption is greater than if the mean value were used. It is important that not only the magnitude but also the distribution of inelastic effects be taken into consideration in making any approximations.

In the case of prestressed concrete, the range of loading for which the member is uncracked and hence the range for which moment and curvature may be linearly related at any section, may be more considerable as compared to reinforced concrete. This still does not mean however, that all sections in any member will have the same flexural rigidity. The inadequacy of using a constant flexural rigidity, even in this case, based only on the behavior of the member in the immediate vicinity of the critical section, has been experimentally demonstrated by both G. Macchi¹⁰ and E. W. Bennett.¹¹

Using a mean value for the flexural rigidity for the elastic behavior of a member will of course, preclude the possibility of calculating strains by elastic methods. In general, it may be considered that strains and deformations are linearly incompatible for reinforced concrete members in flexure.

In addition to these comments on the moment curvature relationship before yield, it is well known that after yield it is practically impossible to predict a curvature value in the region of the critical section. This is especially true for under reinforced members with mild steel reinforcement, where the behavior is very nearly plastic (i.e., the moment curvature relationship is horizontal).

The difficulty of using the moment curvature relationship in analysis procedure is thus clear.

MOMENT ROTATION RELATIONSHIP

Returning to Eq. 6; this generalized compatibility relationship may be expressed in terms of rotations rather than curvature, as follows:

At any section, $K = K_E + K_I$, in which K_E is equivalent to the mean elastic curvature and K_I the equivalent inelastic contribution to the total curvature at this particular cross section, therefore

$$\int_{\text{span CD}} \frac{x}{l} K_E dx + \int_{\text{span CD}} \frac{x}{l} K_I dx + \int_{\text{span DE}} (1 - \frac{x}{l}) K_E dx + \int_{\text{span DE}} (1 - \frac{x}{l}) K_I dx = 0 \dots (7)$$

However, $K dx = d\theta$ where $d\theta$ is equivalent to the rotation of the length dx such that $K dx = K_E dx + K_I dx$, therefore

$$\int_{\text{span CD}} \frac{x}{l} d\theta_E + \int_{\text{span CD}} \frac{x}{l} d\theta_I + \int_{\text{span DE}} (1 - \frac{x}{l}) d\theta_E + \int_{\text{span DE}} (1 - \frac{x}{l}) d\theta_I = 0 \dots (8)$$

¹⁰ Macchi, G., discussion, Proceedings, Session I, Third Congress, Federation Internationale de la Precontrainte, Berlin, West Germany, 1958.

¹¹ Bennett, E. W., discussion of "Ultimate Load Design of Concrete Structures," Proceedings, Institution of Civ. Engrs., October, 1962, pp. 225-269.

In any trial and error solution, with a knowledge of the mean constant flexural rigidities, the elastic contributions to these equations may be obtained. The inelastic effects can be considered as rotations—in fact, the moment of rotation. Not only must the moment rotation relationship be known but it is also necessary at this stage to assume that for the purposes of calculation the inelastic rotation is concentrated at the particular critical section. This will be correct only for those bending moment distributions which are symmetrical about this critical section. However, it has been shown⁸ that any resulting error would be small and will have no real significance, especially with under-reinforced members and in regions of hogging moment.

$$\int \frac{x}{I} d\theta_E + \sum \frac{x}{I} \theta_I + \int \left(1 - \frac{x}{I}\right) d\theta_E + \sum \left(1 - \frac{x}{I}\right) \theta_I = 0 \dots\dots (9)$$

span CD span CD span DE span CD

In any statically indeterminate system if there are s possible critical sections and n statical indeterminacies, s will be greater than n . There will be a set of n Eq. 9 and although there are only n unknowns i.e., the value of the bending moment at n critical sections, these equations are not linearly independent of each other and the solution must be found by trial and error.

Using Eq. 9, there are at this stage two approaches to the problem:

The Flexibility Method.—A bending moment distribution is assumed for the structure—choose values at n critical sections, the remaining $(s-n)$ values being determined from equilibrium criteria. For the solution to be valid, the n compatibility equations must be satisfied. This method is discussed in the report¹² to the Institution of Civil Engineers on Ultimate Load Design of Concrete Structures as a means of rigorous analysis.

The Stiffness Approach.—In this case, the nonelastic behavior is introduced into the structure which is considered to be wholly elastic, as an imposed discontinuous rotation at the relevant critical section. Bending moment diagrams may be obtained for each critical section which, when added to the bending moment diagrams for the loaded elastic structure, will give a bending moment distribution which must satisfy the behavioral criteria for each critical section. This is effectively what is done in the method of Imposed Rotations developed by Macchi.¹³

Both of these methods will yield the same result and will also suffer from the same limitation, in that being trial and error methods they can only be applied to simple systems. The stiffness approach is possibly the easier to apply. The main value of these methods is that they can deal with reinforced concrete members whose inelastic behavior is not plastic. In this event, it is unlikely even for the simplest statically indeterminate systems that at failure the structure will have complete adaptation i.e., maximum or ultimate moment at every critical section. This is significant especially if as Guyon³ has suggested, it becomes necessary to classify structural systems into those which can approach complete adaptation and those which cannot.

¹² "Ultimate Load Design of Concrete Structures," Research Committee Report, Proceedings, Institution of Civ. Engrs., February, 1962, pp. 399-442.

¹³ Macchi, G., "Proposed Method of Analysis Based on the Theory of Imposed Rotations," C. E. B. Bulletin No. 21, January, 1960.

DEVELOPMENT OF DESIGN METHODS

Because, in normal practice, most reinforced concrete sections are under-reinforced, it can be assumed that there is little moment variation after yield. This has been demonstrated from tests carried out at Imperial College and other laboratories participating in the C.E.B. moment rotation work (see conclusions). Typical moment rotation curves are shown in Fig 4. Thus, it is seen that for the purpose of calculation the critical section may be assumed as a frictionless hinge after it has yielded. Hence, we have the concept of plastic hinge in reinforced concrete, which allows for considerable simplification of calculation as will be discussed in the following paragraphs.

Although it has become common practice to use the terms "plastic hinge" and "critical section" it must be appreciated that they are representative of the behavior of either the whole of a member or a certain length of it. The properties of a plastic hinge are not the properties of the individual critical sections and correspondingly the moment-rotation relationship for any critical section is dependant on much more than just the properties of the cross-section involved.

Any analytical problem will always be a question of trial and error but in devising a method of design it is possible to capitalise on one's experience and consider a structure to be proportioned in such a way that a solution at or near ultimate load may be obtained. Bearing in mind the assumptions previously made, ultimate load will be either when excessive deformations occur at one or more critical sections or when sufficient hinges form to cause kinematic instability in either a part of or the whole of the structure.

Based on the simplification on the moment rotation curve as discussed above, design methods can be developed such as the Simplified Limit Method developed by A.L.L. Baker,^{14,15}

Essentially a flexibility approach, this simplified method is solely a means of design. The structural system is considered at that stage just prior to complete collapse, i.e., one hinge has still to form for the structure to become a mechanism. For a structure with n statical indeterminacies, n hinges are considered to have formed such that the structure is still kinematically stable and statically determinate. If for this set of hinges a bending moment distribution can be found which satisfied the requirements of equilibrium and deformational compatibility then it will be a valid solution. Furthermore, it will be a Lower Bound solution. However, the structure will only be one hinge away from collapse and for all practical purposes may be considered to be the ultimate condition. The derivation and application of this method has been well documented elsewhere^{14,15} but the compatibility equations are essentially the same as Eq. 9 except they are usually in more familiar terms.

This method is still a trial and error procedure but with experience and the use of design charts it is applicable to most structural frameworks. Apart from the present lack of experimental information about the behavior of individual members the main limitation of this method is that it does not involve any serviceability criteria. The structure is designed solely on its behavior near collapse. A check on working load behavior may be done either

¹⁴ Baker, A. L. L., "The Ultimate Load Theory Applied to the Design of Reinforced and Prestressed Concrete Frames," Concrete Publications, London, England, 1956.

¹⁵ Baker, A. L. L., "Ultimate-Load Theory for Concrete Frame Analysis," Transactions, ASCE, Part II, Vol. 127, 1962.

by using a relaxation technique or by elastic analysis. In addition, any solution will be dependant on the initial choice of those n sections that are to form plastic hinges, these sections being proportioned accordingly. However, there could be more than one set of n hinges that could produce a structure with different proportions which can support the same ultimate load. Whatever solution is arrived at will be valid, but the choice of hinges is to some extent arbitrary, but for most typical frameworks a practical set of hinges may be readily obtained.⁸ As no elementary mechanisms are to be allowed, there will be in each elementary mechanism one critical section at which a hinge does not form. For m elementary mechanisms, $m = s - n$.⁸ From practical experience it is not difficult to choose m critical sections which are unlikely to become hinge sections. In any event the structure chosen, if not the optimum will at least be a logical solution as far as ultimate strength is concerned. However, it is doubtful whether this is sufficient and it is of interest to enumerate the requirements necessary for the design of any structure.

DESIGN CRITERIA

The criteria governing design are numerous and although their relative importance varies it is essential that each particular requirement be satisfied in the actual structure, irrespective of the method of design.

Under the action of the working loads or any combination thereof, the structure must fulfill those requirements, functional, structural, ornamental or otherwise, for which it is intended. Cracking, surface spalling, deflection and other deformational effects must be kept within reasonable limits. The effects of creep, temperature, permanent set and fatigue weakening should also be considered.

Primarily in the interests of safety the structure must be capable of taking an overload such that together with the effects of creep, strain hardening, etc., there is an adequate safety factor against failure of the structure. Values for this safety factor are usually given in terms of the working loads. This load factor must not only incorporate the physical behavior at this ultimate load, but should also in some way reflect the probability of overload, the consequences of failure, and the indications (if any) of impending failure.

Working load criteria are essentially deformational, while failure criteria are for concrete structures dependant on both the strength and the deformational capacity of the structure. Relative to previous behavioral assumptions, an additional criterion may be applied and that is that within the working range, no plastic hinge should be allowed to form. Even though the deformation may be small the presence of inelastic strains will mean that under load reversals different bending moment will result because the basic structure will have changed. In addition to this it is assumed that, to obtain an ultimate load solution, the load is continuously increasing such that prior to the application of the ultimate load no inelasticity had occurred in the structure. Thus, the working loads must lie within the elastic range of behavior of the structure.

From the above discussion it is readily seen why a design method such as the simplified limit method of design quoted is not entirely satisfactory.

OPTIMUM METHODS OF DESIGN

With the development of methods for finding the ultimate strength of a structure, it has become possible to think in terms of an optimum solution, i.e., that which most efficiently satisfies both the serviceability and ultimate load criteria. The elastic theory is effectively only a working load design. Most ultimate load methods of design, of which the simplified limit method is perhaps the most useful, are based solely on ultimate load criteria.

Now the load distribution for the critical working load conditions need not necessarily be the same as that for ultimate load. For each and every load condition both the ultimate strength and the method of failure of the system may vary. For these reasons it is unlikely that a method of design which includes both working and ultimate load requirements can be developed for structures in general. However, for certain simple statically indeterminate systems it is possible to arrive at what may be termed an optimum solution. Possibly the most interesting method is that developed by M.Z. Cohn,^{16,17}

Because the original publications are not easy to understand, a clarified concise version is outlined below to facilitate discussion.

For reasons which become apparent later, these methods apply only to those systems which consist solely of beam mechanisms. In this type of system the failure mechanism will be either one beam mechanism or a combination of them failing simultaneously. There are no elementary sway or joint mechanisms which will ensure that no combination of elementary mechanisms can form so that no load factor less than that for the elementary mechanisms may be obtained. Thus this method, with one notable exception, is limited to continuous beams.

The following assumptions are necessary:

(1) Prior to yield the structure behaves linearly elastically while after yielding the critical sections will rotate plastically.

(2) The load factor, (i.e., the ratio of the actual load to the working load) applies to the live loads only.

For working load conditions the various arrangements of live load must be taken into account in evaluating the maximum bending moment at a particular critical section. Take a particular critical section i . Some arrangement of live load will produce maximum moment and the first plastic hinge at this section. For example, alternate spans may be loaded in a continuous beam. The magnitude of this maximum moment for this particular live load when expressed in terms of the actual working loads will take the form:

$$X_i = a_i G l^2 + b_i \lambda P_i l^2 \dots\dots\dots (10)$$

in which G and P_i represent respectively the dead load with $\lambda = 1$ and the particular live load acting. Where a_i and b_i are elastic coefficients obtainable

¹⁶ Cohn, M. Z., "Limit Design of Reinforced Concrete Structures for Maximum Yield Safety," Indian Concrete Journal, June, 1962.

¹⁷ Cohn, M. Z., discussion of "Ultimate Load Design of Concrete Structures," Proceedings, Institution of Civ. Engrs., October, 1962, pp. 225-269.

either from handbooks or by analysis, due consideration being given to the relative stiffness of the members.

For the same distribution of live load but for a load less than the working load, is less than l .

If this load is increased so that the section at i yields prior to any other plastic hinge forming then at yield the moment at i will be

$$X_{i(I)} = a_i G l^2 + b_i \lambda_{i(I)} P_i l^2 \dots \dots \dots (11)$$

in which $\lambda_{i(I)}$ must be greater than l if the section is not to yield at working loads.

If, for this particular arrangement of loading, the loads are increased until failure occurs, either from partial or complete kinematic instability, then the magnitude of the moment at this section i for this particular ultimate load will be

$$X_{i(u)} = a_i G l^2 + \bar{b}_i \lambda_{i(u)} P_i l^2 \dots \dots \dots (12)$$

The structure no longer being elastic will have a coefficient $\bar{b}_i$ which is not equal to b_i . From the assumed moment rotation characteristic it is known that $X_{i(I)} = X_{i(u)}$ and hence $b_i \lambda_{i(I)} = \bar{b}_i \lambda_{i(u)}$ and therefore

$$\bar{b}_i = b_i \frac{\lambda_{i(I)}}{\lambda_{i(u)}} \text{ or } \bar{b}_i = b_i x_i \text{ for } x_i = \frac{\lambda_{i(I)}}{\lambda_{i(u)}}$$

The smaller the value of the ratio $\lambda_{i(I)}/\lambda_{i(u)}$ the greater will be the moment redistribution and the factor of safety against kinematic instability for that particular system of loading. Therefore, as far as possible, minimum values for x_i are required.

Similarly, j for some critical section, j for example, for a different live load arrangement the maximum possible moment is given by:

$$X_{j(u)} = X_{j(I)} = a_j G l^2 + b_j x_j \lambda_{j(u)} P_j l^2 \dots \dots \dots (13)$$

There will be theoretically as many equations as there are critical sections. In general, neither $\lambda_{i(I)}$ nor $\lambda_{i(u)}$ nor x_i are necessarily equal to $\lambda_{j(I)}$, $\lambda_{j(u)}$ or x_j .

However, in a particular beam mechanism in any continuous beam all the critical sections in this mechanism will have the same ultimate or collapse load factor.

For each of the elementary beam mechanisms there will be an equilibrium equation. Applying the principle of virtual work each equation will be as follows:

For the beam mechanism A, referring to Fig. 5.

$$\gamma_p^A X_{p(u)} + \gamma_q^A X_{q(u)} + \gamma_r^A X_{r(u)} = \psi^A (G l + \lambda_u^A P_i l) l \dots (14)$$

in which γ_p^A , γ_q^A etc. are dimensionless parameters defining the location and

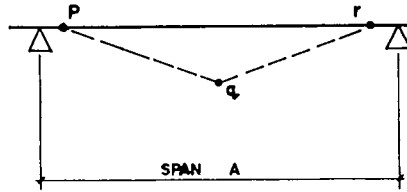


FIG. 5.--TYPICAL BEAM MECHANISM

contribution from each plastic hinge, and where ψ is a dimensionless parameter defining the contribution from the total load ($G_l = \lambda_u^A P l$).

$$\text{i.e.} \quad \sum_{i=p, q, r} \gamma_i^A X_{i(u)} = \psi^A (G + \lambda_u^A P) l^2 \dots \dots \dots (15)$$

Substituting in values for X_p , X_q and X_r .

$$\sum \gamma_i^A (a_i G + b_i x_i \lambda_u^A P) l^2 = \psi^A (G + \lambda_u^A P) l^2 \dots \dots \dots (16)$$

$i = p, q, r$.

Since Eq. 16 is an equilibrium relationship the dead and live loads may be separately considered, hence

$$\sum \gamma_i^A b_i x_i \lambda_u^A P l^2 = \psi \lambda_u^A P l^2 \dots \dots \dots (17)$$

and

$$\sum \gamma_i^A b_i x_i = \psi A$$

or

$$\sum \gamma_i^A b_i \lambda_{i(I)} = \lambda_u \psi A$$

As λ_u^A , P and l are common to both sides of the equation.

In any continuous beam with y spans say, there will be $(2y-1)$ critical sections. However there are only y beam mechanisms (assume that they are all different) and hence only y linearly independent equations. Hence, to solve for $(2y-1)$ values for x_i it is necessary to impose certain conditions so that $(2y-1-y)$, i.e., $(y-1)$ values for x_i may be otherwise obtained.

Previously, in arriving at the ratio $\lambda_i(I)/\lambda_{i(u)}$, it was noted that for maximum redistribution and thus maximum ultimate or collapse load factor, the value x_i should be a minimum. In addition, for the most efficient behavior at working loads all the critical sections should have the same yield load factors i.e., $\lambda_i(I)$ which are to be greater than or equal to unity but which are to be as small as possible. In theory, a structure with $\lambda_i(I)$ everywhere equal to 1, would be the ideal serviceability solution—this is in effect the object of most elastic designs provided of course, that deformations are not excessive. Thus, in general, it may be inferred that the minimization of x_i values and the equalization and minimization of $\lambda_i(I)$ values will produce an optimum solution.

At this juncture, there are two alternative procedures that may be applied.

Case I.—A logical criterion for the beam would be for all beam mechanisms to have equal collapse load factors i.e. common λ_u . Therefore a minimization and equalization of values of $\lambda_i(I)$ (and x_i) is the optimum condition.

From the mathematics involved for y different independent equations there must be at least y different values for $\lambda_i(I)$. Hence the maximum number of equal values for $\lambda_i(I)$ must be y , i.e., in one beam mechanism only are all the values for $\lambda_i(I)$ equal. To find this minimum common value of $\lambda_i(I)$ for y sections is merely a question of arithmetic. By taking each of the y equations in turn and by equating the $\lambda_i(I)$ (or x_i) values, the minimum value is determined. Using this minimum value it is a relatively simple matter to solve the remaining values for $\lambda_i(I)$ (or x_i).

Case II.—An alternative approach would be to provide equal values for $\lambda_i(I)$ at particular critical sections such that:

(1) A minimum ultimate safety factor is ensured for the structure but each mechanism need not necessarily have the same ultimate load factors.

(2) All support critical sections are to have equal values for $\lambda_i(I)$; all span critical sections are to have equal values for $\lambda_i(I)$. Span sections are to have a greater safety factor against first yield than support sections, as in general, the rotation at support sections is critical. Eq. 17 becomes

$$\sum \gamma_i^A b_i \lambda_i(I) = \lambda_{i(u)} \psi^A$$

$i = p, q, r$ for beam mechanism A.

In each of these y equations $\lambda_p(I) = \lambda_r(I) = \lambda_s(I)$ at supports and for the span sections $\lambda_q(I) = \lambda_m(I) > \lambda_s(I)$ and $\lambda_i(u) \geq \lambda_u$, the safety load factor for collapse. Therefore,

$$\gamma_p^A b_p \lambda_s(I) + \lambda_q^A b_q \lambda_m(I) + \gamma_r^A b_r \lambda_s(I) \geq \lambda_u \psi^A$$

therefore

$$\lambda_s(I) \geq \frac{\lambda_u \psi^A - \gamma_q^A b_q \lambda_m(I)}{\gamma_p^A b_p + \gamma_r^A b_r} \dots \dots \dots (19)$$

There will be y of these equations which, for a chosen value for $\lambda_m(I) > 1$ and the requisite value for λ_u will give y values for the right hand side of the equation. For a safe solution $\lambda_s(I)$ is taken as equal to the maximum value obtained.

This second approach is inferior to the first in that there is more than one possible solution and depending on the value assigned to $\lambda_m(I)$, the value for $\lambda_s(I)$ need not necessarily be a minimum. In the first method the solution is unique and a true optimum.

Whichever approach is used to arrive at values for $\lambda_i(I)$ it remains to substitute these values in the appropriate Eq. 9 for the structure to be proportioned and accordingly reinforced.

Whatever the relative merits of these two approaches, they are both theoretically superior to any method (known to the authors) for the design of continuous beams. They are both true design methods in that it is not necessary to know the sectional properties or dimensions of the individual sections and the most efficient solution in terms of serviceability and ultimate load criteria have been incorporated.

The significance of the value of $\lambda_{i(I)}$ was initially appreciated by V. Petcu^{18,19} and Cohn.¹¹ Cohn defined the ratio $\lambda_{i(I)}/\lambda_{i(u)}$ as the yield safety parameter and working in terms of this parameter, has been mainly responsible for the development of the methods of optimum design which have here been briefly outlined. Petcu on the other hand, incorporated the value for $\lambda_{i(I)}$ in an expression involving the total failure load and the load at first yield, that is

$$r_i = \frac{G + \lambda_{i(u)} P}{G + \lambda_{i(I)} P}$$

in which r_i is the redistribution factor for the critical section i . Of the two methods, Cohn's approach is possibly the easier to apply, although both methods will yield the same result.

For the usual types of loading, these optimum design methods may be applied to continuous beams in general, irrespective of their end conditions. For 2, 3, 4 span beams with standard loading systems, both Cohn and Petcu have drawn up tables, while the former has also given tables for beams of non-uniform section.

For any optimum solution to be valid, it remains to check that deformations are not excessive, i.e. deflection at working loads and rotation capacity of the critical sections near ultimate load. The former may be done by conventional elastic methods, while Cohn has suggested an approximate method of checking the latter.

These optimum methods are limited to continuous beams. In more complex systems, apart from the practical difficulties of considering the critical loading condition for each critical section, the fact that the collapse mechanism may be a combination of various elementary mechanisms means that $\lambda_{i(u)}$ in each elementary mechanism could be $\geq \lambda_u$, the actual load factor, and $\lambda_{i(I)}$ need not necessarily be related to λ_u . In dealing with fairly simple single storey portal frames, if a design condition is imposed such that the individual elementary mechanisms are to have the same value for the ultimate load factor as that for the combined collapse condition with all load present, then it is possible to arrive at a solution similar to that for a continuous beam. i.e. by designing the structure so that at collapse all the elementary mechanisms form simultaneously. In the writers' view, only under exceptional circumstances will this be practically possible.

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

To sum up the above discussion, there are two basic concepts from which analysis and design of reinforced concrete linear structures can be developed.

¹⁸ Petcu, V., "The Optimum Redistribution Cycle," Indian Concrete Journal, July, 1961.

¹⁹ Petcu, V., "Optimum Plastic Moments of Reinforced Concrete Continuous Beams," Indian Concrete Journal.

They are: (1) moment curvature concept and (2) moment rotation concept.

As emphasized in preceding paragraphs, there is great difficulty in using moment curvature relationship in analysis. There is also a severe lack of knowledge of the fundamental properties of moment curvature relationship of reinforced concrete members. Therefore, before any analytical method based on this concept can be acceptable in practice, much fundamental experimental work on the subject is necessary. However, some allied computer work on this subject has already been started. Ferry Borges²⁰ has attempted, by means of computer, to use different trial moment curvature relationships for different sections along a structural member, to check calculated deformations against those obtained from tests. So far only theoretical moment curvature curves have been used. It is hoped that in the future actual moment curvature curves obtained from tests will be used. This will no doubt throw further light on the validity of this approach.

Although there are experimental evidence, as discussed in preceding paragraphs, that the moment rotation concept is safe and satisfactory, nevertheless there is yet a considerable amount of experimental work to be done in this field. For instance, more examples of statically indeterminate structures with the worst combination of variables should be tested to compare with analytical results to check the errors involved. The degree of adaptation for different type of structural system should also be experimentally investigated. Since the mean EI values in the calculation, according to the moment rotation concept, are obtained from the moment rotation characteristics of the member, a greater range of variables must be covered in the experimental work to ensure general application.

Furthermore, the latter part of the moment rotation curve should be carefully studied so that a more universal definition of the "practical ultimate" capacity for rotation can be defined.

It must be emphasized again that to arrive at the deformation of a structure and to make use of the simplifications of the moment rotation characteristics in design methods, it is necessary to use the mean flexure rigidity of the various sections. These mean values are, therefore, not compatible with the strain occurring in the sections.

It should be mentioned that a fairly extensive experimental program is being carried out at the moment on the moment rotation characteristics of reinforced and prestressed concrete members. The European Concrete Committee (C.E.B.) has organized such a program on an international basis. Moment rotation characteristics of statically determinate and prestressed beams are being studied with quantity of reinforcement, concrete strength, type of reinforcement, type of loading, shape of section and influence of compression reinforcement as parameters.

Although a fair amount of analytical work has been done since the last review on the subject,²¹ and advancement such as the optimum design method has recently been achieved, there are still a number of analytical problems requiring early investigation. The instability of framework in the inelastic

20 Borges, J. Ferry, and Arantes e Oliveira, E. R., "Non Linear Analysis of Reinforced Concrete Structures," Ministerio Das Obras Publicas, Laboratorio Nacional de Engenharia Civil, Lisbon, Portugal, January, 1963.

21 Yu, C. W., and Hognestad, E., "A Review of Limit Design for Structural Concrete," *Journal of the Structural Division*, ASCE, Vol. 84, No. ST8, Proc. Paper 1878, December, 1958.

stage and the behavior of structural systems subjected to oscillating load in the inelastic stage, are just a few examples to mention. The writers also believe that sufficient knowledge is available to enable simple design rules for simple statically indeterminate structures to be developed, so that practical designers can proportion ordinary structures realistically and economically, without excessive drawing office time.

It is seen that the amount of experimental and analytical work yet to be done is extensive and if research workers would cooperate to ensure no overlapping of investigation, this could be accomplished without undue waste of effort and time.

DISCUSSION

EDWARD G. NAWY,²²—This paper highlights the importance of the moment-rotation relationship in studying the behavior of reinforced concrete frames and indeterminate beams, versus the more cumbersome and less accurate moment-curvature relationship. It shows that the moment-rotation characteristic varies significantly from section to section, making the use of the moment-rotation more realistic particularly as the limit load is approached. However, the authors are asked to comment and amplify on the effects of creep and stability on the moment-rotation criteria presented, and in what manner a quantitative evaluation of the long-term effects can be incorporated.

HERBERT A. SAWYER,²³—This paper attempts to compare various proposals for non-linear analysis and design, and, at their present stage of development, comparative studies of such proposals needed. However, the comparisons presented in this paper are often invalid because of misconceptions of one of the methods considered.

The authors' argument is that "to apply the moment-curvature concept to any design method would require a knowledge of the moment-curvature relationship at every section", and that since we do not have this knowledge for most cases, "the difficulty of using the moment-curvature relationship is thus clear." Instead, they propose to use a moment-rotation relationship in which the inelastic rotations for any moment less than the maximum design moment are ignored and assumed to be zero.

However, as the writer has stated,²⁴ the basic purpose of the moment-curvature relationship when used for bilinear design is not the accurate representation of M versus ϕ at a cross section, but rather the accurate representation, upon integration, of the moment-rotation relationship.

²² Asst. Prof. of Civ. Engrg., Rutgers Univ., New Brunswick, N. J.

²³ Prof. of Civ. Engrg., Univ. of Florida, Gainesville, Fla.

²⁴ Sawyer, H. A., Jr., "Design of Concrete Frames for Two Failure Stages," Internatl. Symposium on the Flexural Mechanics of Reinforced Concrete, ASCE-ACI, 1965.

Actually, the triangular inelastic curvature areas of the sloping bilinear $M-\phi$ relationship, Fig 6(b) proposed most recently in paper No. 15, represent the simplest possible mathematical model which adequately represents inelastic behavior, Fig 6 (a) for design purposes, the triangle being the simplest geometric area. Certainly the inelastic "peaks" of Fig. 6(a) and Fig. 3 can be reasonably well represented by such triangles, and it is even more certain that, for pre-maximum loading stages, such representation is far superior to ignoring these peaks entirely, Fig. A(c), as proposed by the authors.

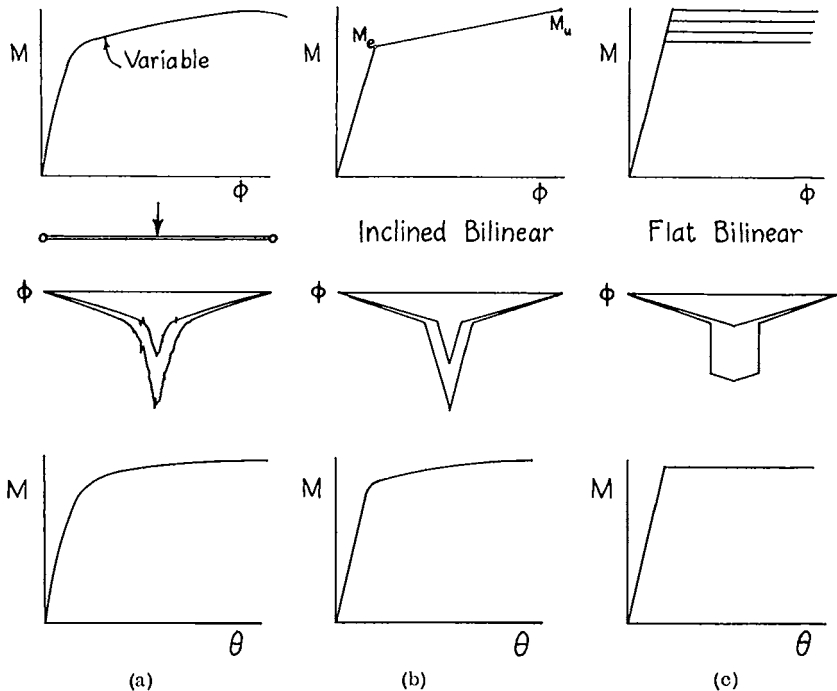


FIG. 6.—COMPARISON OF ACTUAL, INCLINED BILINEAR, AND FLAT BILINEAR $M-\phi$ RELATIONSHIPS

The triangular inelastic curvature area of Fig. 6(b) is analogous to the rectangular (or parabolic) stress distribution used in calculation of the ultimate flexural strength of beams. We do not know the actual stress distributions at ultimate strength for, for example, various concretes or various rates of loading, but we should not therefore discard the concept of a compression zone, or assume it to be zero in size, and thus use a moment arm for the flexural couple equal to the total effective depth!

This analogy is closer than it might seem, because the error in neglecting inelastic rotations for less-than-maximum moments is usually ²² on the unsafe side and it is of a similar magnitude. The maximum magnitude of this error can be easily shown to be $1 - M_e/M_u$, in which M_e is the moment defined by the intersection of the two lines of the sloped bilinear $M-\phi$ approximation,

and M_u is the maximum design moment at the critical section. The apparent value of this quantity for the results reported in Fig. 3 is approximately 0.40. The average value of this quantity is approximately 0.13 for the C.E.B. tests referred to by the authors (agreeing with Fig. 6 of Paper 15), and with slower rates of loading this quantity is larger. Therefore, one may question the authors' statement (p. 39) that "for the purpose of calculation the critical section may be assumed as a frictionless (presumably "constant-friction" rather than "frictionless" was intended) hinge;" the increase in load factor necessary to compensate for this error imposes too severe a penalty on the method advocated by the authors.

It should be emphasized that Eqs. 1-9 merely express the geometric conditions which must in effect be satisfied by all the methods mentioned. Although Eqs. 7-9 and certain techniques for satisfying these equations are discussed in the section "Moment Rotation Relationship," this section can, if desired, be equally applied to the moment-curvature approach and the moment-rotation approach. However, in practice it is believed that these geometric conditions can be most conveniently satisfied by use of numerical methods related to the Cross moment distribution method for elastic structures, as previously discussed.²²

M. Z. COHN,²⁵—The writer appreciates the critical review of the optimum limit design (OLD) methods and the interpretation given in the paper to one of his related contributions.¹⁷ It is hoped that principles and features of OLD methods will be more apparent in a comprehensive study²⁶ from which the reference¹⁷ has been abstracted.

The following comments to the paper are intended to clarify possible misunderstandings concerning the applicability of OLD:

1. The assumptions mentioned (model of inelastic behavior and definition of ultimate load) are indeed proper to OLD solutions published up to date. However, they are not restrictive since the basic principles of the method would retain their validity even for assumptions differing from those mentioned. This has been proved for the bi-linear, strain-hardening type of material and for practically all current definitions of ultimate load in terms of adopted load factors.²⁷

2. It should be emphasized that serviceability criteria suggested previously¹⁵ and reproduced in the paper are only possible criteria. Other optimum solutions would result by adopting other criteria for serviceability,²⁵ the choice of the most appropriate ones for a specific problem being a problem open to further study. In any event, it is clear the designer disposes of a broad spectrum of service conditions to choose from to fit the best his design.

3. The authors suppose the following conditions to represent the ideal for serviceability:

... "all the critical sections should have the same yield load factors, which are all to be greater than or equal to unity, but which are to be as small as

²⁵ Assoc. Prof. of Civ. Engrg., Univ. of Waterloo, Waterloo, Ontario.

²⁶ Cohn, M. Z., "Optimum Limit Design for Continuous Reinforced Concrete Beams," Proceedings, Institution of Civ. Engrs. (to be published).

²⁷ Cohn, M. Z., "Optimum Solutions in the Limit Design of R. C. Structures," (to be published).

possible. In theory a structure with $\lambda_i(\eta)$ everywhere equal to 1 would be the ideal serviceability solution--this is in effect the object of most elastic designs . . . Thus in general, it may be inferred that the minimization of x_i values and the equalization and minimization of $\lambda_i(\eta)$ values will produce an optimum solution."

This interpretation is somewhat confusing because:

- a) $\lambda_i(\eta)$ should always be greater than (and not equal) to unity, if the principle of avoiding inelastic behavior at service load is accepted;
- b) the yield load factors should be as large (and not as small) as possible for better service conditions;
- c) a structure with $\lambda_i(\eta)$ everywhere equal to the unity would not be the ideal solution for serviceability since this would imply all critical sections are yielding at working load; and
- d) the object of elastic design is to produce structures with the $x_i = 1$ for all critical sections, but with $\lambda_i(\eta) = \lambda > 1$.

4. It is difficult to agree with the authors' opinion that application of OLD is limited to continuous beams only, though this indeed is the simplest example of its possible use. In fact, essential to OLD is the expression of design plastic moments as fractions of the elastic envelope for the factored loads and the adoption of some rational limit and serviceability criteria. There is no reason why these basic ideas could not be integrated in a design of structures involving more complex mechanisms than those of beam type failures. Relating $\lambda_i(\eta)$ to λ_u as the yield safety parameters (x_i) of sections is, in the writer's opinion, one of the most positive features of his OLD approach. A similar view is expressed by Clyde,²⁸ in a paper illustrating the possibilities for extension of OLD to R.C. frames along with the present difficulties of such extension.

The authors should be congratulated for their comprehensive coverage of many essential aspects of limit design. Perhaps it is worth to emphasize the following remarks in the paper, the importance of which cannot be overestimated:

- 1. The behavior of a critical section is not representative of the behavior of a member as a whole;
- 2. the difficulties in accurately evaluating the actual flexural rigidity of R. C. members;
- 3. the properties of plastic hinges are not those of individual critical sections, but of the inelastic zones assumed as concentrated at these sections;
- 4. the need to consider the service conditions in limit design--the absence of plastic hinges at working loads as an explicit serviceability requirement.

It is believed the authors have thus brought a positive contribution to a realistic approach of the problem by denouncing both trends towards a fictitious excessive rigour and a dangerous oversimplification in limit design.

²⁸ Clyde, D. H., "Conservatism in Reinforced Concrete Frame Theory," Internat'l. Symposium on the Flexural Mechanics of Reinforced Concrete, ASCE-ACI, 1965.

E. F. P. BURNETT AND C. W. YU.—The writers would like to thank Sawyer for his interest and for raising a number of important points. However, from the discussion, it would appear that one of the main points made in the paper has been missed. Nowhere in the paper was it proposed to use a moment-rotation relationship in which, for any bending moment less than the maximum design moment, the inelastic rotations are ignored and assumed equal to zero. On the contrary, both in this paper and in previous work,^{8,9} the authors have gone to some lengths to emphasize that if the member is assumed to behave linearly elastically prior to yield, then a mean value for the flexural rigidity must be used. This value will include the inelastic peaks shown in Fig. 3 and moreover will ensure that deformations prior to yield are correct, and that after yield the equivalent elastic and inelastic effects are realistically proportioned.

Sawyer has proposed to use as a basis for design a bi-linear moment-curvature relationship which is not an accurate representation of the moment v curvature at a cross section but rather the accurate representation, upon integration, of the moment-rotation relationship.

This bi-linear empirical moment curvature relationship would have to be derived from the actual moment-rotation curve and as this moment-curvature relation is then used to evaluate the relevant rotations it would appear that it is of no real consequence which concept is used. Much of what has been said about rotation applies equally to curvature and vice versa. However, the use of the more fundamental moment-curvature relationship introduces a number of assumptions that may be unnecessary and misleading.

Firstly, it is known that moment and curvature are not linearly related after initial cracking, and that their relationship varies along the length of the member. Secondly, to assume a linear relationship after yield implies that bending moment distribution is a major parameter in the calculation of rotations. For example, the inelastic rotations which would occur in identical members with either a point load, a uniform load or a constant bending moment distribution could be expected to be very different from each other. The work of both Macchi¹⁰ and Baker¹² indicates otherwise. With reference to the latter's empirical expression for the inelastic rotation i.e.

$$\theta_p = \frac{(e_{cu} - e_{ce}) l_p}{h_u d}$$

where $l_p = K_1 \cdot K_2 \cdot K_3 (z/d)^{1/4} \cdot d$ and z is the distance from the critical section to the point of contraflexure. All other parameters remaining constant, θ_p is proportional to $z^{1/4}$ whereas from a bi-linear moment-curvature diagram the resultant inelastic rotation would be at least proportional to z .

Rather than introducing these contradictions, would it not be preferable to use the actual moment-rotation relationship rather than the empirical moment-curvature relation?

In reply to Nawy, it may be briefly stated that, because of their greater stiffness in comparison with the equivalent steel members, stability is often not of prime importance in the design of reinforced concrete structures. Over-all stability is dependent on criteria such as the geometry of the structure and secondary and incremental effects both of the loading and the behavior of reinforced concrete; as a result it is difficult to make a general comment on

its effects on moment-rotation criteria. One point we would like to make, however, is that if the falling branch of the moment-rotation characteristic is significant, this will have to be incorporated in any stability calculation.

With regard to creep it may be said that axially loaded specimens have long term strengths of about 80% of the short term strengths. G.C. Ernst²⁹ has shown that the faster the rate of loading the less the redistribution. For more detailed comment the authors would refer Nawy to the paper on creep deflection by Benjamin, Gabrielson and Cornell.

Cohn's comments are very much appreciated and the publication of a more comprehensive study of Optimum Limit Design methods is eagerly awaited. He is indeed correct in pointing out that for ideal service conditions all critical sections should have the same yield load factor, the value of which is to be greater than but not equal to unity. In the paper this sentence should be amended accordingly. In minimizing λ_{i1} , an optimising condition, the value should tend to unity but not equal unity as this would imply yielding at working load. However for $\lambda_{i1} > 1$ and provided that all attendant service conditions not covered by this criterion, e.g. deflection, are satisfied, then it is debatable whether a system with yield load factors greater than the minimum allowable value would result in better service conditions. The intention is surely to optimise this condition and any increase in the value of λ_{i1} over and above the minimum would be unnecessary. In elastic design it is current practice to introduce a stress factor of safety and to design each critical section so that for $\lambda = 1$ the maximum allowable stresses are attained. For this condition $\lambda_{i1} = \lambda = 1$ at all critical sections. Using load factors such that $\lambda_u > 1$, then, as Cohn has pointed out, for the ideal elastic design $\lambda_u = \lambda_{i1}$ or $x_i = 1$ at all critical sections. The authors are in agreement with Cohn about the significance of the yield safety parameter. For the purposes of this paper it was decided to work in terms of the individual load factors as this does permit some simplification.

²⁹ Ernst, G. C., "Plastic Hinging at the Intersection of Beams and Columns," Proceedings, Amer. Concrete Inst., June, 1957.

SAFETY OF REINFORCED CONCRETE FRAMED STRUCTURES

By Milík Tichý¹ and Miloš Vorlíček²

With discussion by Theodore Zsutty, Jack R. Benjamin, C. Allen Cornell,
and Milík Tichý, and Miloš Vorlíček

SYNOPSIS

Because the ultimate strength and deformation ability of critical sections are random variables, the ultimate strength of a structure must likewise be a random variable.

If the structure is subjected to load from one source and there is only one possible collapse mechanism, the determination of the ultimate strength Z_U of the structure is simple.

If the structure is subjected to load from one source but there are m possible collapse mechanisms, it becomes necessary to analyze the structure with the aid of equations of the type given herein. The ultimate strength Z_{Uj} , for $j = 1, 2, \dots, m$ of the structure is determined by means of each of these equations assuming the occurrence of the j -th collapse mechanism. The probability p_{Uj} that the structure will change into the j -th mechanism may be ascertained for a definite value of the load for each random variable Z_{Uj} . But the actual probability of failure must be expressed with the aid of the so-called conditional probabilities since the individual mechanisms are not always statistically independent.

If the structure is subjected to load from ν sources and there are m possible collapse mechanisms an equation for the j -th mechanism will graphically be represented by an interaction diagram. For a given population of structures, identical according to the design, there exists a number of possible combinations of load with a corresponding probability of failure p_{Uj} . Geometrically speaking, they are points in the ν -dimensional space. Their locus is the so-called boundary of the safe domain $\Pi_{\min}$.

When the deformation ability of a structure is considered, the system of equations forms the starting point. In this instance the random variable Z_{Uj} is a linear combination of ultimate moments M_{Uj} and the ultimate plastic rotation θ_U of the section. The statistical solution is analogous with the previous one.

It may be demonstrated that the variability in ultimate strength of a redundant structure is lower than that of a statically determinate one in all cases.

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Consequently, the application of the statistical method must result in savings of material in redundant structures.

INTRODUCTION

The statistical method of analysis of reinforced concrete framed structures to be described is based on the condition of structural safety expressed by

$$S_{\max} \leq U_{\min} \dots\dots\dots (1)$$

In this condition, $S_{\max}$ denotes the maximum value of a function describing the effect of load, determined by the statistical or other method, and $U_{\min}$ the minimum value of a function defining the ultimate strength of the structure, ascertained for the probability p_U of occurrence of values $U < U_{\min}$.

Expressed in other terms, Eq. 1 states the following (Condition 1):

If the structure is subjected to the effect of load $S_{\max}$, it may fail with a probability never exceeding p_U .

The present paper does not deal with the question of random variability of the load; it is assumed that the value of $S_{\max}$ is a known, constant quantity.

From the aspect of a rigorous statistical approach these conditions cannot be considered entirely correct since they do not treat the system "structure-load" as a statistical entity. On the other hand, this condition enables a far clearer analysis of structural safety than the existing modern methods of calculation likewise based on statistical concepts (the method of overload and undercapacity factors) already incorporated in some of the design codes. The advantages of the stated condition are mainly asserted in complex, statically indeterminate, structures.

The method to be described is not considered the last word in the problem of safety of framed structures. It is expected, however, that its application will bring forth valuable results. Although the method is directed to the analysis of reinforced concrete structures, it is certainly feasible to apply its results to prestressed concrete and steel structures.

The problem of safety of statically indeterminate, reinforced concrete structures has been examined from the statistical aspect by Sandi³ and the writers⁴ and from the point of view of the variability of the load, by Horne.⁵

ULTIMATE STRENGTH OF A CONTINUOUS STRUCTURE AS A RANDOM VARIABLE

The ultimate strength of a statically determinate structure is directly given by the ultimate strength of its critical section. The overall ultimate

³ Sandi, H., "Proprietati statistice ale capacitatii portante a structurilor static nedeterminate" (Statistical Properties of Statically Indeterminate Structures), Studii si cercetari de mecanica aplicata, No. 3, 1961.

⁴ Tichý, M., and Vorlíček, M., "Calcul statistique des constructions en béton armé et béton précontraint," Acta technica CSAV, No. 2, 1961.

⁵ Horne, M. R., "Some Results of the Theory of Probability in the Estimation of Design Loads," Proceedings of the Symposium on the Strength of Concrete Structures, London, England, May, 1956, Cement and Concrete Assn., London, England, 1958.

strength of a statically indeterminate structure is, however, determined by both the ultimate strength of several critical sections and by the deformation ability of these sections (or, in some cases, by the deformation ability of those parts which play no direct role in the failure).

Since the ultimate strength and the deformation ability of the individual sections of a structure are random variables owing to the random behavior of both basic materials (concrete and reinforcement) as well as dimensions of the sections, the ultimate strength of the structure must also be a random variable defined by its statistical distribution which is dependent on a certain number of statistical parameters.

Thus the random behavior of a statically determinate structure is given by the random behavior of a single section, while that of a statically indeterminate structure is the resultant of a number of effects (ultimate strength and deformation ability of critical sections as well as the possible occurrence of various collapse mechanisms).

If the type of the statistical distribution and the parameter of the ultimate strength of the structure are known, one can—for a given load—estimate the probability of failure, or for a chosen probability of failure determine the dimensions of the structure.

Considerations herein are based on two main assumptions:

1. The ultimate moments of critical sections are mutually statistically independent and their statistical distribution and parameters are known may be determined e.g. on the basis of a theoretical analysis;⁴ and
2. the structure as a whole or its part do not attain their ultimate strength until complete redistribution of moments occur, i.e. when all critical sections attain their ultimate moments and the structure becomes a collapse mechanism.

As experience indicates, the first assumption is fully justified from the practical standpoint. Although the second assumption is not always perfectly satisfied in reinforced concrete structures, it is used in the present study both for its simplicity, which facilitates the elucidation of the problem, and because its application rests only on the values of ultimate moments and not on the deformation quantities, and the variability of which has not been satisfactorily clarified so far.

Although the normal forces were assumed to exert no effect on the results the method may easily be extended to include the effect of normal forces.

The solution takes no account of the vicinity of the critical section although—in view of the fact that the ultimate strength of the section is a random variable—the ultimate strength of the structure might also be affected by the sections in this vicinity. Such considerations are, however, outside the scope of the present study and have been considered elsewhere.

STRUCTURES SUBJECTED TO LOAD FROM ONE SOURCE AND WITH ONE POSSIBLE COLLAPSE MECHANISM

Minimum Value of Ultimate Strength.—Often the necessity arises to analyze a structure subjected to a single load or to a group of loads mutually dependent, with one possible collapse mechanism. An example is the case of a fixed-ended beam subjected to a concentrated load V acting in the mid-span, or to a uniform

load q , or to both loads related in the manner that $q = \lambda V$, in which λ is a constant independent of the magnitude of the load (Fig. 1a). Similar cases are also encountered in framed structures (Fig. 1b).

Denoting a general load by Z , it follows that $U \equiv Z_U$. Under the assumptions stated previously

$$Z_U = k_0 \sum_{i=1}^n k_i M_{Ui} \dots \dots \dots (2)$$

in which k_0 is a constant depending on the type of load, k_i (for $i = 1, 2, \dots, n$) are constants depending on the collapse mechanism of the structure (arrangement of plastic hinges) and M_{Ui} the ultimate moments in the individual critical sections.

As is well known, quantities M_{Ui} are random variables whose random behavior may be described by a statistical distribution which depends on several

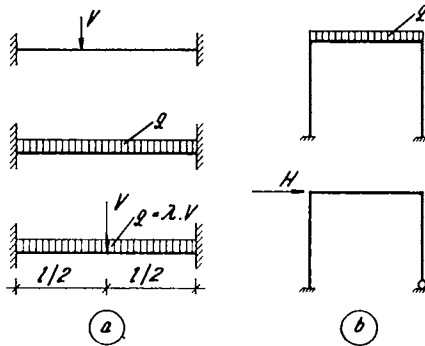


FIG. 1.—EXAMPLES OF STRUCTURES WITH ONE COLLAPSE MECHANISM AND LOAD FROM ONE SOURCE

statistical parameters, such as the mean $\bar{M}_{Ui}$, variance s_{Mi}^2 (or coefficient of variation v_{Mi}) coefficient of skewness a_{Mi} , etc.

Now if the statistical parameters of quantities M_{Ui} are given, the statistical parameters of quantity Z_U may be determined using the methods of mathematical statistics. Assuming that the type of distribution of the former quantity is known (Pearson's type III curve being best suited for practical applications), the minimum value of ultimate strength $Z_{U,min}$ to meet the condition of safety may be found for a chosen probability p_U of occurrence of values $Z_U < Z_{U,min}$.

Because the random variables M_{Ui} appearing in Eq. 2 are—according to the assumption—mutually statistically independent, the statistical parameters of quantity Z_U may easily be derived. Thus, the following expressions are obtained for the mean

$$\bar{Z}_U = k_0 \sum_{i=1}^n k_i \bar{M}_{Ui} \dots \dots \dots (3)$$