

TORSIONAL PHENOMENA ANALYSIS AND CONCRETE STRUCTURE DESIGN

By K. G. TAMBERG and P. T. MIKLUCHIN

The analysis of structures with respect to torsion is discussed by considering the interrelationships between applied loads, types of structure, generated forces and moments, and resulting stresses. It is emphasized that torsional analysis requires that a distinction be made between solid and thin-walled members and that it is an integral part of the overall structural analysis process. The state of the art of analysing structures in which torsion is generated is established by referring to existing literature, pertaining to theoretical and experimental methods of analysis, important theoretical formulations, and practical analysis results.

The analysis of concrete structures using methods based upon elasticity is discussed. The importance of employing model and field testing in the analysis of structures, and the augmenting aspects of theoretical and experimental methods are elaborated upon. The need for further research is stressed in order to clarify the behavior of concrete structures subjected to torsion and to furnish data for the analysis and design of such structures.

It is proposed that clauses pertaining to the problem of torsion be incorporated into existing engineering codes of practice, making distinctions between the behavior of solid and thin-walled type constructions as appropriate.

Keywords: cracking (fracturing); creep properties; deformation; *elastic analysis*; flexural strength; loads (forces); models; prestressed concrete; reinforced concrete; shells (structural forms); shrinkage; structural analysis; *structural design*; structural forms; *torsion*.

□ Employing concrete, the engineer is now able to design structures of virtually any desired shape to meet requirements imposed by a wide range of conditions of use, applied loads, and aesthetics. It is now possible to create forms of great structural efficiency and architectural beauty. The conception and utilization of structures which more fully take advantage of the ability of reinforced and prestressed concrete to resist complex stresses requires an explicit consideration of torsional effects. In certain types of structures torsion is of paramount importance.

The approach adopted in this presentation is to identify how the occurrence of torsion relates to the overall process of design, of which the process of analysis is an integral part. It is indicated that both theoretical and experimental methods of analysis are part of the arsenal of methods at the disposal of the engineer.

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Considering theoretical methods, however, it is emphasized that classical elastic analysis still dominates. Present theoretical procedures assume that structures behave elastically, since limit design and stochastic procedures have not yet been developed to sufficient generality that they constitute alternatives. Therefore, only methods of analysis based on the elastic behavior of structures need consideration.

Structures are grouped according to their torsional behavior, revealing that a large variety of structures develop torsional stresses. Torsional phenomena must be analysed in the context of the general problems of static and dynamic equilibrium and stability. Methods which permit the elastic analysis of structures over the full range of torsional behavior, embracing warping, Saint-Venant and mixed forms of torsion must be used as appropriate, and can be modified in many cases to consider effects of cracking and creep. Since model analysis and field testing complement theoretical methods in the analysis of structures, they are briefly discussed. Practical examples are presented to illustrate effects of torsion.

Although further research must be undertaken to clarify the torsional behavior of both reinforced and prestressed concrete structures, the information presented in this paper together with the material in the references quoted ought to permit the reader to acquire an insight into the principles of a rigorous approach when analysing structures subjected to torsion. No timidity need be

shown by the engineer in the torsion-oriented analysis of structures, in view of the analytical and experimental methods of analysis available.

THE PROCESS OF DESIGN AND TORSION

Design of structures is a creative process. It is an art and science of searching for an ideal structural form in response to an expressed need or desire, and in harmony with social goals and individual objectives, applied loads, materials, framings, and methods of construction. Design thus entails a sequence of decisions and actions which lead to structures which can be built at some point in time.

This paper is concerned principally with torsion as it may have to be considered as part of the design process. Whether a structural unit* is subjected to torsional action depends upon types of applied loads, materials, framings and methods of construction. The severity of that torsional action depends on the action of the loads in relation to the geometry, physical properties, and boundary conditions of the structural unit, as well as the modes of connection and structural behavior of adjacent units connected to it. The deformations and stresses resulting from that torsion must be considered in conjunction with those generated by axial load, shear, and flexure in order to establish resultant stress effects which the unit must be capable of resisting.

In order to design an adequately performing structure, then, the structural engineer must understand how loads act on a structural unit and how the unit behaves under the specified loading. With such understanding, he can use his ingenuity and knowledge of structural framing systems to reduce or avoid undesirable load effects. For example, he may conceive systems to minimize torsion, if he is uncertain of the generated structural response when a unit is subjected to torsion. On the other hand, it has been found that the mobilization of the torsional strength of structures can effect significant savings in total costs. Excellent examples of this effect are provided by modern bridge, viaduct, and shell type structures, and as a consequence the structural engineer must now consider explicit effects of torsion in the process of structural design and analysis. The aim of this paper is to gather together the best of the available information and to place it in perspective. In the past, many engineers have shown great reluctance in attempting to deal with problems of torsion because of conservativeness of methods of design used and the scarcity of information available.

The process of designing structures is however subject to continuous development, and the ultimate proof of the adequacy of a design is its performance measured in terms of the total cost incurred to the owner and society. Progress made in design methodology takes place through the continual interaction between the various phases of design, construction, performance and research as depicted in Fig. 1-1. The process of design, its component parts and their

*The term structural unit is used in this paper to designate any component part of a structure, or the entire structure, depending on which is under consideration.

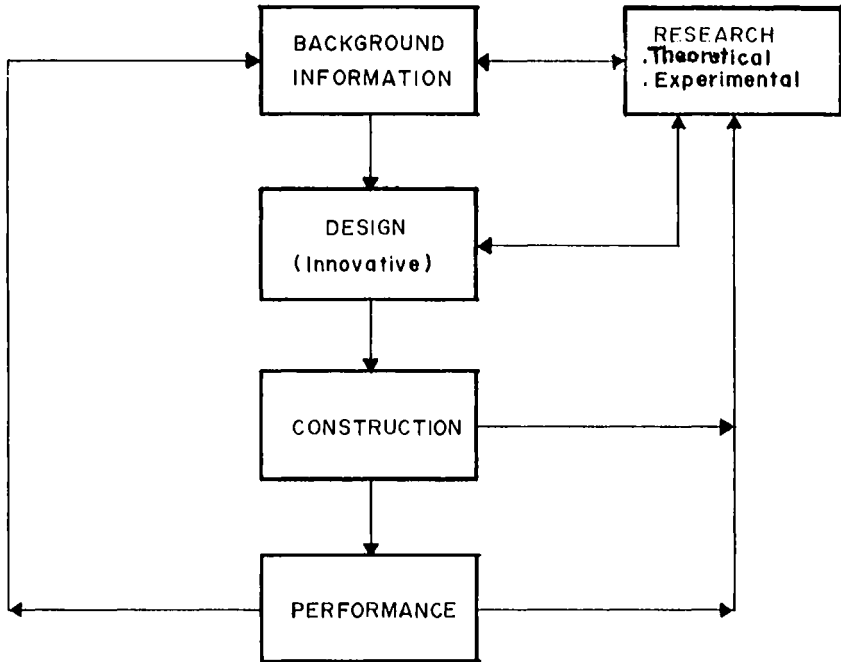


Fig. 1-1 – Development of structural design

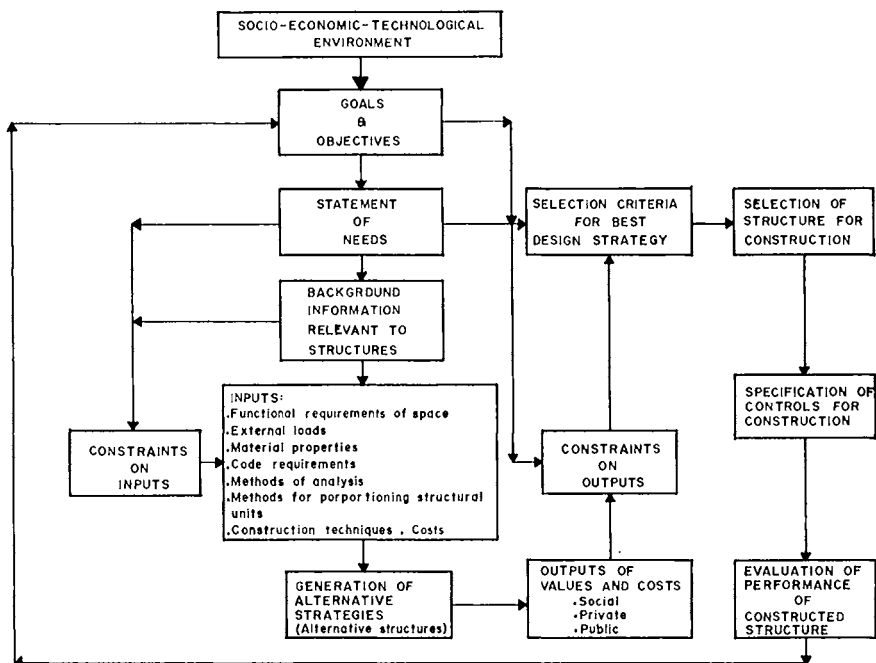


Fig. 1-2--Process of design

interrelationships, can be described as shown in Fig. 1-2. Although many variables influence design, it should be understood that the essential contribution of the structural engineer to the conception and realization of satisfactory structures of functional and aesthetic form is to minimize costs of design, construction, and maintenance while maximizing benefits. An integral part of the process of design therefore is structural analysis which can be described as shown in Fig. 1-3. It is indicated there that the designer may resort to either analytical or experimental methods of analysis, or a combination of both.

The logistics of analysis and proportioning of concrete structures is outlined in Fig. 1-4. As shown there, the methods of analysis based on limit strength and on stochastic principles have not reached the same state of development as the classical elasticity-based methods. The engineer, therefore, is presently restricted to elastic analysis and, for certain structures, to ultimate strength design.

The 1960's were a period of intensive research on torsion. A large number of publications appeared, covering primarily the areas of elastic analysis and design. ACI Committee 438—Torsion has encouraged the undertaking of research, has scrutinized and digested available information on the proportioning of concrete subjected to torsion, and has published tentative recommendations¹ for the ultimate strength design of reinforced concrete members to resist torsion. With this latter document as a reference, ACI Committee 318 has incorporated provisions for combined torsion and shear for straight non-prestressed members, in the ACI 318-71 building code requirements for reinforced concrete.² T, L, I, and closed compact single-cell, rectangular, hollow box-sections are considered. No provisions have yet been furnished to cover non-rectangular or multi-cellular box sections or other open type cross-sections, apart from the fact that prestressed concrete construction is not covered. The information with respect to effects generated by warping, axial forces, creep, and the dynamic application of loads is not complete. Data relating to torsional rigidities which should be used to determine torsional moments in structures is scant. It is expected that future research will fill in missing information and that ultimate strength criteria will become available to cover a wider range of structures. It is noteworthy that imaginative design engineers, using elastic methods of analysis, meanwhile have succeeded in the design of bridge, viaduct, and shell type structures which are subjected to torsion,³ notwithstanding these gaps in information. It is worth mentioning that theoretical elastic methods can be extended to cover creep, if creep is viewed as a delayed elastic response to stress.⁴

It should be noted also that the applicability of theoretical methods of elastic analysis can be enhanced by resorting to experimental model analysis⁵ and field testing.⁶ Specific structures can be tested and test information can be incorporated to modify analytical procedures. For example, the behavior of concrete structures is significantly affected by the reinforcing provided. If cracking occurs, a significant reduction in both the flexural and torsional rigidities takes place which redistributes internal forces; this may influence the proportioning of reinforcement in structures such as spandrel beams^{7,8} and grids.⁹ Consequently, reduced flexural and torsional rigidities, determined by experimental methods, can be used in the elastic analysis procedure. On the other hand, model analysis can be used exclusively to design specific structures.⁹

STRUCTURES SUBJECTED TO TORSION

To put then the problem of torsion into further perspective, the interrelationships between applied loads, types of structure, generated internal forces, deformations, and dynamic behavior are considered next in qualitative terms. Fig. 1-5 is intended to show that structures which are subjected to loads and contain varieties of materials, employing various framing systems and methods of construction, can be categorized into three main types: A, B, and C, as is shown in column 4. Of these, types A and B are of special interest. That is, type A embraces structures which may be considered to consist of single structural units resisting torsion; for example, a beam or shell unit. In turn, type B embraces structures consisting of component parts resisting torsion, such as, for example, a space frame or grid-type structure. It is indicated that axial, shear, flexural, torsional stresses and deformations, and their various combinations must be considered, as appropriate, in order to allow prediction of the behavior of structures in their stressed state. It is clear that the degree of importance of torsion for concrete structures may vary widely, but it must be correctly estimated to permit rational design.

The problem of structural analysis involves the establishment of behavioral characteristics and associated structural constants for individual structural units of various types, support and boundary conditions, and the analysis of complex structures in the light of individual unit behavior.

Fig. 1-6 shows basic forms of structures of type A. It is possible to conceive of a large number of different configurations which consist basically of singular structural units having differing alignments, support and boundary conditions, and cross-sections. Such structures occur in the construction of bridges, viaducts, and buildings. Fig. 1-7 shows some structures of type B, each of which may be conceived as containing individual structural units.

Referring to types A and B it is now possible to identify another type categorization which distinguishes between structures with open thin-walled cross-sections and those with solid, compact, bulky-type cross-sections (Fig. 1-8). Depending upon the types of cross-sections encountered, lengths of span, modes of support, and boundary conditions, analysis may range from elastic warping torsion theory to the classical Saint-Venant torsion theory. For a large number of frequently occurring structures, modified versions of these theories for mixed torsion, accounting for both warping and Saint-Venant torsion effects, must be used if an analytical analysis is to yield reliable estimates of torsional stiffness, deformations, normal and shear stresses, and deflections. Analysis of thin-walled sections, either open or closed, or of open-closed types, must also take account of the effects generated by changes in the geometry (distortion) of cross-sections. In addition, effects due to cracking, temperature, shrinkage, and creep must be determined, if significant. By the proper incorporation of warping and Saint-Venant torsion effects into the elastic analysis procedure, the practical analysis of many structures becomes possible.

The relative importance of the warping and Saint-Venant torsion effects is initially demonstrated by referring to three cases of simply supported beams,

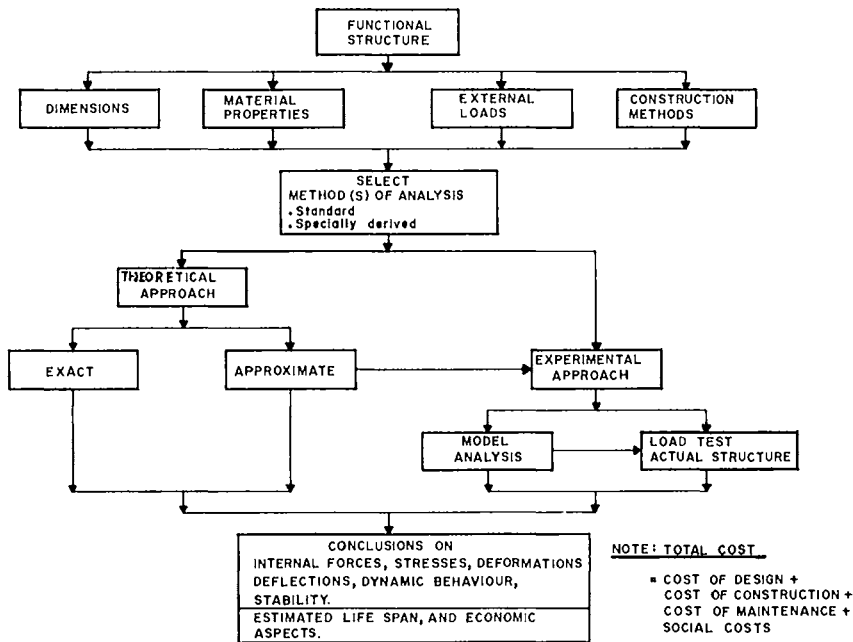


Fig. 1-3—Process of analysis for structural design

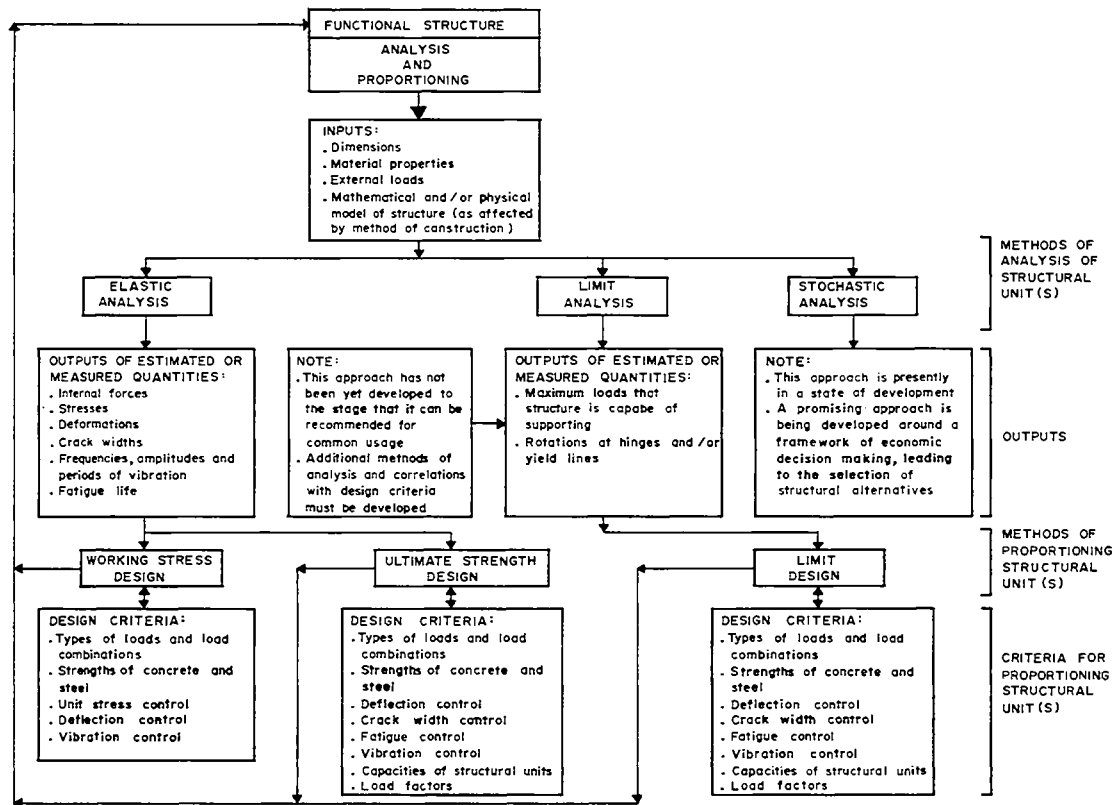


Fig. 1-4—Logistics of analysis and proportioning of concrete structures

STRUCTURES SUBJECT TO GENERALIZED TORQUE						
LOADS WHICH DETERMINE GENERALIZED FORCE FIELDS	STRUCTURE		CATEGORIZATION OF STRUCTURES FOR PURPOSES OF EVALUATING TORSIONAL EFFECTS	ESTIMATE QUANTITIES FOR DESIGN OF STRUCTURES DUE TO ACTION OF GENERALIZED FORCE FIELDS		
	MATERIAL	METHOD OF CONSTRUCTION		INTERNAL FORCES	DEFORMATIONS	VIBRATIONS
D - dead load SD - superposed dead load L - live load LF - longitudinal force from L CF - centrifugal force W - wind load on structure WL - wind load on L EQ - earthquake SF - stream flow pressure ICE - ice pressure E - earth pressure B - buoyancy S - shrinkage C - creep T - temperature F - longitudinal force from friction R - rib shortening BL - blast load	Plain concrete Reinforced concrete Prestressed concrete (pre and post-tensioned) Ferro concrete	Cast in situ: generally in stages (monolithic) Precast: structural units are erected and connected either mechanically or chemically Cantilever technique Segmental technique: polylithic structures Lift slab technique Tilt-up technique	A Force field applied to structural unit ↓ Torsion B Force field applied to structural unit (consisting of an assembly of components) ↓ Torsion in component units C Force field applied to structural unit (consisting of an assembly of components) ↓ No Torsion in component units	Axial Shear Flexure } and comb. Torsion + Axial Torsion + Flexure Torsion + Axial + Flexure Torsion + Flexure + Shear Torsion + Axial + Flexure + Shear	Axial Shear Flexure Torsion } and comb. NOTE: CHECK STABILITY OF STRUCTURE AS REQUIRED	Axial Shear Flexure Torsion } and comb.
1	2	3	4	5	6	7

Fig. 1-5 – General synthesis of relevant relationships to be considered in evaluating torsional effects

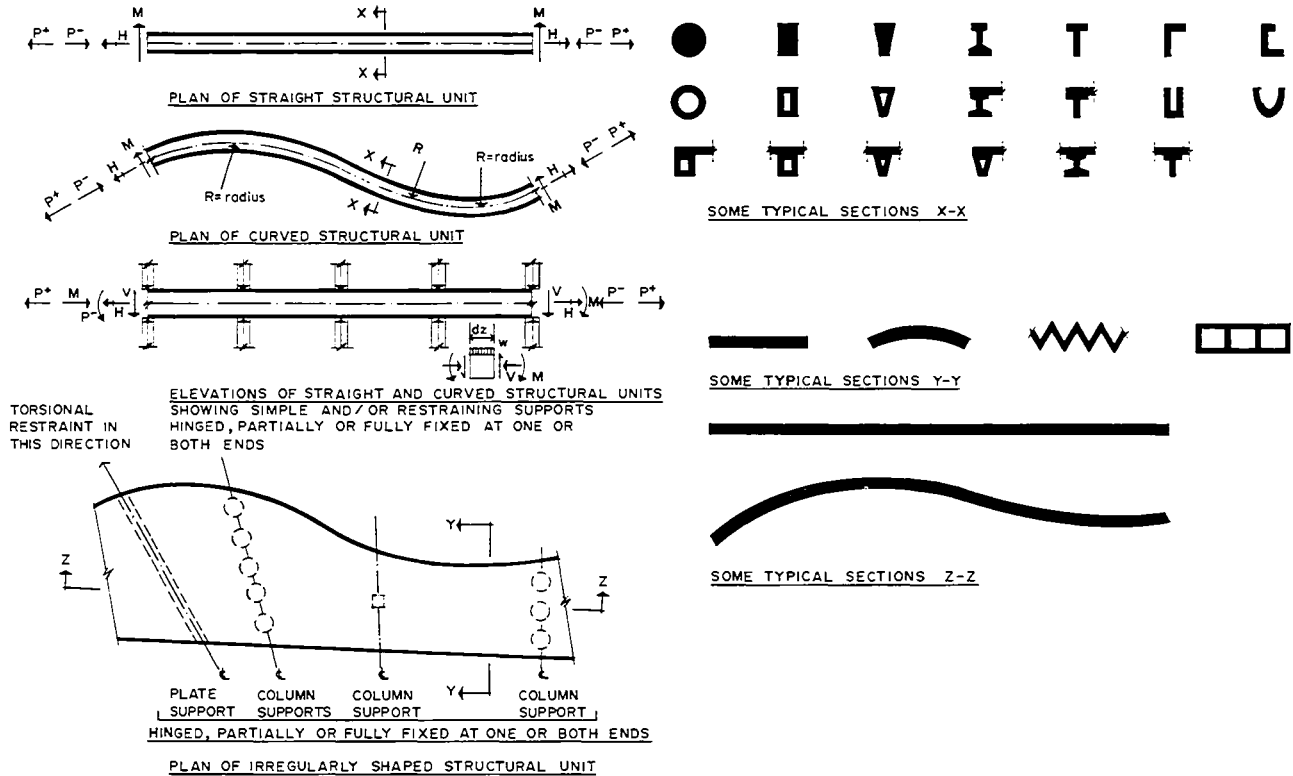


Fig. 1-6 – Varieties of structural units in which torsional forces are generated

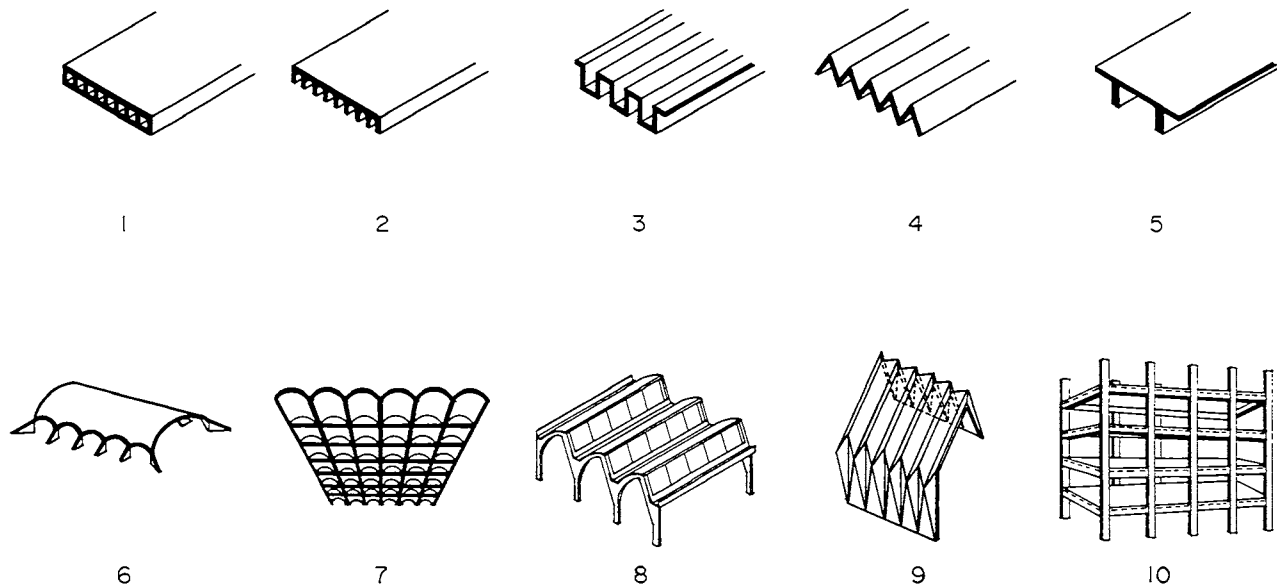


Fig. 1-7 – Varieties of structural units with torsion acting in component parts

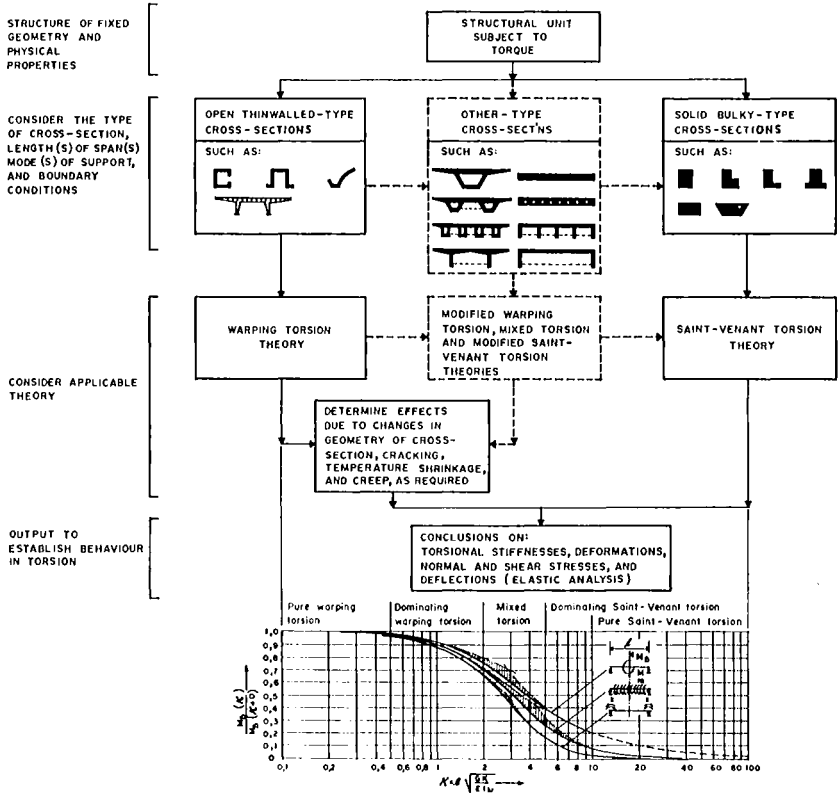


Fig. 1-8—Determination of torsional properties of a structural unit

with rigid cross-sections, identified at the bottom of Fig. 1-8. It is shown that the magnitudes of the warping moment M_B at center span are directly dependent on the parameter κ which is equal to the span length l times the square root of the Saint-Venant torsional rigidity GK divided by the warping torsional rigidity El_w . If the magnitude of κ is small, warping torsion dominates, whereas for large values of κ the Saint-Venant torsion provides the governing criterion. The division of the κ -range into identified regions of dominating torsion is somewhat arbitrary but serves to illustrate the concepts involved in dealing with the full range of torsional behavior. The methods of calculation used to derive the three graphs are discussed under "Theoretical Methods of Analysis."

To further direct the thinking of the reader when attempts are made to evaluate torsional effects, Fig. 1-9 is used to describe changes in structural action when the span length is changed. Warping tends to be of greatest importance for the cylindrical shell shown when beam action prevails, and tends to become less significant as the length of the structure is reduced. Such effects are negligible in the case of short shells, and for small magnitudes of L_2 arch action prevails. Analogous or similar arguments may be developed for other types of structures in order to investigate the relative importance of warping and Saint-Venant torsion effects or whether torsion is altogether negligible.

THEORETICAL METHODS OF ELASTIC ANALYSIS

To discuss analytical methods of analysis which explicitly consider and permit the evaluation of torsional effects, fundamental aspects of structural analysis are first examined. Following this, the torsion of structural units with solid, open, closed, and open-closed type cross-sections is considered, and the general analysis of structures is discussed. The position and importance of torsion in structural analysis is highlighted by the identification of methods of analysis which consider torsional action, the presentation of some key equations, and the provision of numerical results. The reader may complement the presented information by referring to the references listed at the end of this paper to gain a thorough understanding of specific methods of analysis.

FUNDAMENTAL ASPECTS OF STRUCTURAL ANALYSIS

The fundamental problem of elastic structural analysis entails the determination of deformations and stresses in a body subjected to a set of external forces which are in a state of equilibrium. All methods used in the analysis of elastic structures derive from the examination of relationships between applied loads and generated deformations and stresses. The basic conditions which must be satisfied are those due to equilibrium, geometry, physical characteristics, compatibility of deformations, and boundary and support conditions. The types of strategies adopted to satisfy these conditions determine the analytical formulations which can be used, the techniques which can be employed to analyse a structure or to solve particular equations or systems of equations.

Considering an elementary rectangular parallelepiped with its sides parallel to the planes defined by the x-y-z axes of a cartesian co-ordinate system, it can be easily ascertained that 15 unknowns must be determined in order to obtain magnitudes for normal strains, shear strains, and stresses¹⁰ in any given structural unit, namely:

u, v, w - the three components of displacement in the x, y and z directions respectively,

$\epsilon_x, \epsilon_y, \epsilon_z$ - unit strains in the x, y and z directions,

$\sigma_x, \sigma_y, \sigma_z$ - normal stresses in the x, y and z directions,

$\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ - unit shearing strains in the xy, yz and zx planes respectively,

$\tau_{xy}, \tau_{yz}, \tau_{zx}$ - shear stresses in the xy, yz and zx planes,

These unknowns can be determined by simultaneous solution of equations defining equilibrium, equations defining the geometry of deformations, and equations defining the relationships between stresses and strains. The equations utilize material constants and satisfy requirements of compatibility and boundary conditions. Generally speaking, there are three basic methods of formulating of the fundamental equations for analysis.

The Deformation Method takes the displacement of points within a structural unit to be the principal unknowns.

Three unknown displacement functions:

$$u = f_1(x, y, z), \quad v = f_2(x, y, z), \quad w = f_3(x, y, z) \quad (1)$$

after suitable mathematical transformations, reduce to the following system of three principal equations:

$$\begin{aligned} \psi_1(u, v, w) &= 0 \\ \psi_2(u, v, w) &= 0 \\ \psi_3(u, v, w) &= 0 \end{aligned} \quad (2)$$

Solution of these equations determines u-v-w, from which corresponding unit strains and stresses can be derived.

The Force Method considers forces or stresses as unknowns. There are six unknown stress functions.

$$\begin{aligned} \sigma_x &= \phi_1(x, y, z), \quad \sigma_y = \phi_2(x, y, z), \quad \sigma_z = \phi_3(x, y, z) \\ \tau_{xy} &= \phi_4(x, y, z), \quad \tau_{yz} = \phi_5(x, y, z), \quad \tau_{zx} = \phi_6(x, y, z) \end{aligned} \quad (3)$$

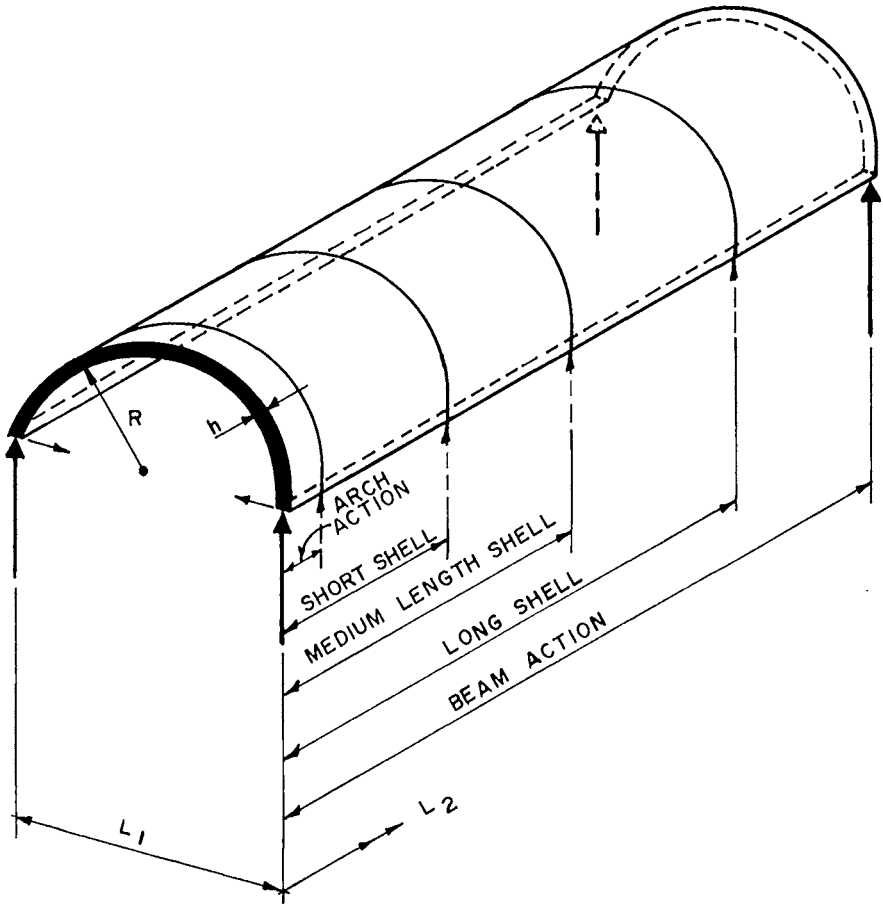


Fig. 1-9—Changes in structural action as function of span length

STRUCTURE OF FIXED
GEOMETRY, PHYSICAL
PROPERTIES, SUPPORT
AND BOUNDARY
CONDITIONS .

CONSIDER AVAILABLE
STRATEGIES FOR
THE CRITICAL EVALUATION
OF STRUCTURE TO MEET
SPECIFIED PERFORMANCE
CRITERIA

CONSIDER AVAILABLE
TECHNIQUES FOR
ANALYSIS AND THE
SOLUTION OF EQUATIONS
USING MANUAL AND/OR
COMPUTER OPERATIONS

OUTPUT AND SYNTHESIS
OF CALCULATED QUANTITIES

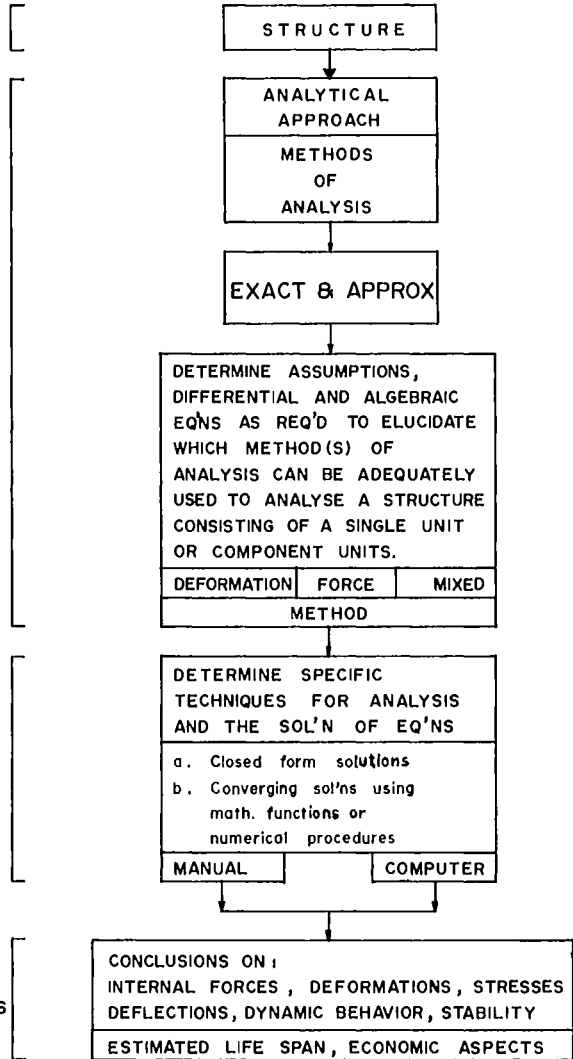


Fig. 1-10—Methodology for analysis-theoretical approach

which can be transformed to obtain six principal equations:

$$\begin{aligned}
 F_1 (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}) &= 0 \\
 F_2 (\sigma_x \text{-----} \tau_{zx}) &= 0 \\
 F_3 (\sigma_x \text{-----} \tau_{zx}) &= 0 \\
 F_4 (\sigma_x \text{-----} \tau_{zx}) &= 0 \\
 F_5 (\sigma_x \text{-----} \tau_{zx}) &= 0 \\
 F_6 (\sigma_x \text{-----} \tau_{zx}) &= 0
 \end{aligned} \tag{4}$$

Solution of these equations determines the stresses acting in a structure.

The Mixed Method takes as unknowns some displacements and some stresses, in whatever combination is required to derive principal equations which lend themselves to efficient solution.

These three approaches provide the basis for the solution of both static and dynamic problems of structural analysis. The entire field of structural analysis originates from manipulations along the lines described above. However, the important thing to note in connection with torsion is that, in order to understand the general problem, it is mandatory to understand how torsion contributes to changes in the magnitudes of the displacements u , v , and w and the corresponding stresses σ and τ . With that relationship, it is possible to develop methods of structural analysis which explicitly incorporate torsional effects as special terms in Eq. (4). These methods are parallel to those with torsional effects excluded and are not significantly more complicated. Fig. 1-10 broadly outlines the methodology for structural analysis.

In dealing with specific structures, the engineer must decide which one of the three basic approaches he will use to derive applicable equations and techniques of analysis. A comprehensive discussion of the relative merits of various strategies and techniques of analysis is too extensive to be included in this paper. Only those particulars directly relating to torsion are considered. It may be mentioned, however, that the use of electronic computers becomes practically necessary in the analysis of complex structures¹¹ in which even the choice of strategy and technique of analysis may be governed by the characteristics of digital computation. In view of the lengthy arithmetical work involved and the increasing availability of computer programs, long-hand methods of calculation can be considered obsolescent, except for checking purposes. For example, the design of highway bridges in France has been computer-automated, enabling large savings to be realized in the time and the cost of design.¹²

SAINT-VENANT TORSION AND STRUCTURAL CONCRETE-SOLID SECTIONS

For elastic, homogeneous structural units of uniform cross-sections subjected to pure torsion, the Saint-Venant torsion theory^{13,14} allows the calculation of

torsional stiffnesses, shear stresses, and resulting twists. The theory assumes that cross-sections are free to warp and that all cross-sections along the span warp in the same manner. The applicable differential equation, using the cartesian x-y co-ordinate system, is:

$$\frac{\partial^2 \phi_{(x,y)}}{\partial x^2} + \frac{\partial^2 \phi_{(x,y)}}{\partial y^2} = -2 G \theta' \quad (5)$$

where ϕ = Saint-Venant stress function,

G = shear modulus of elasticity,

θ' = angular twist per unit length of structural unit.

The slope of the ϕ -surface in any given direction is proportional to the shear stress in the perpendicular direction, and the torque carried at a specific cross-section is equal to twice the volume under the ϕ -surface. At the cross-section boundary, ϕ must equal zero. Thus, it is possible to determine the shear stresses τ acting in a cross-section perpendicular to the longitudinal axis z of a structural unit, the torsional stiffness K , and the generated torsional moment M_K , when θ' and G are known.

$$\tau_y = \frac{\partial \phi_{(x,y)}}{\partial x} \quad (6)$$

$$\tau_x = \frac{\partial \phi_{(x,y)}}{\partial y}$$

$$\begin{aligned} M_K &= 2 \iint \phi_{(x,y)} dx dy \\ &= 2 \int_A \phi_{(x,y)} dA \\ &= KG \theta' \end{aligned} \quad (7)$$

where τ_y = shear stress in y-direction,

τ_x = shear stress in x-direction,

$A = \int dx dy$ = area of cross-section,

M_K = torque

K = torsional stiffness

KG = torsional rigidity.

Eq. 5-7 were solved exactly by Saint-Venant by the application of various mathematical techniques, often involving infinite series, for various cross-section shapes, such as rectangular, triangular, semicircular, and elliptic. Many irregularly

shaped cross-sections however defy exact mathematical analysis. In 1903 Prandtl developed his famous membrane analogy to facilitate the calculation of torsional constants.^{14,15} It is based on the mathematical identity between Eq. (5) and the differential equation which relates the deflected surface of a membrane to the inflating pressure and membrane tension. This technique provides an extremely simple and clear manner of giving visual form to the ϕ -surface, and when it is possible and convenient to describe the surface by means of mathematical functions, torsional constants can be calculated. Subsequently, numerical techniques of analysis have been employed to solve Eq. (5) for complex sections. Fig. 1-11 illustrates the ϕ -surfaces pertaining to solid I-beams of uniform cross-sections, with and without a slab attached, for which torsional constants can be calculated using finite differences.¹⁶

If the cross-sectional dimensions of a structural unit vary along its span, the K -values may be approximated by those for a substitute unit having discrete parts of constant cross-section as shown in Fig. 1-12. Equation (7) then can be modified to yield:

$$M_T = \frac{\ell}{\sum_{i=1}^n \left(\frac{\Delta \ell}{K} \right)_i} G \theta' \quad (8)$$

and approximate shear stresses can be calculated from the slopes of ϕ -surfaces corresponding to the $K_1 \dots K_n$ -values. Exact mathematical solutions are, of course, available for certain cases,¹⁰ but, for other cases, the division of a structural unit into discrete parts and the use of Eq. (8) provides the only approach to calculation. Some information can be found for structural units of variable rectangular cross-sections, as well as graphs for easy determination of K -values.¹⁷

If it is assumed that the principle of superposition applies (when effects due to axial forces, shear and bending are taken into consideration) and that the restraining of warping (due to loading, boundary and support conditions) produces negligible effects, the Saint-Venant torsion theory can be used to consider the torsional action of solid units.

Now the total strain energy of an elemental structural unit of span $\ell = z$ is:

$$U = \int \frac{P^2 dz}{2AE} + \int \frac{M_x^2 dz}{2EI_x} + \int \frac{M_y^2 dz}{2EI_y} + \int \frac{M_K^2 dz}{2KG} + \int C_1 \frac{V_x^2 dz}{2AG} + \int C_2 \frac{V_y^2 dz}{2AG} \quad (9)$$

where

P = axial force,

E = modulus of elasticity,

M_x, M_y = moments about x and y axes respectively,

I_x, I_y = moments of inertia about x and y axes respectively,

C_1, C_2 = coefficients,

V_x, V_y = shear forces in x and y directions respectively,
 A, M_K, K, G = as defined previously.

It follows, then that the deformation, force or mixed method of formulation can be used to derive equations for the analysis of structures subjected to torsion, and that various techniques of analysis can be used to determine unknowns. A structural unit subsequently can be designed by proportioning it to resist, at every location, maximum longitudinal stresses σ and shear stresses τ . Longitudinal stresses are due to P and M , and shear stresses are due to M_K and V ; the maximum values are determined from vectorial summations. For an overall elastic analysis of concrete structures, it is necessary to determine applicable values of AE , EI , and KG .

While valid KG -values can be determined for perfectly elastic structures, considerable difficulties are experienced in the determination of these values for concrete structures. In the past, many engineers have indiscriminantly used Saint-Venant KG -values but it should be realized that this approach is certainly not correct for all structures. Apart from the fact that the stress-strain curves of concrete are not linear, magnitudes of KG are affected by cracking, creep, and rise times in load application. The estimation of valid KG -values for structural concrete is difficult because of its heterogeneous and viscoelastic nature.

In the event that cracking occurs at low loads and significant reductions in KG take place, it may not be logical to adopt an ultimate strength approach to torsion design as has been proposed.^{1,2} Instead, it may only make sense to provide torsional reinforcement to control cracks caused by twist.

If magnitudes of KG -values are overestimated, the magnitudes of actual torsional moments M_K occurring in a structure are overestimated and corresponding angular twists θ are underestimated, with the result that distress may take place. Or certain load effects in adjacent structural components may be underestimated. Alternatively, the underestimation of KG -values may result in too low M_K -values and in overestimation of some load effects in adjacent components.

For prestressed concrete construction using structural units with solid cross-sections, Mukherjee and Warwaruk¹⁸ provide useful information on the behavior of such units subjected to combined loadings. They tested (6 x 12 in.) rectangular beams and investigated their behavior. It was established that, for the loading sequence used in the study, the initial torsional stiffness is not significantly affected by the level of prestress, torque-moment ratio, or flexural shear. The point of departure and rate of departure from the initial torsional stiffness is dependent on the torque-moment ratio. It appears, from examination of the torque versus angle-of-twist curves contained in the report, that the Saint-Venant torsion theory can be used with confidence in the analysis of structures, provided that the occurrence of cracking is prevented; in the design of prestressed concrete structures subjected to torsion, it is common practice not to permit cracking at working load levels. It was also found, among other things, that the torsional rigidity of the test beams was greatly reduced in the post-cracking range, but the reduction occurred more gradually and smoothly when

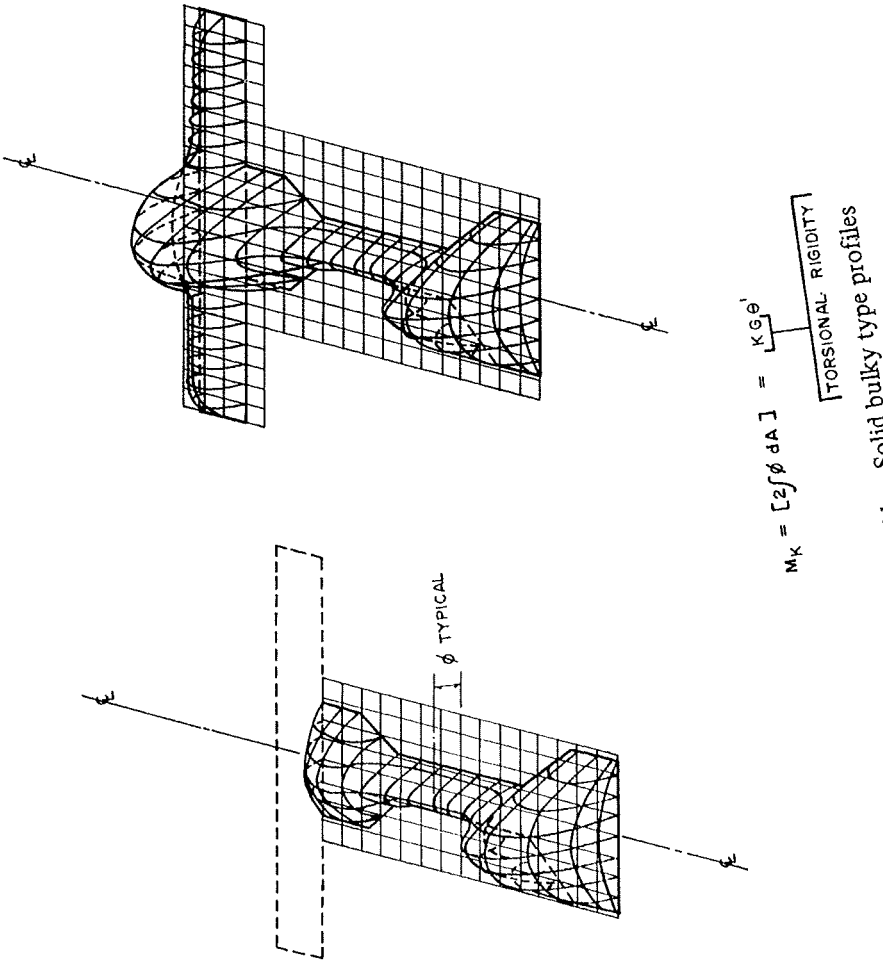
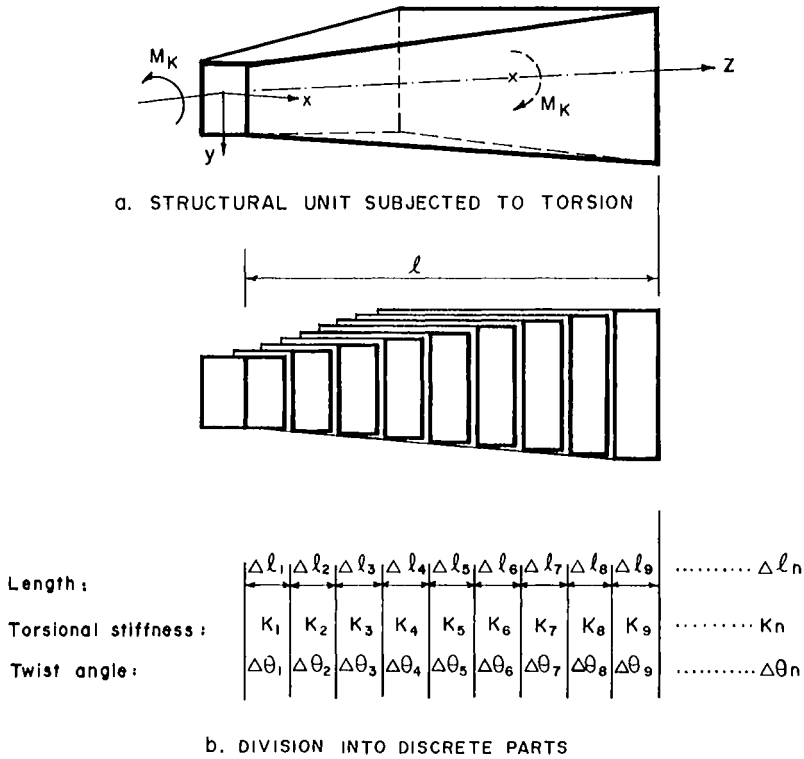


Fig. 1-11 -- Solid bulky type profiles



$$\theta_l = \theta_l = \frac{M_K l}{G K}$$

$$\theta_l = \Delta \theta_1 + \Delta \theta_2 + \dots + \Delta \theta_n = \frac{M_K}{G} \left(\frac{\Delta l}{K_1} + \dots + \frac{\Delta l_n}{K_n} \right)$$

$$K_E = \frac{l}{\left(\frac{\Delta l_1}{K_1} + \frac{\Delta l_2}{K_2} + \dots + \frac{\Delta l_n}{K_n} \right)}$$

Fig. 1-12—Equivalent torsional stiffness K_E of structural unit with varying cross-sectional dimensions and moments of inertia along span

the torque-moment ration was low. Pending the publication of additional test results, it will be possible to furnish definite recommendations with regard to KG -values which can be used for various types of cross-sections whose dimensions are either constant or variable along the span. Effects exerted by creep and the dynamic loading of structures await further clarification.

It seems appropriate to mention also at this juncture that various approximate techniques have been developed to calculate elastic torsional constants and that the engineer must satisfy himself that he understands the implications introduced by the approximations. Tamberg¹⁶ has evaluated magnitudes of ϕ , K , and τ for prestressed concrete AASHTO girders, with and without attached concrete slabs. He used finite differences to determine ϕ -values, and, by evaluating the volume contained under the ϕ -surface, K was established. Newton's interpolation function was used to calculate torsional shear stresses. More recently, Jackson¹⁹ developed an even simpler technique for calculating the constant K by the utilization of Saint-Venant's results for rectangular sections. Rowe²⁰ has proposed the use of Bach's approximation. Table 1-1 compares the results obtained for the K -values of four AASHTO girders using the three different approaches. It is seen that the results obtained by Tamberg and Jackson are in close agreement whereas the computed values in accordance with Rowe are much smaller in magnitude.

Bach's approximation entails the division of a cross-section into component rectangles, the separate evaluation of the torsional stiffness of each rectangle, and the summation of these stiffnesses to determine the torsional stiffness of the whole section. This is the reason for the presence of the smaller magnitudes in Table 1-1; the continuity effect between component rectangles is neglected and the volume under the ϕ -surface is underestimated as can be readily detected from an examination of Fig. 1-11. K -values may be underestimated by approximately 50 percent for many sections occurring in practice (Table 1-1). The errors introduced are large, and, frequently, this approach does not furnish realistic values for the torsional stiffness K . It is recommended that, in general, more accurate methods^{16,19} be used to calculate K -values. Bach's approach is only practicable when errors introduced by neglecting continuity effects are inconsequential, as for a cross-section consisting of thin-walled components.

For reinforced concrete construction, in contrast with prestressed concrete, the determination of KG -values presents some difficulties, some of which have been alluded to already. Under conditions of static short term loading, prior to the occurrence of cracking, it would be reasonable to use Saint-Venant KG -values but because cracking takes place at low loads, only a fractional magnitude of the elastic torsional rigidity KG can be developed. Osburn et al.⁴ have tested rectangular and L-shaped reinforced concrete beams designed by ultimate strength principles,²¹ in combined torsion, shear, and bending, and discovered that the torsional rigidity at service load levels is approximately one-half the torsional rigidity of the uncracked section. It has been noted also that the presence of bending and shear causes cracks to occur at a lower torque than if loaded in pure torsion only, and the post-cracking torsional rigidity of structural units subjected to combined loads is greater than in pure torsion. In the latter case, the post-cracking rigidity may be only about 10 percent or less of the initial uncracked

TABLE 1-1—COMPARISON OF CALCULATED K-VALUES

SECTION	TORSIONAL STIFFNESS K in. ⁴		
	Finite Difference Method (Tamberg)	Approximate Method (Jackson)	Approximate Method (Rowe)
AASHO Girder I	4900	4995	2860
AASHO Girder I with 40 x 7 in. Slab	12300	12350	6920
AASHO Girder IV	34500	35500	21950
AASHO Girder IV with 40 x 7 in. Slab	54100	52100	26010

rigidity.^{5,22} To develop design criteria, torsional rigidity interaction curves involving bending, shear and torsion could possibly be derived.

Lampert²³ recently has developed a method for calculating the post-cracking rigidity of reinforced and prestressed concrete beams subjected to torsion or bending and torsion, arguing that torsional rigidity is independent of shear forces present; the combination of prestressed and nonprestressed reinforcement is covered if the mild steel reinforcement and tendons reach their yield stress at about the same additional strain. On the other hand, Collins and Lampert²⁴ have shown, in the case of certain reinforced concrete structures consisting of intersecting members, that it is possible to analyse and design such structures by assuming that component units have zero torsional rigidity, providing only minimal amounts of torsional reinforcement to control cracks as caused by twist. While supplementary testing is required to furnish comprehensive design information with respect to minimum amounts of reinforcement needed for various structures, the cardinal rule that concrete structures tend to behave the way they are reinforced is exemplified.

If it is deemed feasible and justifiable to neglect torsional rigidity ($KG = 0$) in the elastic analysis procedure, the determination of magnitudes of governing design parameters is somewhat simplified. But, if reliance on torsional action proves unavoidable or economical, applicable KG -values must be determined for use in the overall elastic analysis of structures.

To expound further on the difficulties of determining KG -values, Pandit²⁵ has found by carrying out a few pilot tests that a significant loss in torsional rigidity occurs when solid, plain and reinforced concrete beams are subjected to sustained pure torsion. A plastification of the cross-section takes place. It appears that correction factors should be derived to modify elastic K and short term G -values. Szilard²⁶ has carried out a limited number of pilot tests on solid, plain, and reinforced concrete specimens, subjected to dynamic loading in pure torsion. He established that the test specimens did not behave linearly elastically under rapidly applied torque, and that allowance should be made for the visco-elastic behavior of concrete for short rise times in load application. He recommends the use of complex shear moduli of elasticity.

In the meantime, the engineer is confronted with the task of having to analyse and design structures in the face of imperfect information, with consequent risk and uncertainty. He must therefore use his ingenuity and experience to circumvent difficulties by conceiving structural systems whose behavior he can understand and control. In the absence of specific information, values of KG must be selected in such manner that safe design is achieved. He may choose, when feasible, to use prestressed in lieu of reinforced concrete to control cracking. Information derived from project-oriented test programs may also be relied upon to proportion specific structures. To advance the state of the art with respect to the elastic analysis of structures and the proportioning of elemental component units, it is mandatory to investigate further not only the strength of individual components but their behavior as part of the total structure.

SAINT-VENANT TORSION AND STRUCTURAL CONCRETE—HOLLOW SECTIONS

The uncertainties involved in the calculation of torsional constants of hollow structural units, subjected to static and dynamic conditions of loading, are similar to those for units with solid cross-sections. Additional studies, of course, must be undertaken in order to be able to recommend applicable KG-values for general use, although Lampert's analysis²³ appears to yield adequate values for post-cracking rigidities of compact single-cell structural units of uniform cross-section. Test information of interest may be found in References 27-29. For these sections, as for others previously discussed, estimated KG-values may be viewed as modified Saint-Venant values for purposes of formulating Eq. (9).

If conditions of loading and reinforcement are such that a structure behaves linearly elastically at working loads, and effects due to warping are insignificant, the pure Saint-Venant theory can be used to calculate K- and τ -values. Whenever it is not obvious that the assumptions are justified, the analyst must resort to explicit calculations based on the general theory of thin-walled sections, as discussed later. The assumptions of elasticity and free warping are frequently justified for box units which act as supporting beams in buildings and bridge decks of short spans.³⁰⁻³⁴ Torsional constants may then be calculated for compact sections (Fig. 1-13) by employing finite differences.¹⁶ In the case of rigid thin-walled sections, the approximation that the shear stress $\tau(s)$ is assumed to be constant over the wall thickness $h(s)$ permits the efficient determination of K and τ .

Shown in Fig. 1-14 is a rigid single-cell thin-walled structural unit, the associated ϕ -surface, and the approximation of the ϕ -surface which permits the efficient determination of K and $\tau(s)$ by means of Prandtl's membrane analogy. The shear stress $\tau(s)$ is shown to be constant over the wall thickness $h(s)$ in Fig. 1-14c, and the key thought provided by Prandtl is that:

$$2 G \theta' = \frac{p}{T(s)}, \quad (10)$$

where p = air pressure acting on the membrane,

$T(s)$ = membrane tension.

Equating the upward force exerted by the membrane pressure p and the downward component $T_v(s)$ of the membrane pull integrated along the periphery yields:

$$pA = \oint T(s) \frac{t}{h(s)} ds \quad (11)$$

and, since the torque about any point O is $M_K = \int_s \tau(s) h(s) ds n(s) = \tau(s) h(s) \int_A dA$, then

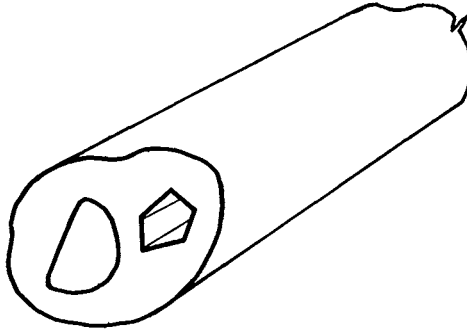


Fig. 1-13—Bulky structural unit containing voids

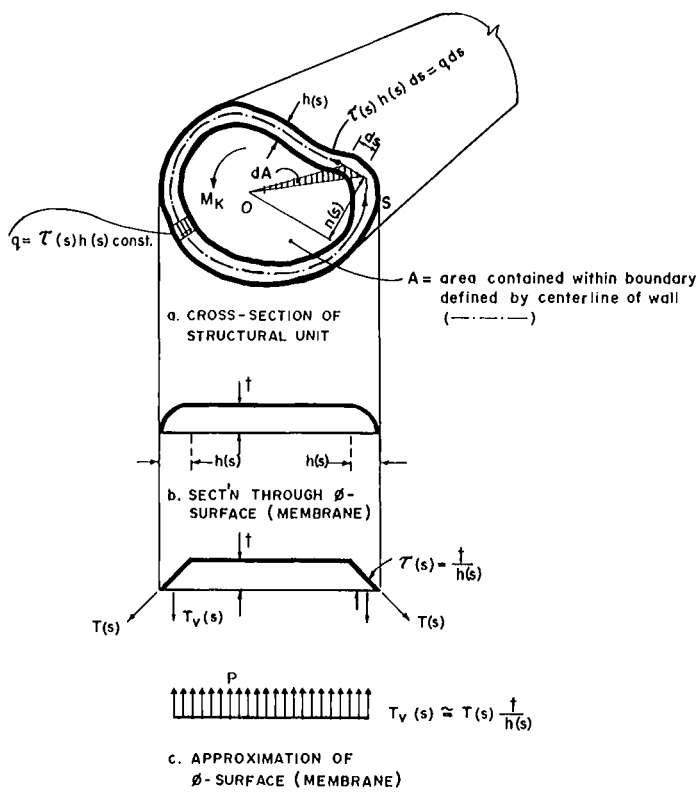


Fig. 1-14—Single-cell thin-walled structural unit

Exact Saint - Venant Theory: $\tau_E = 2M_K \frac{R_1}{\pi (R_1^4 - R_2^4)}$

Bredt's Formula : $\tau_B = M_K \frac{1}{2 \pi R^2 h}$

$$\frac{\tau_B}{\tau_E} = \frac{R_1^4 - R_2^4}{4R_1 R^2 h} = \frac{1 + (R_2/R_1)^2}{1 + (R_2/R_1)}$$

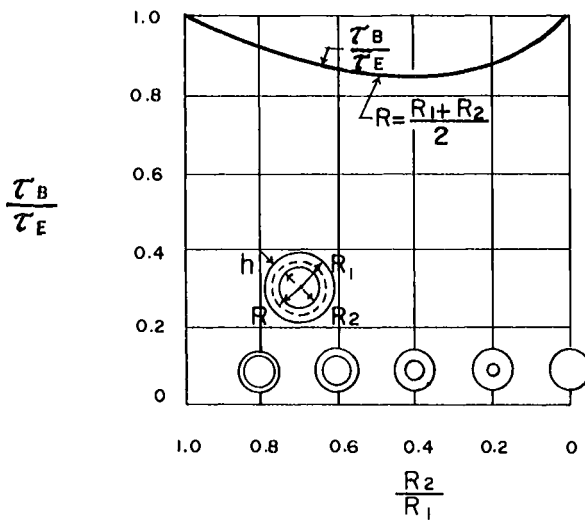


Fig. 1-15—Comparison of τ -values calculated by exact Saint-Venant theory and Bredt's formula for circular sections

$$\tau(s) = \frac{M_K}{2Ah(s)} \quad (12)$$

where $A = \frac{1}{2} \oint n(s) ds$ = area contained within boundary defined by center line of wall,

substitution of Eq. (12) into (11), with $p/T(s) = 2G\theta'$ and $t/h(s) = \tau(s)$, provides:

$$M_K = \frac{4A^2}{\oint \frac{ds}{h(s)}} G\theta', \quad (13)$$

which corresponds to Eq. (7). Consequently,

$$K = \frac{4A^2}{\oint \frac{ds}{h(s)}} \quad (14)$$

This formula has been extensively used in the analysis of structures subjected to torsion and was given for the first time by Bredt.³⁵

An indication of the reliability of results obtained from the use of Bredt's formula can be ascertained by comparing calculated shear stresses, using Eq. (12), to the "exact" values pertaining to circular sections with wall thicknesses $h(s)$. Fig. 1-15 identifies the error which occurs when the wall thickness h is changed. It is seen that the assumption of constant shear stresses acting over the wall thickness introduces small errors. Bredt's formula may be considered to furnish adequate results for rigid hollow sections encountered in concrete structures, when warping effects are negligible.

The arguments pertaining to single-cell cross-sections can be extended to cover rigid multi-cell sections. The fundamental problem entails again the evaluation of shear stresses τ and the torsional stiffness K . As a first step towards the derivation of generalized equations, consider the double-cell unit shown in Fig. 1-16.

The application of a torque M_K causes a rotation of the cross-section per unit length of the structural unit θ' and generates a shear flow as indicated in Fig. 1-16b. To determine torsional constants, the membrane surfaces of cells 1 and 2 must be considered in conjunction. Employing Eq. (10) to (14), with $q = \tau(s)h(s)$, it follows that

$$\begin{aligned}
 \frac{P}{T} A_1 &= 2G\theta' A_1 = \frac{2M_K A_1}{K} = q_1 \int_0^{s_1} \frac{ds}{h(s)} + q_1 \int_0^{s_2} \frac{ds}{h(s)} + q_1 \int_0^{s_3} \frac{ds}{h(s)} \\
 &\quad - q_2 \int_0^{s_3} \frac{ds}{h(s)} + q_1 \int_0^{s_4} \frac{ds}{h(s)} : \frac{2M_K A_1}{K} \\
 &= q_1 \oint_{\text{cell 1}} \frac{ds}{h(s)} - q_2 \int_{\text{cell 2}} \frac{ds}{h(s)} \quad (15)
 \end{aligned}$$

for cell 1, and for cell 2:

$$\frac{2M_K A_2}{K} = q_2 \oint_{\text{cell 2}} \frac{ds}{h(s)} - q_1 \int_{\text{cell 1}} \frac{ds}{h(s)} \quad (16)$$

Eq. (15) and (16) may be solved simultaneously to furnish relative magnitudes of q_1 and q_2 , provided $2M_K/K$ is set equal to 1, a procedure which is necessary since K is not yet known. Designating the relative magnitudes by $\bar{q}_1$, and $\bar{q}_2$, the absolute magnitudes for q are then $q_1 = \bar{q}_1 (2M_K/K)$ and $q_2 = \bar{q}_2 (2M_K/K)$. Since the torque, M_K , equals twice the volume between the membrane and the plane of the cross-section, K , can be calculated from the relationship $M_K = 2(q_1 A_1 + q_2 A_2) = (4M_K/K)(\bar{q}_1 A_1 + \bar{q}_2 A_2)$, which results in $K = 4(\bar{q}_1 A_1 + \bar{q}_2 A_2)$.

Examining the form of Eq. (15) and (16) and remembering how q and K were determined, it is now possible to conclude that to derive torsional constants for cross-sections consisting of n cells, n linear equations must be solved to find the unknown shear flows q . The general format of these equations is:

$$\frac{2M_K A_i}{K} = q_i \oint \frac{ds}{h(s)} - \sum_k q_k \int_{ik} \frac{ds}{h(s)}, \quad (17)$$

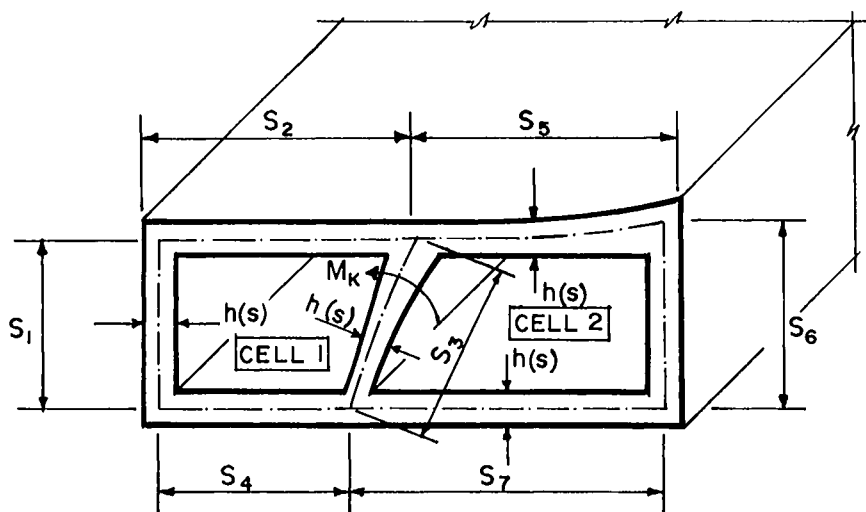
where i - designates cells 1, 2, 3, - - - n of the cross section,

k - designates the adjacent cells which share a common wall with i .

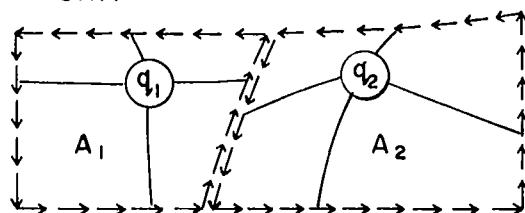
The corresponding shear flows are:

$$q_i = \bar{q}_i \frac{2M_K}{K}, \quad (18)$$

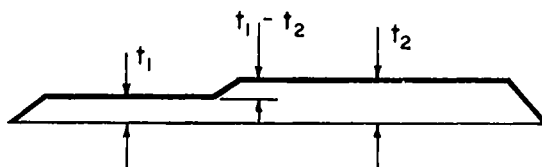
and the torsional stiffness:



a. CROSS-SECTION OF STRUCTURAL UNIT



b. SHEAR FLOWS $q = \tau(s) h(s)$
DUE TO M_K IN CELLS 1 AND 2



c. SECT'N THROUGH Ø-SURFACE
(MEMBRANE)

Fig. I-16 – Double-cell thinwalled structural unit

$$K = 4 \sum_{i=1}^n A_i \bar{q}_i \quad (19)$$

Assuming elastic behavior, Eq. (17) to (19) can be used for the torsional analysis of multi-cellular cross-sections (Fig. 1-6 - 1-8).

When a rigid cross-section contains closed cells which are separated from each other by stretches of slab, as is the case in some bridge decks (Fig. 1-8), the second term on the right hand side of Eq. (17) becomes zero and the contribution of each closed cell to the total torsional stiffness K_T may be calculated separately, using Eq. (14). If all cells and stretches of slab rotate through the same angle θ' , their contributions may be added:

$$K_T = \sum_{i=1}^n K_i + K_{\text{slab}} \quad (20)$$

where K_i = torsional stiffness of cell i

K_{slab} = torsional stiffness of stretches of slab which are not part of the closed cells.

If cross-sections contain concretes of differing consistencies, an allowance for this can be made in the analysis procedure by accounting for the differences in shear moduli of elasticity.¹⁶ In the case of hollow sections, the allowance can be made by modifying wall thicknesses as required. An exhaustive discussion of the calculation of Saint-Venant torsional constants for rigid hollow sections can be found in Reference 36 but effects arising due to the variability in the dimensions of cross-sections along the span of a structural unit are not considered.

Under conditions of combined loading, the shear stresses caused by resultant shear forces acting at a section must be added vectorially to the torsional shear stresses. To evaluate the magnitude of the shear stresses caused by resultant shear forces, the condition prescribing vanishing shear deformations γ around all cells can be used to advantage. That is:

$$\oint_i \gamma \, ds = \oint_i \frac{\tau(s)}{G} \, ds = \oint_i \frac{q}{Gh(s)} \, ds = 0 \quad (21)$$

Assuming that cuts are introduced in the walls of closed cells to transform them into open cross-sections, the resulting shear flows q_0 due to shear forces can be readily calculated by using well known formulas. Introducing unknown shear flows q_i which are required to close the cells and to restore continuity in accordance with Eq. (21), the prevailing shear flows in the walls of cells are:

$$q = q_0 + q_i \quad (22)$$

Assuming that the unknown shear flows q_i follow the same flow pattern as shown in Fig. 1-16. Eq. (21) yields:

$$\frac{q_0}{G} \oint \frac{ds}{h(s)} + \frac{q_i}{G} \oint \frac{ds}{h(s)} - \sum_k \frac{q_k}{G} \int_{ik} \frac{ds}{h(s)} = 0$$

which, after rearrangement of terms and the cancellation of G , becomes:

$$- q_0 \oint \frac{ds}{h(s)} = q_i \oint \frac{ds}{h(s)} - \sum_k q_k \int_{ik} \frac{ds}{h(s)} \quad (23)$$

It is interesting to note that with the exception of the left hand sides, Eq. (17) and (23) are identical. The latter equation permits the calculation of shear flows, and therefore shear stresses, caused by resultant shear forces in rigid, arbitrarily shaped, multi-cellular, hollow cross-sections, provided no cell is completely enclosed by others.

To take into account effects exerted by the flexibility of cross-sections, further modification of the calculated shear stresses and corresponding angular twists is required. This can be accomplished by superposing on the calculated stresses and twists the secondary stresses and twists caused by the distortion of cross-sections from their initial shapes. If thin-walled cross-sections are flexible and undergo distortion, significant secondary shear stresses, τ ; twists, ϕ ; and longitudinal normal stresses, σ , are generated.

Neglecting torsional warping effects, Tung³⁷ has developed the rationale for the elastic torsional analysis of single-cell, thin-walled, trapezoidal or rectangular concrete box-girder bridges which are symmetrical about the vertical axis of cross-sections which are non-variable along the span. He states that external loads can be resolved into symmetrical and antisymmetrical components. The symmetrical loading produces ordinary bending, whereas the antisymmetrical loading produces torsion and is responsible for the distortion of cross-sections. To evaluate the effects of the latter loading Tung argues that the shear flows developed in cross-sections are those due to pure Saint-Venant torsion and a self-balanced pure shear system which distorts cross-sections. The shears and twists caused by pure torsion are calculated by the use of Bredt's formula, but corresponding magnitudes due to the shear system are calculated by formulating equations which recognize that these shears are resisted by the longitudinal bending of individual plate elements of cross-sections in their own planes and the transverse bending of the cross-sections as rigid frames. That is, the distortion of cross-sections produces additional twist, shear, and longitudinal stresses. Tung's procedure of analysis can also be used when intermittent diaphragms are introduced to stiffen cross-sections but specifics are not presented. No attempt has been made to extend the analysis to cover multi-cell sections.

Tung's analysis has definite practical value but is still a special case of the comprehensive problem of torsion in the sense that it neglects warping. In the design of large bridge structures using single-cell sections, cross-sectional

dimensions may be variable along the span and it may be unconservative to neglect torsional warping effects.³⁸ Therefore, it may be required to use more general procedures of analysis.

GENERAL ANALYSIS OF THIN-WALLED STRUCTURES SUBJECTED TO TORSION

To summarize, the general elastic analysis of thin-walled structures postulates that the Bernoulli-Navier law of plane sections is not generally valid. The cross-sections of thin-walled structural units tend to warp (due to torsion and the distortion of cross-sections), resulting in significant deviations from the planar in many cases. The distinctive feature of thin-walled beams is that they undergo longitudinal extension as a result of the warping of cross-sections and it can be shown that there is a direct relationship between their behavior in flexure and in torsion. Corresponding to longitudinal extensions, there are consequently generated longitudinal normal stresses, proportional to the prevailing strains, which lead to an internal equilibrium of the longitudinal forces in each cross-section. These complementary longitudinal normal stresses, which occur as a result of the relative warping of a section and are not considered in the theory of pure (Saint-Venant) torsion, can attain very large values in thin-walled beams with open (rigid or flexible) cross-sections and also in beams with closed flexible cross-sections. Departure from the law of plane sections plays an important role not only in problems of strength but also in problems of stability and vibration.

In the field of structural analysis, elemental structural units can be classified according to their special characteristics as: (1) massive bodies, (2) plates and shells, (3) solid beams and (4) thin-walled beams. It is impossible to draw sharp limits between the various types of structural units since the same structure can be classified differently depending on the external loading conditions, on the manner in which the structure is to be employed in the given structural problem, and on the degree of accuracy demanded from the calculation. Such broad classifications, however, are practically useful. With respect to torsion, it may be stated categorically that all structures which resemble thin-walled beams are suspect of not behaving according to the law of plane sections. Among these structures are certain types of reinforced or prestressed concrete girder-bridges, arched bridges, suspension bridges having a roadway with a channel cross-section, long cylindrical or prismatic shells, long ribbed, cylindrical or prismatic shell-arches, tunnels, and pipe lines. Each specific problem encountered ought to be formally analyzed to establish prevailing stresses. As illustrated in Fig. 1-8, for example, an insight may be developed into the importance of warping torsion occurring in certain rigid beam structures; Fig. 1-9 demonstrates qualitatively how it is possible to assess changes in structural action and to visualize thereby the torsional behavior of structures.

More than anyone else, Vlasov has provided the practical foundation for the analysis of structures which consist of thin-walled units subjected to torsion.^{50,61,62,64} The general theory of thin-walled beams, which permits a quantitative assessment of specific torsional effects, was evolved by Vlasov from the theory he developed for the analysis of orthotropic cylindrical and

prismatic semi-momentless (partial membrane-state) shells. A knowledge of Vlasov's theory is desirable in order to understand subsequent developments in the analysis of thin-walled structures, and to judge the importance of basic assumptions made in the development of other techniques of analysis. Of particular interest among recent works are those published by Kollbrunner et al.^{36,76,78,87,100} Dabrowski,¹⁰² Bažant,^{83,94} Křístek,³⁸ and Scordelis et al.^{77,89,90,104,106,108,113,115,117} The special interest of Scordelis' work lies in the manner in which he takes advantage of the characteristics of digital computation in formulating the analysis of thin-walled structures.

When confronted with the need to design a thin-walled structural unit which is to support certain externally applied loads, the engineer first must select a suitable type of cross-section for the unit. A large number of different geometrical arrangements may be conceived, but, *a priori*, it must be decided whether to employ an open, closed, or open-closed section of the rigid or flexible types. Then, the deformations and stresses must be predicted to ensure the design of a safe structure. The discussion of thin-walled sections which follows has been organized with these primary considerations in mind. Since this paper is torsion-oriented, the structural units with $(h/d) < 0.1$ and $(d/l) < 0.5$ (approximately) are of interest; "h" designates wall-thickness, "d" is any characteristic dimension of the cross-section (its width or depth), and "l" its length.

Structural Units with Open Rigid Cross-Sections

Open rigid cross-sections may have the shape of any arbitrary smooth or broken curve. In concrete structures, rigidity of cross-sections may exist because of their dimensions, or, alternatively, transverse ribs may be provided separately or together with longitudinal ribs to prevent the distortion of sections from taking place. Assuming that the sections under scrutiny are rigid, it is possible to analyse such structural units by resorting to techniques of analysis derived from the general theory of shells.^{61,63}

Considering first cylindrical shells of uniform thickness, the basic eighth-order differential equation which establishes the relationship between the external forces and the internal stresses and strains for a shell is:

$$\Omega \Omega \phi(z,s) + \frac{12}{h^3} \cdot \frac{\partial^4 \phi(z,s)}{\partial z^4} = P(z,s) \quad (24)$$

where

$$\Omega = \frac{\partial^2}{\partial s^2} \left(R \frac{\partial^2}{\partial s^2} \right) + \frac{\partial}{\partial s} \left(\frac{1}{R} \cdot \frac{\partial}{\partial s} \right)$$

$\phi(z,s)$ = stress function,

h = thickness of shell,

z and s = coordinates of a point on the middle surface of the shell (z is directed along the generatrix s is directed along the contour of the shell),

$P(z,s)$ = a given load function expressed through the components P_z, P_s, P_n , of the vector of intensity of external forces acting on the shell,

$R = R(s)$ = radius of the middle surface of the shell.

The stress function $\phi(z,s)$ which satisfies the differential Eq. (24) and the boundary conditions of the structural unit under scrutiny provides the solution. The longitudinal normal stresses $\sigma(z,s)$ and corresponding transverse bending moments $G(z,s)$ may be determined from the relationships:

$$\sigma(z,s) = \frac{12}{h^3} \cdot \frac{\partial^2 \phi(z,s)}{\partial z^2} \quad (25)$$

$$G(z,s) = \Omega \phi(z,s) \quad (26)$$

In context here, it is also important to note that Eq. (24) is equivalent to a system of two symmetrical differential equations:

$$\frac{\partial^2 [\sigma(z,s)h]}{\partial z^2} + \Omega G(z,s) = P(z,s) \quad (27)$$

$$\Omega \sigma(z,s) - \frac{12}{h^3} \cdot \frac{\partial^2 G(z,s)}{\partial z^2} = 0 \quad (28)$$

Eq. (27) may be derived directly by considering the equilibrium conditions of a shell element, whereas Eq. (28) originates from the consideration of the compatibility of deformation of the middle surface of the shell. Transformation of Eq. (28) into terms of strain gives:

$$\Omega \epsilon(z,s) + \frac{\partial^2 \chi}{\partial x^2} = 0, \quad (29)$$

and the well-known equations of elasticity:

$$N(z,s) = Eh\epsilon(z,s); \quad G(z,s) = -\frac{Eh^3}{12} \chi(z,s) \quad (30)$$

where

$N(z,s)$ = normal force,

E = modulus of elasticity,

$\epsilon(z,s)$ = longitudinal tensile strain,
 χ = transverse bending strain.

Consequently, if the shell is rigid and its contour remains unchanged, Eq. 28-30 can be combined to yield:

$$\Omega \sigma(z,s) = \frac{\partial^2}{\partial s^2} \left(R \frac{\partial^2 \sigma(z,s)}{\partial s^2} \right) + \frac{\partial}{\partial s} \left(\frac{1}{R} \cdot \frac{\partial \sigma(z,s)}{\partial s} \right) \quad (31)$$

which implies that the longitudinal normal stresses, being a function of the contour coordinates z and s , follow the law of sectorial areas and that warping of the originally plane cross-sections of the shell occurs.

The format of the solution for Eq. (31) is:

$$\sigma(z,s) = F_1(z) + F_2(z)y(s) + F_3(z)x(s) + F_4(z)w(s) \quad (32)$$

where

$F_1(z); F_2(z); F_3(z); F_4(z)$ = functions of integration,

x, y = Cartesian coordinates of the middle surface of the shell,

$w(s) = \int_s r ds$ = sectorial coordinate representing twice the area enclosed between the initial and moving radius vectors and by the shell contour,

Giving exact geometric interpretations to the functions $F_1(z) \dots F_4(z)$, Eq. (32) can be transformed (see for example, Vlasov⁵⁰ to become:

$$\sigma(z,s) = \frac{P(z,s)}{A_s} + \frac{M_x(z)}{I_x} y(s) + \frac{M_y(z)}{I_y} x(s) + \frac{M_B(z)}{I_W} w(s), \quad (33)$$

where

$P(z,s)$ = total normal force parallel to z -axis,

A_s = cross-sectional area of shell,

$M_x(z)$ = bending moment with respect to x -axis,

I_x = moment of inertia with respect to x -axis,

$M_y(z)$ = bending moment with respect to y -axis,

I_y = moment of inertia with respect to y -axis,

$M_B(z)$ = bimoment,

$I_W = \int_s w^2(s)h ds$ = sectorial moment of inertia.

The first three terms of Eq. (32) are nothing else but the well-known stress components of axial force and moments about the x and y -axes respectively, as defined in ordinary beam theory. The fourth term, constructed to be of the same form as the second and third terms, defines the contribution of warping torsion to σ . To compute the magnitudes of $M_B(z)$, $w(s)$, and I_W appearing in that term, the location of the flexural or shear center of a structural unit must first be determined.

The flexural center (better known in the English language as the shear center) of a cross-section of a structural unit has the property that if the external transverse load (reactions included) passes through it, the unit will be in the condition of ordinary transverse bending; that is, plane sections remain plane. The loci of shear centers establish the line of shear centers. If principal cartesian coordinates are used to define the location of shear centers, their coordinates may be evaluated according to the formulas:

$$x = \frac{\int y(s) h w(s) ds}{\int y^2(s) h ds} \quad (34)$$

$$y = \frac{\int x(s) h w(s) ds}{\int y^2(s) h ds} \quad (35)$$

and it can be proven that the line of shear centers is also simultaneously the line of principal sectorial poles.⁵⁰ Fig. 1-17 shows the line of shear centers for a structural unit which is a cylindrical shell.

Once the flexural center has been located, it is possible to compute the bi-moment

$$M_B(z) = -EI_W \theta''(z) \quad (36)$$

where both I_W and $\theta(z)$ must be referred geometrically to the flexural center. As will be described later, $\theta(z)$ can be computed by relating it to the external loading through Eq. (42).

By a similar approach, the tangential shear stresses also may be determined⁵⁰ as:

$$\tau(z,s) = \frac{E}{h} [\xi''(z)h ds + \xi'''(z)S_y(s) + \eta'''(z)S_x(s) + \theta'''(z)S_w(s)] \quad (37)$$

where

$$S_y(s) = \int x(s) h ds$$

$$S_x(s) = \int y(s) h ds$$

$$S_w(s) = \int w(s) h ds$$

ξ, η, ζ = deformations of the shell along x, y, z directions at any point of the shell (Fig. 1-17) for a given loading and characteristics of the structural unit under consideration.

The solution for Eq. (37) is:

$$\tau(z,s) = F'_1(z) + F'_2(z)S_y(s) + F'_3(z)S_x(s) + F'_4(z)S_w(s), \quad (38)$$

where

$F'_1(z); F'_2(z); F'_3(z); F'_4(z)$ = functions of integration.

Giving exact geometric interpretations to the functions $F'_1(z) \dots F'_4(z)$ results in:

$$\tau(z,s) = \frac{M_K(z)}{I_p} h + \frac{V_x(z)}{I_{yh}} S_y(s) + \frac{V_y(z)}{I_{xh}} S_x(s) + \frac{M_W(z)}{I_{wh}} S_w(s) \quad (39)$$

where

$M_K(z)$ = torsional moment,
 I_p = torsional (polar) moment of inertia,
 $V_x(z)$ = shear force in xoz plane along x -axis,
 $V_y(z)$ = shear force in $yo z$ plane along y -axis,
 $M_W(z)$ = torsional warping moment.

The first term is the contribution of Saint-Venant torsion to the total shear stress τ . The second and third terms are the shear stresses arising from ordinary beam action. The fourth term, as before, permits the calculation of the stress due to warping of the cross-section. The moments $M_K(z)$ and $M_W(z)$ can be computed by utilizing the relationships:

$$M_K(z) = GI_p \theta'(z) \quad (40)$$

$$M_W(z) = M_B'(z) = -EI_W \theta'''(z) \quad (41)$$

Examination of Eq. (33) and (39) should make it clear that the *ordinary beam theory is nothing else but a special case of the more general flexural-torsional theory*. Fig. 1-17 identifies positive quantities for the evaluation of Eq. (33) and (39); also shown is the corresponding distribution of the variables w and S_w .

Although the description of the flexural-torsional theory commenced with the consideration of a cylindrical shell unit, whose profile is defined by smooth curves, it has been shown that Eq. (33) and (39) are applicable to open rigid cross-sections of any shape,^{61,62} including for example those with longitudinal

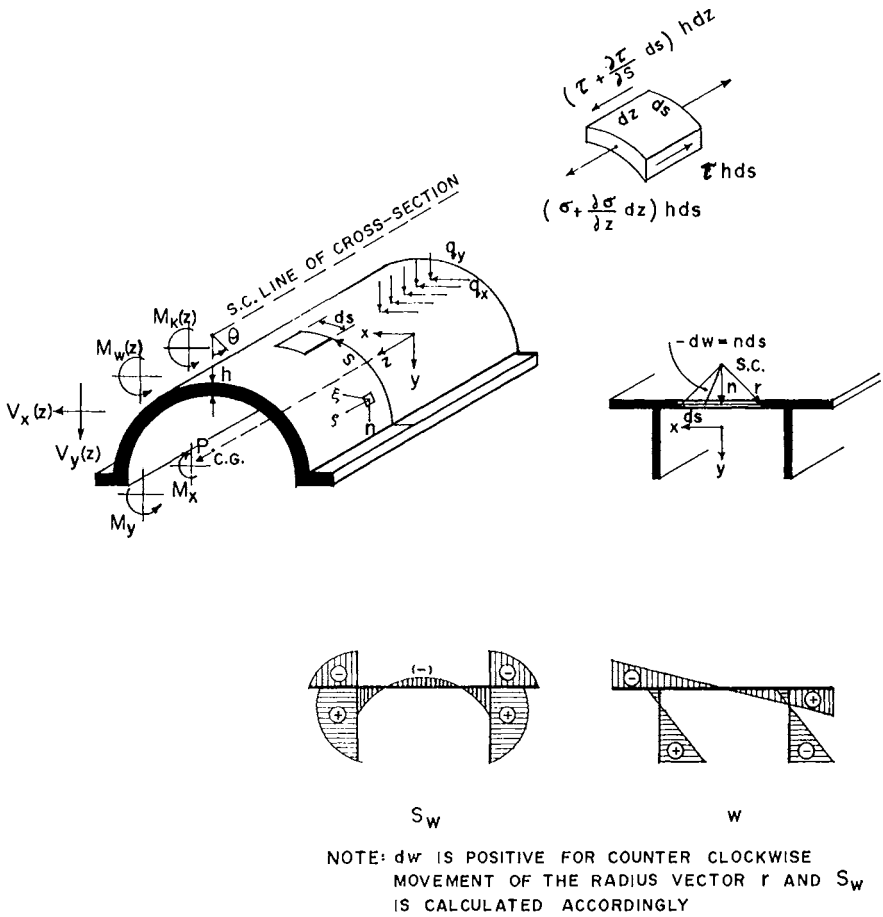
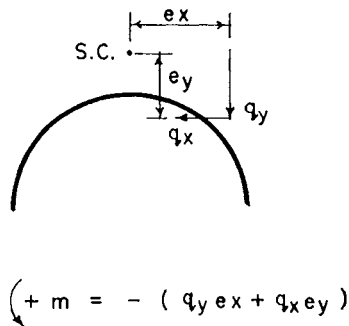


Fig. 1-17—Definition of positive quantities

Fig. 1-18—Definition of positive m

ribs and variable thickness h along the contour of the cross-section. Furthermore, the integral signs which appear in the various expressions may be replaced by summation signs whenever more convenient.

The determination of the angle of rotation $\theta(z)$ and its derivatives, required for the solution of Eq. (36), (40), and (41), is best discussed for the case of transverse loading, since such loading is common and frequently responsible for the generation of torsion. The differential equation for the torsional angle $\theta(z)$, derived from the consideration of equilibrium under transverse load only,^{36,50} is:

$$EI_W \theta^{iv}(z) - GI_p \theta''(z) - m(z) = 0 \quad (42)$$

where

EI_W = sectorial rigidity,

GI_p = torsional rigidity = GK

$m(z)$ = external torsional moment per unit length of beam
 = $-(q_y e_x + q_x e_y)$ as shown in Fig. 1-18,

and its solution is:

$$\theta(z) = C_1 + C_2 \frac{z}{\ell} + C_3 \sinh \frac{\kappa}{\ell} z + C_4 \cosh \frac{\kappa}{\ell} z + \bar{\theta}(z) \quad (43)$$

where

C_1, C_2, C_3, C_4 = functions of integration

$$\kappa = \ell \sqrt{\frac{GI_p}{EI_W}}$$

ℓ = span

$\bar{\theta}(z)$ = particular solution of the differential equation

It follows that in the light of Eq. (43), and recognizing that any section the torsional moment acting on the section is $H(z) = M_K(z) + M_W(z)$, Eq. (36), (40), and (41) may be rewritten to become:

$$M_B(z) = -GI_p \left[C_3 \sinh \frac{\kappa}{\ell} z + C_4 \cosh \frac{\kappa}{\ell} z + \frac{\ell^2}{\kappa^2} \bar{\theta}''(z) \right], \quad (44)$$

$$H(z) = GI_p \left[\frac{C_2}{\ell} + \bar{\theta}'(z) - \frac{\ell^2}{\kappa^2} \bar{\theta}'''(z) \right]; \quad (45)$$

the functions of integration $C_1 \dots C_4$ may be determined from conditions imposed by given boundary conditions, and particular solutions $\bar{\theta}(z)$,

corresponding to specific conditions of loading, may be derived by the utilization of the technique of initial parameters.⁵⁰ Having evaluated the magnitudes of these quantities, $M_B(z)$, $M_K(z)$, and $M_W(z)$ can then be determined and Eq. (33) and (39) can be used to evaluate the σ and τ -stresses acting at any point of a cross-section of a structural unit. The detailed analysis and design of structural units with rigid cross-sections, *non-variable* along the span, becomes possible. The reasoning involved in the determination of the unknown quantities will be illustrated with reference to some specific cases:

Case 1: Simply supported beam with concentrated torsional moment M at midspan (Fig. 1-19).

The following boundary conditions apply:

$$\text{At } z = 0 \text{ and } z = \ell, \theta = 0 \text{ and } M_B = 0.$$

The particular solutions are $\bar{\theta}\left(z < \frac{\ell}{2}\right) = 0$, and

$$\bar{\theta}\left(z > \frac{\ell}{2}\right) = \frac{M\ell}{GI_p} \left[\frac{1}{\kappa} \sinh \kappa \left(\frac{z}{\ell} - \frac{1}{2} \right) - \left(\frac{z}{\ell} - \frac{1}{2} \right) \right]$$

For the conditions $\theta = 0$, $M_B = 0$ and $\theta = 0$ for $z = 0$, Eq. (43) becomes

$$0 = C_1 + C_4$$

and Eq. (44) becomes

$$0 = -C_4 - \frac{\ell^2}{\kappa^2} \bar{\theta}''(z = 0), \text{ whence } C_4 = 0$$

For the conditions $\theta = 0$, $M_B = 0$ and $\bar{\theta} = \bar{\theta}\left(z > \frac{\ell}{2}\right)$ for $z = \ell$, together with the fact that C_1 and C_4 are 0, Eq. (43) becomes:

$$0 = C_2 + C_3 \sinh \kappa + \frac{M\ell}{GI_p} \left(\frac{1}{\kappa} \sinh \frac{\kappa}{2} - \frac{1}{2} \right),$$

and Eq. (44) becomes:

$$0 = -C_3 GI_p \sinh \kappa - \frac{M\ell}{\kappa} \sinh \frac{\kappa}{2}, \text{ whence } C_3 = -\frac{M\ell}{GI_p} \frac{\sinh \frac{\kappa}{2}}{\kappa \sinh \kappa}$$

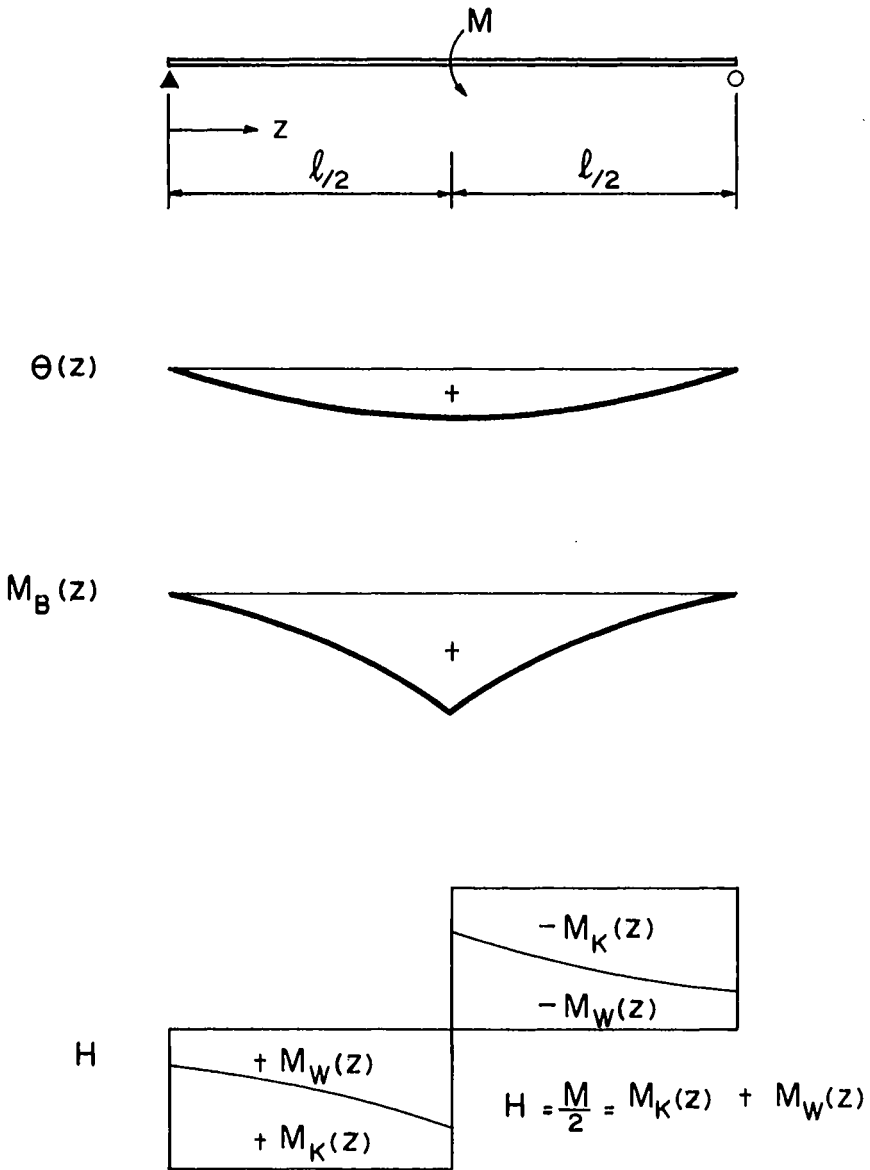
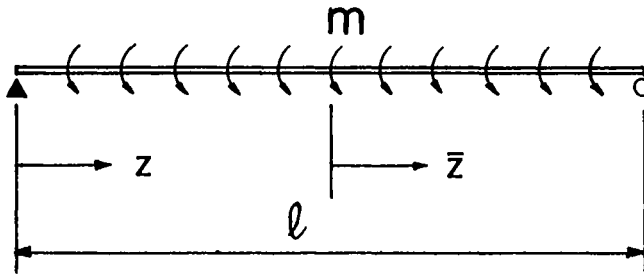
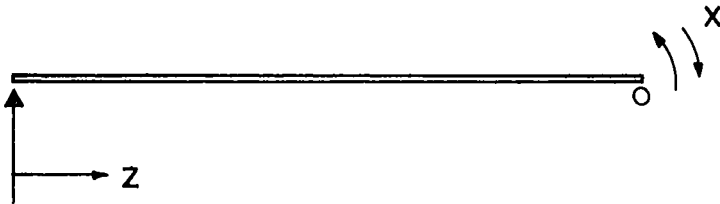


Fig. 1-19—Qualitative delineation of the variation of $\theta(z)$, $M_B(z)$, $M_K(z)$ and $M_W(z)$ and the relative magnitudes of $M_K(z)$ and $M_W(z)$ for beam with a torsional moment at midspan ($\kappa \sim 3$)



- a. BEAM WITH UNIF. DIST. TORSIONAL MOMENT



- b. BEAM WITH BIMOMENT APPLIED AT ONE END CAUSING WARPING OF CROSS - SECTIONS

Fig. 1-20—Loadings applied to beam structures

$$\text{Therefore, } C_2 = \frac{M \ell}{2GI_p}.$$

Using the computed constants C in conjunction with Eq. (41), (43-45), it is now possible to state that:

$$\text{for } 0 < z < \frac{\ell}{2} \quad \theta(z) = \frac{M \ell}{GI_p} \left[\frac{z}{2\ell} - \frac{\sinh \frac{\kappa}{2}}{\kappa \sinh \kappa} \sinh \frac{\kappa}{\ell} z \right],$$

$$M_B(z) = M \ell \frac{\sinh \frac{\kappa}{2}}{\kappa \sinh \kappa} \sinh \frac{\kappa}{\ell} z,$$

$$M_W(z) = M \frac{\sinh \frac{\kappa}{2}}{\sinh \kappa} \cosh \frac{\kappa}{\ell} z$$

$$H(z) = \frac{M}{2},$$

$$\text{for } \frac{\ell}{2} < z < \ell \quad \theta(z) = \frac{M \ell}{GI_p} \left[\frac{1}{2} - \frac{z}{2\ell} - \frac{\sinh \frac{\kappa}{2}}{\kappa \sinh \kappa} \sinh \frac{\kappa}{\ell} (\ell - z) \right]$$

$$M_B(z) = M \ell \frac{\sinh \frac{\kappa}{2}}{\kappa \sinh \kappa} \sinh \frac{\kappa}{\ell} (\ell - z)$$

$$M_W(z) = M \frac{\sinh \frac{\kappa}{2}}{\sinh \kappa} \cosh \frac{\kappa}{\ell} (\ell - z)$$

$$H(z) = -\frac{M}{2}$$

It is clear that the quantities θ , M_B , M_W , and H can be evaluated for any section along the span; their variation ($\kappa \sim 3$) is depicted qualitatively in Fig. 1-19. Corresponding to selected physical dimensions of a structural unit, magnitudes may be determined for these quantities and, accordingly, Eq. (33) and (39) may be used to calculate σ - and τ -stresses for any point on a cross-section.

Case 2: Simply supported beam with evenly distributed torsional moment m (Fig. 1-20a)

This case is commonly encountered in practice. If a transverse load of constant magnitude and direction is applied along a line parallel to the generator of

the beam, the external torsional couple per unit section due to such a loading will be constant; that is $m = qe$.

Since the origin of the coordinate system has been shifted to midspan to take advantage of symmetry, the constants of integration that are coupled with odd functions in Eq. (43) become zero; that is, $C_2 = C_3 = 0$. The remaining constants C_1 and C_4 are determined from the boundary conditions $\theta = 0$ and $M_B = 0$ for $\bar{z} = z - \frac{\ell}{2} = \frac{\ell}{2}$, utilizing the fact that the particular integral

$$\bar{\theta}(z) = -\frac{m\ell^2}{2GI_p} \left(\frac{z}{\ell} - \frac{1}{2} \right)^2.$$

Thus Eq. (43) becomes:

$$0 = C_1 + C_4 \cosh \frac{\kappa}{2} - \frac{m\ell^2}{8GI_p},$$

and Eq. (44) becomes:

$$0 = C_4 \cosh \frac{\kappa}{2} - \frac{m\ell^2}{\kappa^2 GI_p},$$

$$\text{whence } C_4 = \frac{m\ell^2}{GI_p \kappa^2 \cosh \frac{\kappa}{2}}.$$

$$\text{whence } C_2 = \frac{m\ell^2}{GI_p} \left(\frac{1}{8} - \frac{1}{\kappa^2} \right).$$

Proceeding as in Case 1, this means that:

$$\theta(z) = \frac{m\ell^2}{GI_p} \left[\left(\frac{1}{8} - \frac{1}{\kappa^2} \right) - \frac{1}{2} \left(\frac{z}{\ell} - \frac{1}{2} \right)^2 + \frac{\cosh \kappa \left(\frac{z}{\ell} - \frac{1}{2} \right)}{\kappa^2 \cosh \frac{\kappa}{2}} \right]$$

$$M_B(z) = \frac{m\ell^2}{\kappa^2} \left[1 - \frac{\cosh \kappa \left(\frac{z}{\ell} - \frac{1}{2} \right)}{\cosh \frac{\kappa}{2}} \right]$$

$$M_W(z) = m \ell \frac{\sinh \kappa \left(\frac{z}{\ell} - \frac{1}{2} \right)}{\pi \cosh \frac{\kappa}{2}}$$

$$H(z) = -m \ell \left(\frac{z}{\ell} - \frac{1}{2} \right)$$

Case 3: Simply supported beam with bimoment X applied at one end (Fig. 1-20b).

This case is of importance in connection with transversely loaded continuous beam structures, because the moments acting at the supports are frequently considered as the unknowns to be determined by the force method of statically indeterminate analysts.

In the case of continuous thin-walled beams, internal bimoments occur at the supports in addition to the moments and shears ordinarily considered to act in beams with solid cross-sections. These bimoments occur because there must be continuity of warping of cross-sections. Once the magnitudes of bimoments have been established, Case 3 conditions will permit the calculation of corresponding deformation and force effects generated within spans. (The case of bimoments acting simultaneously at each end of a span can be covered by determining the effects of each bimoment separately, using superposition.)

The boundary conditions are: $\theta = 0$, $M_B = 0$ at $z = 0$; $\theta = 0$, $M_B = X$ at $z = \ell$. The particular solution $\theta(z) = 0$. The boundary conditions at $z = 0$, using Eq. (43) together with Eq. (44), establish that $C_1 = C_4 = 0$.

At $z = \ell$, it follows that $C_2 = -C_3 \sinh \kappa$ and $C_3 = -\frac{X}{G I_p \sinh \kappa}$. This means that $C_2 = \frac{X}{G I_p}$. Proceeding as in Case 1, one obtains that:

$$\theta(z) = \frac{X}{G I_p} \left[\frac{z}{\ell} - \frac{\sinh \frac{\kappa}{\ell} z}{\sinh \kappa} \right]$$

$$M_B(z) = X \frac{\sinh \frac{\kappa}{\ell} z}{\sinh \kappa}$$

$$M_W(z) = X \frac{\kappa \cosh \frac{\kappa}{\ell} z}{\ell \sinh \kappa}$$

$$H(z) = \frac{X}{\ell}$$

In the results for cases 1-3, the parameter κ , contained in all but $H(z)$, reflects the interrelationship of the Saint-Venant $[GI_p = GK]$ and warping $[GI_w]$ torsional rigidities for structural units with finite spans ℓ . For a given span length ℓ , with $GI_p = 0$, $EI_w > 0$, and $M_K(z) = 0$, the magnitudes of $M_B(z)$ corresponds to $\kappa = 0$ are those for pure warping with corresponding $M_W(z)$ -values being developed. If, on the other hand, $GI_p > 0$ and $M_B(z) = 0$, no warping takes place and only pure Saint-Venant torsion can occur. Therefore, the bimoment $M_B(z)$ can be considered to provide a direct measure of the influence of warping and the value of the ratio

$$0 < \frac{M_B(z) \text{ for } \kappa > 0}{M_B(z) \text{ for } \kappa = 0} < 1 \text{ at a specified location along the span of a struc-}$$

tural unit is indicative of whether warping torsion or Saint-Venant torsion is dominant, as shown in Fig. 1-8. These graphs were developed by Kollbrunner and Balser,³⁶ the dash-and-dot curves having been derived by the use of approximate solutions. As the parameter κ is the product of span length and the square root of the ratio of the Saint-Venant and warping torsional rigidities, it should also be observed that, for constant ratios of torsional rigidities (in the range of beam action) the warping effect becomes less significant with an increase in the span ℓ . If the contour of the cross-section and the span ℓ are kept constant, it is possible to demonstrate that the smaller the wall thickness h , the smaller will κ be, and vice versa; this is true because the rigidity GI_p is of the third degree in h while the rigidity EI_w is linear in h . It follows that the smaller the thickness of the component plates and/or shells of a cross-section, the slower is the fall-off of warping effects. Significant torsional warping moments can be developed throughout the span, as shown in Fig. 1-19.

A truly remarkable feature of torsion theory is the parallelism between the mathematical forms of the bimoment theory of bending-torsion and the elementary or ordinary theory of beam bending. This parallelism is well illustrated in Table 1-2. It is not surprising, therefore, that Eq. (9) for strain energy due to ordinary bending finds a counter part in the following equation for bending-torsional strain energy:

$$U = \text{Eq. (9)} + \int \frac{M_B^2(z)dz}{2EI_w} + \int \frac{M_K(z)M'_B(z)dz}{GK} + \int \frac{[M'_B(z)]^2 dz}{2GK} \quad (46)$$

Thus, a designer may use deformation, force or mixed methods when deriving the equations for the bending-torsional analysis of structures, in the same manner as when warping torsion is not present. If force methods are used, it follows that the Maxwell-Mohr theory for elastic statically indeterminate structures may be successfully extended to apply to thin-walled structures, utilizing the bimoment concept.

With this theory analogy, it is possible to determine design deformations and forces for a variety of systems characterized as having thin-walled, rigid cross-sections, such as continuous thin-walled beams,³⁶ continuous hipped plate systems