

Stability of Reinforced Concrete Shells: State-of-the-Art Overview

By Egor P. Popov and Stefan J. Medwadowski

Synopsis: Over the last few decades, shell structures have become bigger--they cover larger areas without intermediate supports--and thinner. Because of this, the problem of buckling of shells has grown in importance. This paper contains an overview of the general problem of stability of reinforced concrete shells. The buckling phenomenon is defined and its manifestations in columns, plates, and shells are discussed. The linear critical load concept is reviewed first, followed by a consideration of geometric nonlinearities and of geometric imperfections of the shape of the shell as built. Next, the material properties of reinforced concrete and its response under load are reviewed. The properties of inelastic behavior of concrete and reinforcement, the cracking of concrete, the amount of reinforcement, and the effects of concrete shrinkage and creep are discussed. These factors make the buckling behavior of reinforced concrete shells significantly different from metallic shells and cause a reduction in the load-carrying capacity of the shell. Current approaches to shell stability analysis and design are commented on.

Keywords: buckling; cracking (fracturing); creep properties; loads (forces); reinforced concrete; shells (structural forms); shrinkage; stability; structural analysis; structural design.

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INTRODUCTION

Domical forms have been used by man from time immemorial, for shelters, utilitarian and symbolic structures. Early materials were natural wood, saplings, hides, stone, clay. Methods of construction were largely a matter of tradition, strongly influenced by local conditions and materials. Even though progress was dispersed and slow, evidence of some impressive structures has survived to our own time.

Decisive progress was achieved in Rome; the most prominent example—partly because of its size, partly because of its beauty, and partly because of its having survived to our own time—is the Roman Pantheon, constructed approximately 1850 years ago. The structure consists of a hemispherical dome supported on a cylindrical drum. The diameter of the dome is approximately 43.3 m, resulting in a most impressive space, made possible by the use of the native pozzolean cement in the construction—in effect, the Pantheon is constructed of concrete. The dome is ribbed to lighten the dead load, as is the drum of the walls. At the edge of the large lantern opening at the top of the dome, the shell is approximately 1.2 m thick; the thickness increases toward the springing line. The total span of the dome was not exceeded until modern times (cf. TABLE 1).

The tradition of domical forms continued, first in Roman architecture, in Byzantium (with the magnificent example of the Church of St. Sophia in Constantinople), later in the Romanesque, the Gothic and the Renaissance, with many buildings remaining as testimony to the daring and skill of the builders. Domes on a noncircular plan, domes with pendentives, ribbed and skeletal shells were added to the tradition. The use of concrete, however, so common in ancient Rome, gradually diminished and it became the forgotten material, to be rediscovered at the time of the Industrial Revolution²².

The Industrial Revolution witnessed many changes in the manner in which man built his structures. A rapid development of what is now called structural mechanics brought with it rational methods for the analysis of shells, the first comprehensive theory having been published by Aron in 1874¹⁰, to be followed soon by the seminal writings of Love⁷². At the same time, new materials of construction were developed: cast iron, later to become steel, and reinforced concrete, a material of particular importance because of its moldability and the ease with which it could be formed into surface structures. Not surprisingly, the first reinforced concrete shell structures appeared before 1918. Examples are the Orly airport hangar by Freyssinet, a corrugated cylinder with only a limited spatial action, but a most impressive project, nevertheless, and some barrel vaults by Perret²², delicate and well proportioned, and yet intensely practical at the same time, offering a promise of things to come.

Rapid growth in the construction of reinforced concrete thin shells came in the decade of the twenties. The rather forbidding formulations of the theory of thin shells were simplified, first for spherical caps by Geckeler³⁶, then by Finsterwalder³³ for circular cylindrical barrel vaults; in each case, the motive was the direct need of the construction industry. New forms appeared. Elliptic paraboloids were experimented with, Freyssinet⁵⁷ used conoidal north-light shells for railway sheds, where he achieved structures of great beauty, and finally the hyperbolic paraboloid appeared, first in the realized projects of Baroni, later in the theoretical investigations of Aimond and Lafaille, to re-emerge triumphant in the exuberant architecture of Candela³². At the same time, new construction techniques appeared: guniting and rigid reinforcement (similar to the Melan system of bridge construction, but extended to three-dimensional surface structures) for the Jena Planetarium, precast skeletal shells in the many projects of Nervi⁸⁰, use of cable reinforcement for shells, and finally the use of prestressing.

TABLE 1 COMPARISON OF SHELL DIMENSIONS AND THICKNESS-TO RADIUS-OF-CURVATURE RATIOS

Project	Location and Year	Type of Geometry	Plan dimensions	Radius of curvature R	Thickness h	$\approx h:R$
The Pantheon	Rome ca. 100	hemisphere	43.3 m (dia)	21.65 m	1.2m at top	1:24
Planetarium	Jena 1923	hemisphere	25 m (dia)	12.5 m	.06m	1:200
Factory	Jena 1923	spherical cap	40 m (dia)	28.28 m	.06 m	1:470
Market Hall	Algeciras 1934	spherical cap on 8 supports	47.6 m (dia)	44.1 m	.09 m	1:490
Filter Plant	Minnesota 1939	ellipsoid of revolution	45.7 m (dia)	47.24 to 5.33 m	.09 m to .15 m	1:525 to 1:35
Factory	Bryn Mawr 1947	elliptic paraboloid on rect. plan	19.6 m x 25.3 m	25.0 m to 32.9 m	.09 m	1:300 to 1:400
Auditorium	MIT Cambridge 1955	segment of a sphere on 3 points	48.0 m bet. support points	34.0 m	.065 m	1:520
Shopping Center	Kanehoe, Oahu 1957	groined vault (intersection of 2 tori) or 4 points	39.0 m x 39.0 m bet. support points	39.0 m to 78.0 m	.076 m to .178 m	1:1000 to 1:500
Palazetto di Sport ¹	Rome 1957	spherical cap	58.5 m (dia)	30.9 m	.335 m rib	1:92
CNIT ²	Paris 1957	groined vault (intersection of 3 cylinders) on 3 points	219 m bet. support points	89.9 m to 420.0 m	1.91 m to 2.74 m	1:47 to 1:153
Hen's egg		surface of revolution		2 cm (min)	.2 mm to .4 mm	1:100 to 1:50

1. precast, ribbed construction; material $h:R$ ratio $\approx 1:250$

2. 2-layer shell, flange thickness from .06 m to .12 m; material $h:R$ ratio 1:120

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It is important to note that the anticlastic surfaces such as the hyperbolic paraboloid, whatever the method of construction, are the only new structural form of the modern era⁷⁶. The initial impact for their use came from the need for efficient and cheap methods of covering large areas without having to construct huge scaffolding structures characteristic of the traditional methods of erection of reinforced concrete shells.

Although the span of the Pantheon is very great, modern shell structures achieve much larger column-free areas and, what is even more significant, the thickness of the modern shell is orders of magnitude smaller than the thickness of the traditional domes. The trend toward greater spans and thinner shells can be clearly discerned (cf. TABLE I). To an analyst, the significance of this ever larger radius of curvature-to-shell thickness ratio lies in the realization that it has an important effect on the buckling of concrete thin shells, and on their strength reserve. The thickness of shells, in the case of small spans, is dictated by construction considerations: the need to accommodate reinforcement, the practicalities of placing concrete. Thus, a concrete thin shell of small span possesses an enormous reserve of strength. This is no longer the case if the spans become large.

Early modern designers realized fully the need to protect shells from the possibility of buckling. However, their ability to predict the buckling load was limited by the incomplete state of understanding of the phenomenon and by the practical difficulties, analytical and computational, of performing the necessary calculations. At this stage in the development of the theory, two shell forms were studied at length: the sphere and the cylinder. At first, familiar with the success of the linear buckling theory of Euler in predicting the buckling load of columns, analysts formulated the shell problem in the same linear manner. However, it soon appeared that the critical buckling load obtained in this way did not agree with the values of critical buckling loads obtained for the same shell forms experimentally, the experimental values being many times smaller than the critical load predicted by the analysis.

The efforts to explain the lack of correlation between theory and experiment continued over several decades, and have met with success. It appears that the problem has its root in the fact that a shell is a spatial, surface-type structure. Accordingly, the in-plane state of stress in a differential element of a shell consists not of just one axial stress (as is the case in a column), but of two axial stresses and an in-plane shear stress. If a shell is subjected to an axial compression in one direction, the associated axial stress in the orthogonal direction may be compressive or tensile. If it is tensile, the orthogonal axial stress tends to stiffen the shell, and to reduce the deflections normal to the surface. However, if the orthogonal stress is compressive, it tends to increase the normal deflections, thus weakening the shell and lowering the critical buckling load. The linear shell theory is incapable of predicting this behavior, and recourse has to be made to the nonlinear theory of shells. This presents formidable computational difficulties, in general intractable until the advent of the computer.

An additional complication in attempting to predict the buckling behavior of reinforced concrete thin shells lies in the nature of their material. The cracking of concrete, nonhomogeneity of the material, its creep characteristics, its inelastic properties, all have to be taken into account when studying nonlinear phenomena in shells. Finally, the inevitable imperfections in the geometry of the shell as built compound the difficulties of the problem and reduce the buckling load. These problems, too, are not simple, and can be handled generally only with the aid of computers. At this stage in the development of the theory, only partial success has been attained, but it seems fair to say that the essence of the problem has been captured.

Until recently, the lack of computational facilities precluded either nonlinear analysis, or a detailed consideration of the material or geometric factors. Yet shells were being built, and safely. The designers, guided by their judgment as much as by some approximate analyses, resorted to a number of devices aimed at making rein-

forced concrete thin shells safe in buckling. These devices included:

- Modification of the shape of the shell. This modification could take several forms. To reduce the radius of curvature of the shell in the small, local undulations might be introduced. Alternatively, the shape of the shell might be changed locally—usually along the edges—so that the radius of curvature might become smaller along a strip of the shell; this device consisted essentially of introducing an edge beam and thus stiffening the edge of the shell. An improvement in the buckling behavior of a shell could be obtained through the use of shells of double rather than single curvature.
- Modification in the manner in which the material is distributed throughout the surface of the shell. This can be done by providing a system of ribs throughout the surface of the shell, in one or two directions (usually orthogonal), as might be required in a given case. Alternatively, all of the material might be placed along the lines of the ribs, as in skeletal (or reticulated) shells, thus providing an "effective thickness" of the shell greater than if the material were evenly distributed throughout the surface of the shell. This type of modification is evident in the projects of Nervi⁸⁰ and Torroja¹⁰⁵.
- A variant in the modification of the manner in which the shell material is distributed was used by Esquillan, one of the great pioneer shell designers of the modern era, in the design of the great hall of the CNIT near Paris³⁰. This shell has the form of a groined vault on three supports, formed from the intersection of three parabolic cylinder segments, with the span between the three supports of approximately 219 m—the largest shell ever built. The shell itself consists of two thin surface layers, connected by shear-transferring diaphragms. Thus, the total thickness of the shell varies from 1.9 to 2.75 m, while the average thickness of the material is on the order of only 17 cm to 24 cm.
- Finally, the introduction of appropriate prestressing loads may result in an improved buckling behavior. Possibly the best early example of the use of such devices is the famed Market Hall at Algeciras designed by Torroja¹⁰⁵.

The excellence of judgment of shell designers is evidenced by the fact that, as far as we know, very few, if any, concrete shell failures have been conclusively shown to be due to buckling—although some failures said to be due to creep of concrete or to the steep temperature gradient may have been, in fact, buckling failures. (The only failure of a large shell conclusively shown to have been due to buckling involved a reticulated spherical dome made of metal¹¹⁴.) This is an enviable record, a testimony to the excellence of the designers as well as the inherent strength of shell structures. For this record to be preserved, and in view of the growing number of shells being built, of ever-growing spans and ever-diminishing thicknesses, it is necessary to develop a better understanding of shell buckling behavior.

The purpose of this paper is to provide an overview of the problem of buckling of reinforced concrete shells. The nature of the buckling phenomenon is examined, and its different manifestations possible in shell structures are reviewed. The difference in the qualitative behavior of columns and shells is considered, and the need for a study of postbuckling equilibrium states based on the geometrically nonlinear conditions for certain classes of shells is explained. The influence of geometric imperfections, of the inelastic material properties, of reinforcement, and of cracking of concrete is considered. A brief review of the currently available analytical and experimental methods is made, followed by an examination of current provisions for the study of reinforced concrete shell buckling in two recent recommendations: the ACI^{2,4} and the IASS⁷⁷. The paper is to serve as an introduction to the detailed studies of these topics undertaken in the companion papers contained in this volume.

THE BUCKLING PHENOMENON

Consider a straight column subjected to a concentric compressive load. Let the load increase gradually, starting from zero. Within some range of the values of the load, the column remains straight—there is no lateral displacement normal to the axis of the column; the only deformation experienced by the column is a small shortening of its axis.

However, at some value of the compressive load, the following event may take place: the column may experience lateral displacement, and this displacement will increase significantly without a correspondingly significant increase in the magnitude of the load. If such lateral displacement does occur, the column is said to have buckled; the phenomenon itself is called buckling, and the value of the compressive load at which buckling occurs is called the critical load.

It should be noted that the phenomenon of buckling is by no means restricted to that described above, that is, to transverse displacement of columns under axial compression. As an example, the same column subjected to a gradually increasing axial load might experience at some load level a sudden twist while remaining straight—a case of torsional buckling. Similarly, a beam supporting a bending load in its own plane might experience sudden twisting deformation—an example of lateral buckling of beams. Both these phenomena are well known and documented, and provisions for investigating the buckling of struts under compressive load, and for investigating lateral buckling of beams, are contained in all building design codes.

It turns out that plate and shell elements—in fact all structural elements subjected to loads such that internal compressive in-plane stresses are present—are also subject to buckling. However, because of the two-dimensional nature of shells (as opposed to the one-dimensional nature of columns and beams), the problem of buckling of shells is very much more complex in its physical manifestations, and very much more complex to investigate.

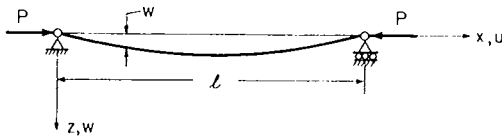


Fig. 1 Simply Supported Buckled Strut

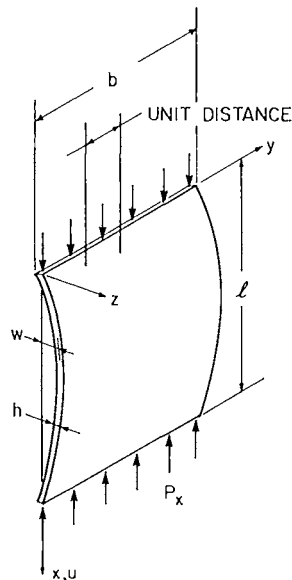


Fig. 2 Buckled Plate Simply Supported at Loaded and Free at Unloaded Edges

PATHS OF EQUILIBRIUM

Introduction

It is of interest to trace the successive positions of equilibrium of structural elements associated with the successive values of slowly increasing loads up to, at, and beyond the critical load. The resulting load vs. deflection curve is called the *path of equilibrium*. The *primary path* is the portion of the curve up to the critical load, while the *secondary path* is the portion beyond the critical load. Columns, flat plates, and shells are all of interest because all three elements occur as parts of thin shell structural assemblies. Because of its relative simplicity, the problem of a column will serve as the starting point.

Ideal Elastic Column

Consider again a column under axial load. The column is supposed to be ideally straight, and subjected to ideally concentric compressive load—an ideal column. The length of the column is l , and it is simply supported at the ends. The external compressive load is P .

In order to determine the successive deflected shape of the column in the successive positions of equilibrium—the problem of *elastica*—the assumptions first made by Euler in 1744 are followed³¹; these assumptions have since then been repeated in virtually all standard texts on the strength of materials (see, for example, Ref. 83). We assume that the strain-displacement relations and the constitutive relations of the technical theory of beams apply, as follows

$$\begin{aligned}\epsilon &= \frac{du}{dx} & \kappa &= -\frac{d^2w}{dx^2} \\ N &= AE \epsilon & M &= EI \kappa\end{aligned}\quad (1)$$

where ϵ and κ are, respectively, the longitudinal strain and the change of curvature of the column, u and w are its axial and transverse displacements, x is the coordinate (cf. Figure 1), N and M are the axial force (positive if tensile) and the bending moment in the column, A and I are the area and the appropriate moment of inertia of the cross

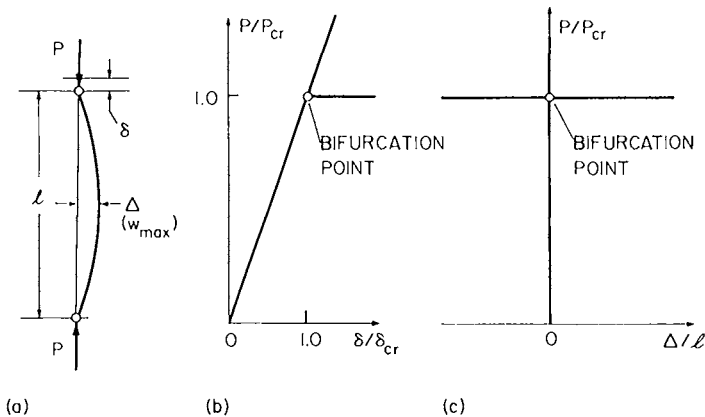


Fig. 3 Buckling Behavior of Ideal Elastic Strut—Linear Theory

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section of the column, and E is the modulus of elasticity of the column material. Next, assuming small deflections and negligibly small shear force components in the z -direction, the equations of equilibrium of the deformed differential element of the column can be written:

$$\begin{aligned}\frac{dN}{dx} &= 0 \\ \frac{dV}{dx} + \frac{dN}{dx} \frac{dw}{dx} + N \frac{d^2w}{dx^2} &= 0 \\ V - \frac{dM}{dx} &= 0\end{aligned}\quad (2)$$

Substitution of eq. (2b) into eq. (2c) together with the use of eqs. (1) and (2a), and assuming $EI = \text{constant}$, results in

$$\begin{aligned}\frac{dN}{dx} &= 0 \\ EI \frac{d^4w}{dx^4} - N \frac{d^2w}{dx^2} &= 0\end{aligned}\quad (3)$$

This is the system of differential equations that governs the problem, written in the two unknowns N and w . It is nonlinear in the sense that these unknowns appear as a product in the second equation. However, eq. (3a) can be solved for N first, and its known value can then be substituted into eq. (3b). In the case of a compressive force P at the ends, $N = -P$, and

$$EI \frac{d^4w}{dx^4} + P \frac{d^2w}{dx^2} = 0 \quad (4)$$

Consulting Figure 1 it is seen that this equation could be derived also by observing that the moment at any point x of the column is $M = Pw$ and hence, using eqs. (1),

$$\frac{d^2w}{dx^2} + \frac{P}{EI} w = 0 \quad (5)$$

Successive differentiation of this equation with respect to x , taking $EI = \text{constant}$, results in eq. (4). Equation (5) is suited to obtaining solutions of problems involving only static boundary conditions, while eq. (4) can be used to obtain solution of problems involving any boundary conditions of a column.

Nontrivial (*i.e.*, $w \neq 0$) solutions of eq. (4) exist only for certain values—the characteristic values—of the axial force P . The smallest of these values is called the critical value P_{cr} , also called the Euler critical buckling load of the column. The value of the critical load for a pin-ended column is given by the expression

$$P_{cr} = \frac{\pi^2 EI}{l^2} \quad (6)$$

For a column of rectangular cross-sectional area A and the smaller side dimension h , the moment of inertia is $Ah^2/12$, and the critical load can be put in the form:

$$P_{cr} = \frac{\pi^2 EA}{12(l/h)^2} \quad \sigma_{cr} = \frac{\pi^2 E}{12(l/h)^2} \quad (7)$$

where the quantity $12(l/h)^2$ is termed the *slenderness ratio*, and $\sigma_{cr} = P_{cr}/A$.

Before considering further the solution of eq. (4), and to simplify a comparison with plates and shells, consider a rectangular column of width b (comparable in magnitude to its length l) shown in Figure 2—in effect a plate, with the loaded edges simply supported and the unloaded edges free. Let the total column load be $P = P_x b$, and the flexural stiffness be $EI = Db$, where

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad (8)$$

and where ν is the Poisson ratio of the column material. Then eq. (4) becomes

$$\frac{d^4 w}{dx^4} + \frac{P_x}{D} \frac{d^2 w}{dx^2} = 0 \quad (9)$$

This equation will be referred to at several points later in this paper.

Solution of eq. (4) can be plotted as is done in Figure 3. As the axial force P increases, axial shortening of the column is observed. However, for the assumed ideal column, no lateral deflection w will occur until the critical load P_{cr} is reached. At that instant, the stable equilibrium configuration of the column is that of a slightly bent element. The point on the path of equilibrium where the magnitude of the applied load is equal to the critical load is called the *bifurcation point*, since at that point the path of equilibrium splits into two branches. The column may take either of two possible equilibrium configurations: the straight one or the slightly bent one. However, the solution of eq. (4) does not provide any information about the magnitude of the lateral displacement Δ at the midheight of the column, nor about the slope and the shape of the branched path of equilibrium, respectively at and beyond the bifurcation point. The lateral displacement Δ can take on any value, subject only to the requirement that it remain small within the meaning of the assumptions made in the derivation of eq. (4). These assumptions are evidently too restrictive to allow the path of equilibrium corresponding to the bent configuration of the column to be plotted beyond the bifurcation point.

A complete solution of the problem was first obtained by Lagrange⁶⁸. Although his formulation was somewhat different (cf. Ref. 102), it was based on exact expressions for strain and curvature, instead of the approximate relation given in eq. (1). The plot of the corresponding path of equilibrium is shown in Figure 4. It is seen that after the bifurcation point has been reached, the column, while in the bent position, can support a load larger than the critical load; the critical load itself is the same as the critical load predicted by the linear theory. However, the lateral deflections become very large, indeed so large as to be inadmissible from the point of view of using the column as a structural element.

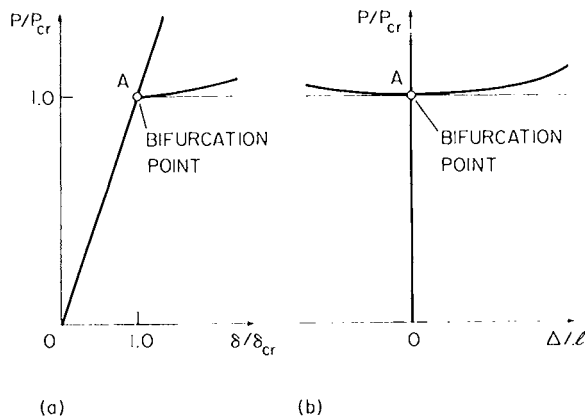


Fig. 4 Buckling Behavior of Ideal Elastic Strut—Nonlinear Theory

The observation that the critical load obtained from the exact solution is the same as the linear Euler critical load leads to the speculation that the nonlinearity included in the analysis need not be strong—approximate nonlinear strain-displacement expressions might suffice if one is interested only in the segment of the path of equilibrium in the immediate neighborhood of the bifurcation point. This turns out to be the case, and the simplest nonlinear theory based on small displacements and moderately large rotations leads to the following approximate expression for the longitudinal strain

$$\epsilon = \frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \quad (10)$$

This relation replaces the equivalent expression in eqs. (1). The practical importance of eq. (10) in the case of columns is clearly limited, since the exact solution due to Lagrange is available, as has been mentioned. However, it is of great importance in the equivalent problem of plates and shells, as will be discussed later in the paper.

Imperfect Elastic Column

Since ideally straight concentrically compressed columns are unattainable in practice, it is important to investigate the effect of geometric imperfections on the shape of the path of equilibrium. Some aspects of the behavior of imperfect elastic columns are illustrated in Figure 5. The column is assumed to have an initial midpoint deflection of Δ_0 . On gradually increasing the applied axial load, the center deflection increases (Figure 5a). By using a linear theory of the type given by eq. (4), the solution shown qualitatively in Figure 5b is obtained. The paths of equilibrium vary depending on the magnitude of the initial eccentricity. However, regardless of the magnitude of Δ_0 , the critical Euler load serves as an asymptote for the solution. As lateral deflections increase, the linear solution (based on small displacements) becomes less reliable. The more accurate solutions for the imperfect elastic column, based on a nonlinear equation more exact than eq. (10), are shown in Figure 5c. The paths of equilibrium suggest that if very large lateral deflections are permitted, the initial eccentricity will play only a very minor role in the final capacity of the column. It must be emphasized again that the resulting deflections are very large. From Figure 5c it is seen that the lateral deflection may be as large as 0.4 of the column length l —this occurs when a pin-ended column is bent into a complete "circle." Such large deflections are completely out of the range of magnitudes admissible in practical applications in structures.

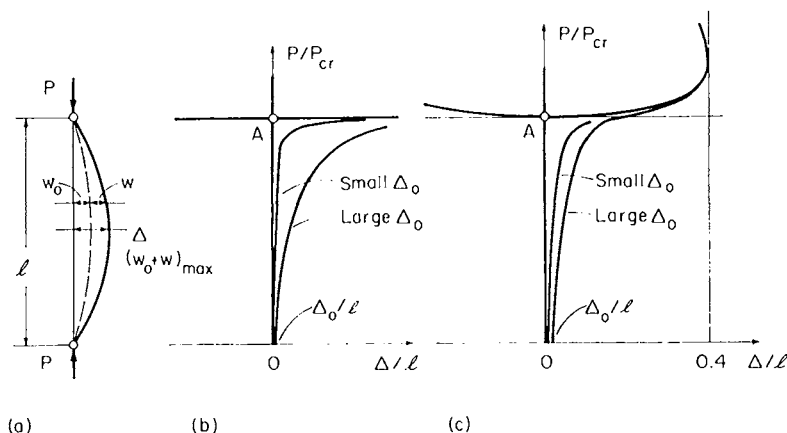


Fig. 5 Buckling Behavior of Initially Bent Struts¹⁰⁷

The important conclusions to be drawn from these considerations are as follows:

- The theory from which the Euler critical load is to be predicted need only be linear.
- The theory needed to predict the path of equilibrium of the column beyond the bifurcation point must be nonlinear.
- The nonlinear theory may be approximate, in that the lateral displacements need to be predicted in a relatively small range of values if they are to remain within practical limits.

Elastic Plate

A plate can be defined as a flat structural element the thickness of which is small compared to its other dimensions. Thus, a plate can be idealized as two dimensional, and the internal stress resultant system consists of eight quantities: three in-plane forces N_x , N_y , and N_{xy} , three bending moments M_x , M_y , and M_{xy} , and two transverse shears V_x and V_y . All these quantities are measured in units of force (or moment) per unit length of the plate section. It follows that whereas a column can be loaded only by forces directed along its axis, a plate can be subjected to two in-plane normal forces, and to in-plane shears. The displacements of the plate are the in-plane displacements u and v , and the transverse displacement w . A number of excellent texts on the theory of plates is available^{15,17,40,99,102,104,107}.

Using the Cartesian coordinate system x , y , and z , the strain-displacement relations and the strain-stress resultant relations of the theory of plates in its simplest formulation are:

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} & \kappa_x &= -\frac{\partial^2 w}{\partial x^2} \\ \epsilon_y &= \frac{\partial v}{\partial y} & \kappa_y &= -\frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \kappa_{xy} &= -\frac{\partial^2 w}{\partial x \partial y} \end{aligned} \quad (11)$$

$$\begin{aligned} N_x &= C(\epsilon_x + \nu \epsilon_y) & M_x &= D(\kappa_x + \nu \kappa_y) \\ N_y &= C(\epsilon_y + \nu \epsilon_x) & M_y &= D(\kappa_y + \nu \kappa_x) \\ N_{xy} &= C \frac{1-\nu}{2} \gamma_{xy} & M_{xy} &= D(1-\nu) \kappa_{xy} \end{aligned} \quad (12)$$

where, with h the uniform thickness of the plate,

$$C = \frac{Eh}{1-\nu^2} \quad D = \frac{Eh^3}{12(1-\nu^2)} \quad (13)$$

and ϵ , γ , and κ are, respectively, strains and changes of curvature at the middle surface.

The equations of equilibrium written for a differential element of the plate in a slightly deformed position and equivalent to eqs. (3) are:

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0 \\ \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} &= 0 \\ D \nabla^4 w - \left[N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right] &= 0 \end{aligned} \quad (14)$$

where

$$\nabla^4 \equiv \nabla^2 \nabla^2 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \tag{15}$$

and the Laplacian ∇^2 is defined by:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{16}$$

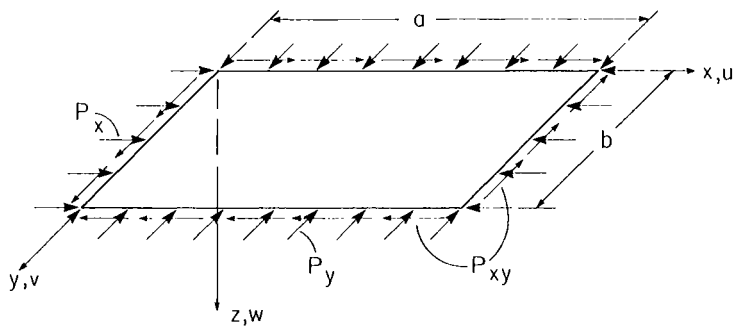


Fig. 6 Rectangular Plate Panel

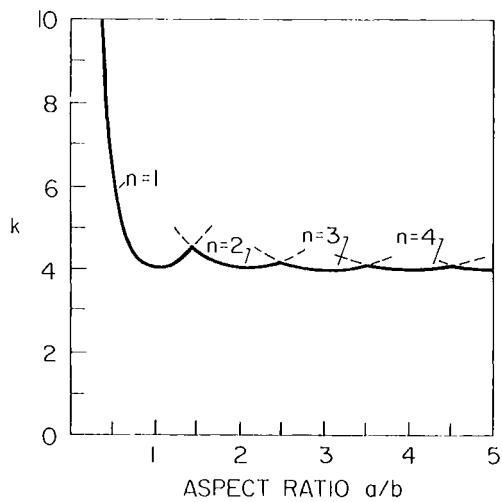


Fig. 7 Buckling Coefficient k for Uniaxially Compressed Simply Supported Rectangular Plates^{37,104}

As in the equivalent case of columns, eqs. (3), the in-plane force system is obtained by integrating eqs. (14a,b). For an edge-loaded plate such as that shown in Figure 6, for small deflections, the internal forces N_x , N_y , and N_{xy} are constant throughout the plate. Then the last of eqs. (14) is a linear differential equation in w , which can be solved relatively simply for the given boundary conditions of the problem. The boundary conditions that have to be used resemble those that occur in the beam problem. A complete discussion of the plate boundary conditions can be found in the references cited previously (cf. Ref. 104).

It is seen that eqs. (14) reduce properly to eqs. (3) if, in the limit, a plate becomes a column.

Solution of eqs. (14) was first obtained by Bryan in 1891¹⁸. The associated critical loads are sometimes called Bryan critical buckling loads.

In general, the buckling stress of an ideal, elastic, isotropic rectangular plate of length a and width b can be written as:

$$\sigma_{cr} = k \frac{\pi^2 E}{12(1-\nu^2)} \left(\frac{h}{b} \right)^2 \quad (17)$$

It is seen that, with $k=1$, this equation can be shown to reduce to eq. (7) for a plate-column. In the case of the uniaxially loaded plate simply supported along all four edges, Bryan obtained the explicit expression:

$$k = \left(\frac{a}{nb} + \frac{nb}{a} \right)^2 \quad (18)$$

where n is the number of half-waves physically significant in a given problem.

A plot of the coefficient k for a sequence of assumed number of waves n as a function of the plate aspect ratio a/b is shown in Figure 7¹⁰⁴. Since the lowest values determine the critical buckling load, only the solid lines of Figure 7 are relevant. Numerous curves of this type are available in the literature for other edge support conditions^{15,37,41}. In general, the critical buckling load is very sensitive to the manner in which a plate is supported. As an example, a plate simply supported along the loaded edges and free along the other two edges buckles in an entirely different mode than a plate simply supported along all four edges. The latter case is illustrated in Figure 8; a part of the plate buckles upward, and the remaining part buckles downward. The number of half-waves depends on the plate aspect ratio a/b . The larger this ratio, *i.e.*, the longer the plate is in comparison to its width, the greater is the number of waves. In the case of the plate shown in Figure 8, the number of half-waves in the x -direction is 3. This pattern of buckling is important in the investigation of buckling of folded plates⁹⁸ in the investigation of structures that have flat plates as one or more of their elements.

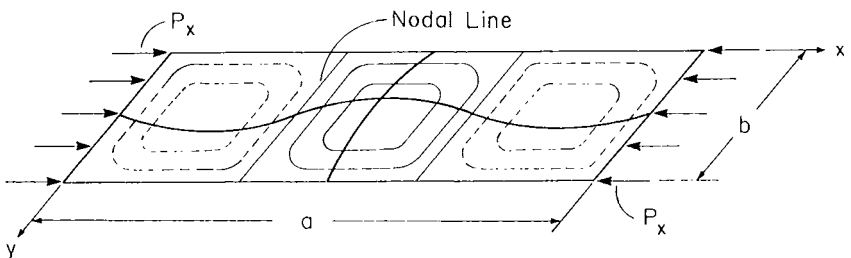


Fig. 8 Buckling Shape of a Simply Supported Plate Panel

As in the case of columns, the linear theory which resulted in eqs. (14) can be used only to predict the magnitude of the critical buckling load of an ideal plate; it cannot be used to predict the path of equilibrium beyond the bifurcation point—a non-linear theory is evidently required for this purpose. Guided by previous considerations, the simplest possible theory is sought which can serve as the basis for the calculation of the path of equilibrium beyond the bifurcation point. The expectation is that such a simplest possible theory will yield results that will describe the postbuckling behavior of plates sufficiently accurately in the relatively narrow zone just after the bifurcation point has been reached, and while the displacements, although moderate rather than small, are still limited so as to remain within the admissible limits. Such a theory was first proposed by von Kármán in 1910⁵⁸. The crucial modification of the linear theory occurs in the strain-displacement relations, equivalent to eq. (10) as follows:

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 & \gamma_{xy} &= \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \\ \epsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \end{aligned} \tag{19}$$

so that the effect of transverse deflections on in-plane strains is approximated. The remaining eqs. (11) and eqs. (12) remain unchanged. After some manipulation, the resulting governing system of differential equations can be written in a particularly simple form with the aid of the Airy stress function, named for the investigator who first proposed its use¹⁰³. The Airy stress function Φ is defined by the relations:

$$N_x = \frac{\partial^2 \Phi}{\partial y^2} \qquad N_y = \frac{\partial^2 \Phi}{\partial x^2} \qquad N_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \tag{20}$$

The result is:

$$\begin{aligned} \nabla^4 \Phi - Eh \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] &= 0 \\ D \nabla^4 w - \left[\frac{\partial^2 \Phi}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 \Phi}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 \Phi}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] &= 0 \end{aligned} \tag{21}$$

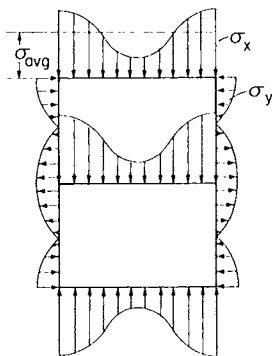


Fig. 9 Postbuckling In-Plane Stress Distribution in a Simply Supported Plate with Uniform End Shortening³⁹

This is a system of coupled nonlinear equations in Φ and w . Solution of this system of equations is by no means simple, and we shall not consider it here. Instead, the results in terms of the distribution of in-plane forces, and the path equilibrium, will be examined.

The type of pattern of stress distribution that develops in a buckled simply supported elastic plate is shown in Figure 9³⁹. Initially, the unloaded vertical sides are assumed free of stress. However, due to buckling, pulling stresses develop at midheight of the plate, and compressive stresses occur near the loaded edges at the top and at the bottom of the plate. As might be expected intuitively, the buckled plate carries smaller vertical stresses near the middle of the plate than along the side supports. Further results of the analysis of nonlinear behavior of rectangular simply supported elastic plates are illustrated in Figure 10. The bifurcation points A in both figures correspond to the Bryan critical buckling load. The results show clearly that, beyond the bifurcation point, the applied load that a restrained plate can carry continues to increase at a rapid rate along the stable curved path of equilibrium (it is important to note that the horizontal scale in Figure 10 is smaller by a factor of approximately ten than the same scale in Figure 5). Figure 10 shows also the path of equilibrium for plates with geometric imperfections—initially bent plates. As in the case of columns, a separate path of equilibrium is obtained for each condition of initial deflection. It should be noted that the behavior of unrestrained plates tends toward that of a column, as already mentioned in connection with the discussion of eq. (9). Finally, it should be noted that a plate can buckle as a result of the action of external shears applied along the boundaries¹⁰². This is a phenomenon that does not arise in the case of columns.

One may conclude that the observations made in connection with the buckling of columns apply also to the buckling of plates. The linear theory can be used to calculate the bifurcation point, nonlinear theory is necessary to study the postbuckling behavior, and an approximate form of the nonlinear theory is adequate for the purpose. The postbuckling behavior of unrestrained plates resembles that of a column. However, restrained plates, *i.e.*, plates supported not only along loaded edges, but also along unloaded edges, exhibit a significantly different behavior after the bifurcation point has been reached, the stable path of equilibrium continuing to rise. This means that a restrained plate can support additional loads beyond the critical which, in turn, means that the factor of safety in the case of restrained plates can be smaller than in the case of unrestrained ones. Similar conclusions hold regarding the difference in the behavior of plates under biaxial load, with the load along one of the axes being compressive, and the load along the other axis being either tensile or compressive. The plate is stiffened by a tensile load, and made less stiff by a compressive one.

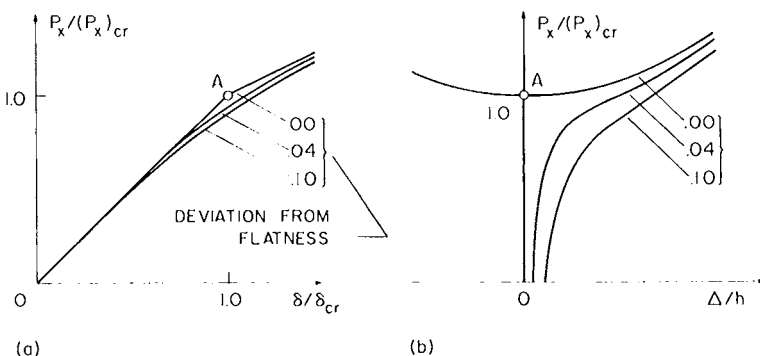


Fig. 10 Postbuckling Behavior of Ideal and Initially Bent Elastic Plates

Elastic Shell

Like a plate, a shell is an element the thickness of which is small compared to its other dimensions. Unlike a plate, a shell is not flat. Accordingly, it can be idealized as a two-dimensional surface curved in a three-dimensional space. At any point of the surface, the curvatures (and the radii of curvature) are defined for a given shell geometry, and the thickness h of the shell is assumed small compared to the radii of curvature. The location of a point on the surface of the shell can be described conveniently with the aid of the intrinsic curvilinear coordinates of the surface, usually those that are associated with the principal curvatures.

Because the geometry of a shell is significantly more complex than the geometry of a plate, no general agreement seems to exist regarding the form of the equations which should be used in the analysis of the shell behavior, including buckling. It appears that different shell geometries and different load conditions require different analytical formulations.

The simplest consistent theory of shells is due to Love⁷², and is usually referred to as *Love's first approximation*. Some inconsistencies were removed from Love's theory by Sanders⁸⁹ and independently by Koiter⁶⁵. The theory is based on the same assumptions as the Kirchhoffian theory of plates¹⁰⁴, and may be thought of as an extension of the plate theory to structures of surface-like geometries. As in the plate theory, there are eight independent stress resultants, some of which, to satisfy the demands of consistency, may be thought of as averaged values⁸⁹, and four conditions must be prescribed at the boundaries. The stress resultants satisfy constitutive relations similar to those of the theory of plates. Unlike the case of plates, the in-plane problem and the flexural problem, even for a linear case, are coupled: as a rule, curvature causes in-plane strain, and in-plane strain causes curvature of a shell.

For some shells, the equations of the Love theory can be significantly simplified. This is the case if the shell is shallow, or if its displacements vary rapidly over its surface. Many shells important in practical applications meet these conditions. The resulting theory is associated with the names of Donnell²⁶, Mushtari⁷⁸, and Vlasov¹⁰⁹; it is the simplest possible general theory of shells. It should be noted that the results obtained from the Donnell-Mushtari-Vlasov (DMV) theory are incorrect if the rigid body motions of the shell are significant, the error becoming larger as the significance of such motions becomes more pronounced.

A further source of error of the DMV buckling theory is as follows. In writing the equations of equilibrium of a differential element of the shell, a deformed position of this element is assumed. The deformed position is taken to be only slightly different from the initial geometry of the shell, since the displacements are assumed small. The prebuckling rotations of columns and plates are, in fact, negligibly small. This is not the case for shells, where prebuckling rotations will, as a rule, occur and may cause the problem to be nonlinear. Thus, depending on the nature of the problem, such prebuckling rotations may have to be taken into account. If they are not, the values of critical load will be erroneous. These rotations are not included in the DMV theory; consequently, it should not be used unless the rotations are small at least over a large portion of the surface of the shell.

In the following, the DMV theory will serve as the basis for a qualitative examination of the buckling behavior of just two types of shells: circular cylinders and panels, and spheres and spherical caps. These are the shell geometries that have been studied most extensively by investigators, both experimentally and analytically. It is probably accurate to say that the origin of the current understanding of the problem of shell buckling lies in these studies, and a consideration of these two geometries will bring out the main characteristics of shell buckling behavior. Finally, a mode of buckling different than the previously discussed bifurcation mode, *i.e.*, snap-through buckling, will also be considered.

Circular Cylindrical Shells

The earliest solutions for symmetric buckling of axially compressed cylinders were obtained by Lorenz⁷⁰ in 1908 and by Timoshenko in 1910¹⁰¹. The asymmetric buckling of such shells was investigated by Lorenz in 1911⁷¹ and by Southwell in 1914⁹⁶. A comprehensive study of buckling of circular cylinders was presented by Flügge in 1932³⁵. A detailed state-of-the-art survey is presented in the paper by Seide and Siev⁹³.

In the case of circular cylinders of radius R , with the coordinate axes x , $y = R\theta$, and z , the equations of the DMV theory are as follows.

Strain-displacement relations are:

$$\begin{aligned}\epsilon_x &= \frac{\partial u}{\partial x} + \left\{ \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right\} & \kappa_x &= -\frac{\partial^2 w}{\partial x^2} \\ \epsilon_y &= \frac{\partial v}{\partial y} + \frac{w}{R} + \left\{ \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right\} & \kappa_y &= -\frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left\{ \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right\} & \kappa_{xy} &= -\frac{\partial w}{\partial x} \frac{\partial w}{\partial y}\end{aligned}\quad (22)$$

The constitutive relations are:

$$\begin{aligned}N_x &= C(\epsilon_x + \nu\epsilon_y) & M_x &= D(\kappa_x + \nu\kappa_y) \\ N_y &= C(\epsilon_y + \nu\epsilon_x) & M_y &= D(\kappa_y + \nu\kappa_x) \\ N_{xy} &= C \frac{1-\nu}{2} \gamma_{xy} & M_{xy} &= D(1-\nu)\kappa_{xy}\end{aligned}\quad (23)$$

and the equations of equilibrium of a differential element, written for a slightly deformed position and corresponding to eqs. (14) for an elastic plate are:

$$\begin{aligned}\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0 \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} &= 0 \\ D\nabla^4 w + \frac{1}{R} N_y - \left\{ N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right\} &= 0\end{aligned}\quad (24)$$

As in the case of a plate, it is possible to introduce a stress function Φ defined by relations

$$N_x = \frac{\partial^2 \Phi}{\partial y^2} \quad N_y = \frac{\partial^2 \Phi}{\partial x^2} \quad N_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}\quad (25)$$

so that the governing system of equations becomes

$$\begin{aligned}\nabla^4 \Phi + \frac{Eh}{R} \frac{\partial^2 w}{\partial x^2} - Eh \left\{ \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right\} &= 0 \\ D\nabla^4 w + \frac{1}{R} \frac{\partial^2 \Phi}{\partial x^2} - \left\{ \frac{\partial^2 \Phi}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 \Phi}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 \Phi}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right\} &= 0\end{aligned}\quad (26)$$

In all these equations, (22) through (26), the terms in curly brackets are the nonlinear terms resulting from admission of moderate rotations into the stress-strain relations. It is useful to reiterate that these equations are valid for shallow shells, or for shells such that their displacements vary rapidly over the surface of the shell.

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It is seen that the constitutive relations, eqs. (23), are the same as the corresponding relations of the plate theory, eqs. (12). However, the stress resultant-displacement relations of the shell theory differ from the corresponding relations of the plate theory because of the presence of terms containing the transverse displacement w in the strain-displacements relations, eqs. (22).

Consider a circular cylinder in an axisymmetric state of stress due to an axial compression by forces P_x . Assume further that, with sufficient accuracy, the in-plane force $N_x = -P_x$, which follows from eqs. (22), (23), and (24) if $w = 0$. Then the second of eqs. (26) becomes

$$\frac{d^4 w}{dx^4} + \frac{P_x}{D} \frac{d^2 w}{dx^2} + \frac{Eh}{DR^2} w = 0 \quad (27)$$

where the last term follows from eq. (24c) on noting that by treating a longitudinal strip of the shell as a beam on an elastic foundation⁴⁵

$$N_y = \frac{Eh}{R} w \quad (28)$$

It is seen that, except for the last term, eq. (27) is identical to eq. (4), provided that EI is replaced by D and P by P_x .

The critical axisymmetric buckling load of the shell (always bearing in mind the assumed linear behavior) is the smallest characteristic value of eq. (27); it is given by the expression

$$P_{cr} = \frac{Eh^2}{R\sqrt{3(1-\nu^2)}}; \quad (29)$$

Expression (29) is valid for $l > 1.72\sqrt{Rh}$; this restriction on the value of l is due to the need for a modification of the value of the critical load if the shell is short, as shown by Batdorf¹³, when the end effects become important. Further, a long cylindrical tube may buckle also as an Euler column. This mode of behavior is not predicted correctly by eq. (29) since, as mentioned, the DMV theory fails in the case when rigid body motions become significant. Subsequent investigations showed that eq. (29) remains valid for axisymmetric buckling modes as well.

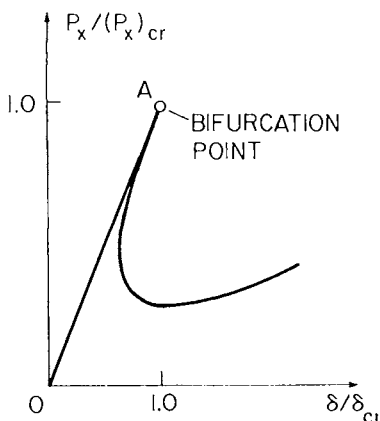


Fig. 11 Path of Equilibrium for an Ideal Axially Compressed Cylinder

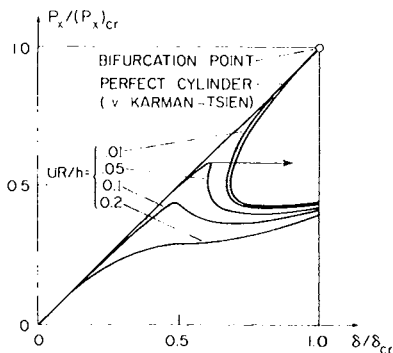


Fig. 12 Effect of Imperfections on Buckling of Axially Compressed Cylinders

As in the case of columns and plates, the linear solution based on eq. (27) yields only the value of the critical load, but no information about postbuckling behavior. A solution of the nonlinear eqs. (26) is required to obtain that information. After several attempts by a number of investigators, von Kármán and Tsien⁶¹ were the first to obtain a meaningful solution of the problem. Their results are shown in Figure 11. It is seen that after the critical load is reached, the secondary path of equilibrium falls markedly down, indicating a significant reduction in the load-bearing capacity of the shell—a behavior that differs from that of columns and plates. The von Kármán-Tsien solution, later improved by others^{9,47}, served as the basis for further developments which clearly pointed to the extreme sensitivity of buckling behavior of axially compressed circular cylinders to geometric imperfections. This conclusion was reached also by experimental investigators who showed that, even for extremely accurately built models, the theoretical critical load and the bifurcation point *A* are never reached.

The effect of geometric imperfections on the buckling behavior of axially compressed circular cylindrical shells was investigated on the basis of a nonlinear theory by Donnell and Wan²⁸, and further modified by Madison and Hoff⁷⁴. The shape of the imperfection was assumed the same as the shape of the deflected surface. An unevenness factor *U* was introduced, assumed to depend on the manner in which the shell is constructed, chosen in a manner analogous to that commonly used in defining the shape of a slightly bent column. The effect of the nondimensional factor UR/h on the buckling of cylinders is shown in Figure 12. The solution for an ideal shell constitutes an asymptote for the paths of equilibrium of imperfect shells. It is of interest to note that even for very small imperfections $UR/h = 0.01$, the maximum load reached is approximately only 83% of the critical load of the ideal shell. For larger imperfections, the largest load that the shell can support may be less than 50% of the critical load of the ideal shell. It is also important to note, as illustrated for the case $UR/h = 0.05$, that a snap through to a larger deflection takes place at the maximum attainable load (refer to the discussion of spherical caps for a qualitative description of the snap-through problem). These analytical results are in full accord with experimental observations. It may be concluded that axially compressed circular cylinders are extremely sensitive to geometric imperfections. The extent of this sensitivity is brought out in Figure 13. It is seen that the buckling capacity of a circular cylinder can be as little as 20% of the critical load of the ideal shell. The range of magnitude of imperfections, *i.e.*, the question whether the deviation from the theoretical shell shape can be of the order of mag-

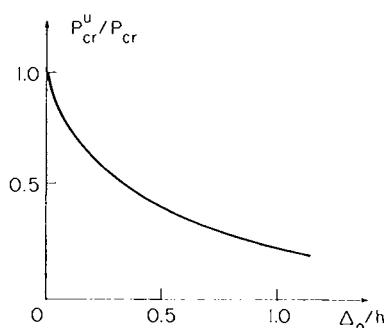


Fig. 13 Approximate Dependence of Ultimate Buckling Load on Initial Imperfections in Axially Compressed Cylinders

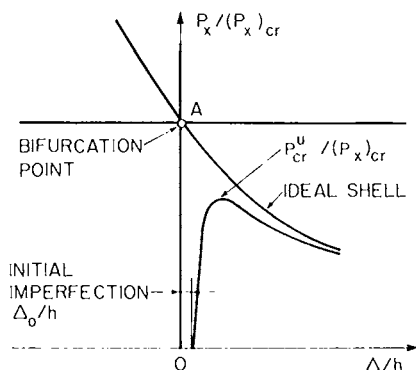


Fig. 14 Path of Equilibrium for an Ideal and an Imperfect Shell³⁸

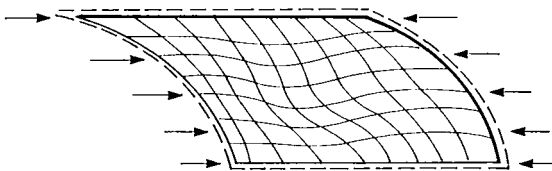


Fig. 15 Tendency of Cylindrical Panels to Buckle Inwards¹⁰⁷

nitude of the thickness of the shell, is rather uncertain. It appears that in the case of reinforced concrete shells used in building construction, the imperfections are small⁴⁴, while in the case of structures built using the slip-form technique, such as hyperbolic cooling towers, the imperfections of shape may be significantly larger²⁴.

A qualitative comparison of postbuckling behavior of ideal and imperfect shells is given in Figure 14³⁸. The downward tilt of the secondary path of equilibrium of the imperfect shell has a clear physical meaning. As shown in Figure 15, a cylindrical shell has the tendency to buckle inward. On the other hand, any outward movement is difficult since it is associated with tensions in the circumferential direction which stabilize the shell and cause it to gain additional load-bearing capacity.

It is useful at this time to consider a circular cylindrical shell subjected to a different kind of load, namely a torque⁷⁹. The paths of equilibrium for the case of the ideal and the imperfect shell are shown in Figure 16. It is seen that the behavior of the shell in torsion differs significantly from its behavior under axial compression. No precipitous decrease in the carrying capacity of the shell in the postbuckling range is observed. This behavior is further accentuated in the case of an internally pressurized cylinder⁷⁹, and becomes reminiscent of the behavior of restrained plates.

Thus, the buckling behavior of a circular cylinder may vary significantly, depending on the type of support, and also on the nature of the applied load.

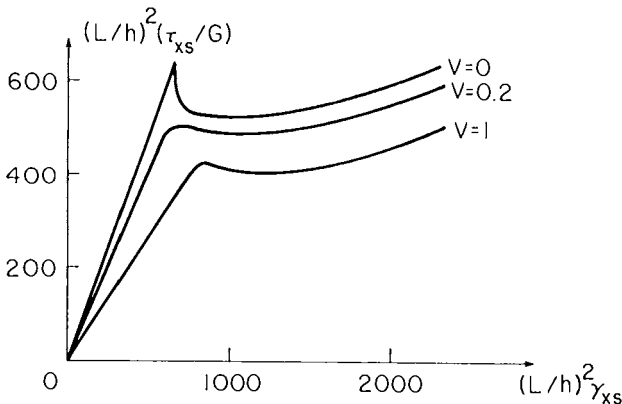


Fig. 16 Postbuckling Behavior of Ideal and Imperfect Cylinders in Torsion⁷⁹

Spherical Shells and Spherical Caps

Shells in the form of a spherical cap have been studied exhaustively with regards to their buckling behavior, both analytically and experimentally¹⁰². This may be due to the frequency with which this type of shell occurs in practical applications, and to the fact that a sphere represents a particularly simple geometry with a constant radius of curvature R . As in the case of cylinders, initial investigations involved the use of a linear theory. In 1915, Zoelly published his results giving the critical elastic buckling stress for a thin-walled sphere subjected to external normal pressure¹¹⁷. Zoelly's solution was based on the assumption that the buckling mode was symmetric. A more general case of asymmetric buckling was analyzed by van der Neut⁸¹ in 1932, resulting in the same value of the critical external pressure p_{cr}

$$p_{cr} = \frac{2E}{\sqrt{3(1-\nu^2)}} \left(\frac{h}{R} \right)^2 \quad (30)$$

The associated critical in-plane buckling force per unit of length of the meridian is

$$(N_x)_{cr} = \frac{Eh^2}{R\sqrt{3(1-\nu^2)}} \quad (31)$$

The same result can be obtained for a shallow spherical cap subjected to external normal pressure. It is seen that eq. (31) is identical to eq. (29) which gives the critical load of an axially compressed circular cylinder of radius R .

As in the case of axially compressed circular cylinders, there was poor correlation between the values of critical pressure predicted by eq. (30) and the buckling pressures observed experimentally. A nonlinear theory based on assumed stress-strain relations similar to eqs. (22) was proposed by von Kármán and Tsien⁶⁰. This theory was an extension of von Kármán's earlier work on large deflection of plates⁵⁸. Their analysis, as well as that of others¹¹¹, clearly showed that the postbuckling path of equilibrium of a spherical shell under external pressure exhibits a significant loss of load-bearing capacity. Also, such shells are very sensitive to geometric imperfections^{53,110}. In both these aspects, the behavior of a sphere is qualitatively identical to that of an axially compressed cylinder (cf. Figure 25).

Important recent results were derived by del Pozo F. and del Pozo V.^{84,85}, who succeeded in obtaining accurate, simple formulae for the value of the critical normal pressure, and the central angle of the buckled portion of the spherical cap, for isotropic as well as ribbed shells.

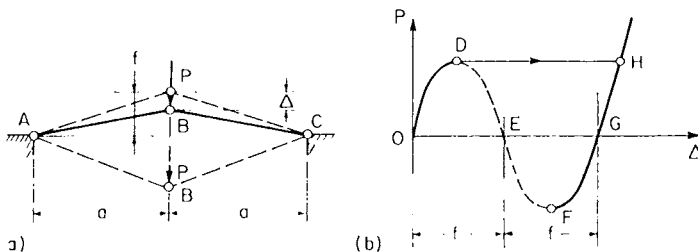


Fig. 17 Snap-Through Buckling of a Two-Bar Assembly

Snap-Through Buckling

A spherical shell can exhibit also a different type of response—snap-through buckling. This mode of buckling can be illustrated on the arrangement of two pin-ended elastic bars shown in Figure 17a, where the initial rise f is assumed small compared to total span. On applying a small force P , the bars compress and shorten, and the applied force moves through the deflection Δ . As the force P increases, it can reach a value such that the bars go through the horizontal position and reach a new position of equilibrium so that they are now in tension (point B' in Figure 17a). The force-deflection path—the path of equilibrium—which corresponds to this mode of behavior is shown in Figure 17b. On reaching the maximum force magnitude at point D , and since the path of equilibrium DF is unstable, the displacement suddenly increases to that which corresponds to point H . This is termed the snap-through buckling phenomenon which must be considered in the case of thin shells in addition to the previously examined bifurcation mode.

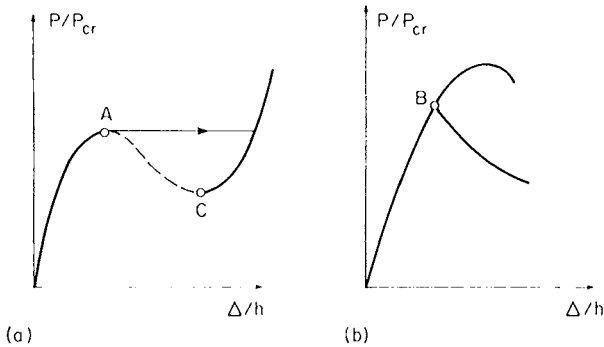


Fig. 18 Equilibrium Paths for Spherical Caps under Normal Pressure: (a) Axisymmetric Case, (b) Asymmetric Case (adopted from *Buckling of Bars, Plates, and Shells*, by Brush and Almroth¹⁷. Copyright by McGraw-Hill, Inc. Used with permission of the McGraw-Hill Book Company.)

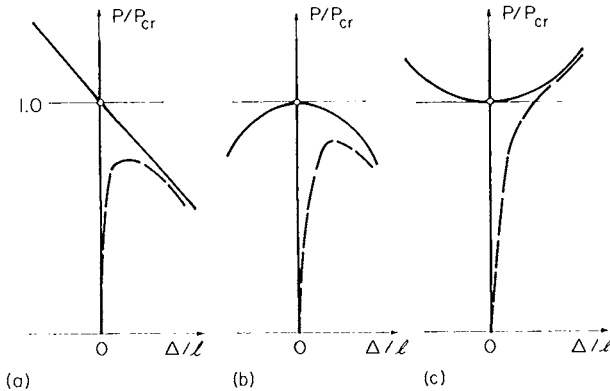


Fig. 19 Koiter Theory: Possible Paths of Equilibrium (adopted from *Buckling of Bars, Plates, and Shells*, by Brush and Almroth¹⁷. Copyright by McGraw-Hill, Inc. Used with permission of the McGraw-Hill Book Company.)

The spherical cap under external normal pressure affords an example of both these modes of buckling. Its physical behavior may be described as follows^{17,34}. For very shallow caps, the shell passes from the initially undeformed cap shape into a dish shape slowly, without the sudden snap-through. For slightly deeper caps, the path of equilibrium exhibits a maximum, and the cap snaps through as shown in Figure 18a. For deep caps, the bifurcation point occurs before the point of maximum on the primary path of equilibrium is reached. This condition is illustrated in Figure 18b, where it is designated as point *B*. Accurate solutions illustrating this behavior were found by Huang⁴⁹, confirming the exceptionally carefully obtained experimental results of Krenzke and Kiernan⁶⁷. Later, this problem was studied by Fitch and Budiansky³⁴ who showed that the mode of buckling—bifurcation or snap-through—depends not only on the geometry of the cap (or, how shallow it might be), but also on the extent of the applied concentric load. The results obtained by the investigators cited above show that the critical loads determined from the linear analysis are significantly too high, with actual capacities perhaps as little as 20% of the critical load of the ideal shallow shell, and somewhat larger for deeper caps. Again, in this the spherical shell strongly resembles in its buckling behavior the circular cylinder under axial compression.

The Koiter Theory

At several points in the preceding paragraphs the observation was made that the linear theory suffices in order to calculate the critical buckling load, but that it is not capable of examining postbuckling behavior: neither the slope of the secondary path of equilibrium nor its curvature in the immediate vicinity of the bifurcation point can be established. At the same time, the importance of postbuckling behavior on the load-carrying capacity of shells was repeatedly stressed, and it was noted that, in general, the study of such behavior requires a nonlinear formulation and solution of the problem. Such solutions are very complex, indeed; they have been obtained, so far, for just a few simple shell geometries.

A significant step in the direction of providing an answer to the problem of postbuckling behavior of shells in the immediate vicinity of the bifurcation point was made by Koiter⁶⁴. It is known as the Koiter initial postbuckling theory¹⁷, and it permits a study of the slope and curvature of the secondary path of equilibrium in the immediate vicinity of the bifurcation point on the basis of a linearized formulation. Because of this linearization, the problem becomes significantly more tractable, and asymmetric buckling of shells of any geometry can be investigated with its aid.

The conclusions that can be drawn from the Koiter theory are as follows. Three types of the path of equilibrium in the vicinity of the bifurcation point of an ideal shell have been discerned: a straight line, a parabola with a maximum at the bifurcation point, and a parabola with a minimum at the bifurcation point. These three paths are shown in Figure 19, where the corresponding paths of equilibrium in the case of shells with initial imperfections are illustrated with broken lines. In the first and second cases, the shell is sensitive to initial imperfections, and experiences a reduction in load-carrying capacity beyond the bifurcation point. In the third case, the shell is not sensitive to initial imperfections, and an increase in the load-carrying capacity beyond the bifurcation point may occur.

It should be noted that the Koiter initial postbuckling theory, because of the linearization process, is exact only at the bifurcation point; it is only approximate beyond it. In the case of some shells, such as an oval cylinder, the secondary path of equilibrium may have the shape shown in Figure 20^{17,52}. Thus, the oval cylinder is seen to be sensitive to the initial imperfections, and buckling may occur at a load significantly smaller than the critical load. However, after this event has taken place, the path of equilibrium takes a turn upwards, and the cylinder experiences an increase in its load-carrying capacity. This second part of the behavior cannot be predicted by the Koiter theory. However, such phenomena do not appear to be common, and the

error is on the conservative side. It seems fair to say that the Koiter initial postbuckling theory represents one of the most important contributions in the field of stability of elastic systems since the initial investigations of Euler and Lagrange.

INELASTIC BUCKLING

So far, the discussion has been limited to elastic columns, plates and shells. The assumption of elastic behavior does not apply to reinforced concrete elements. A typical stress-strain diagram for concrete is nonlinear almost from the beginning of the loading process¹¹³. Similarly, reinforcing steel usually exhibits a pronounced yield point¹¹³. Finally, reinforced concrete assemblies have strongly nonlinear material characteristics, intensified through the cracking of concrete. Thus, it is important to consider the behavior of structural elements—columns, plates and shells—with non-linear mechanical properties of materials.

The basis for determining the carrying capacities of inelastic geometrically imperfect columns was provided in 1899 by Engesser²⁹ and Considère²³, and in 1894 by Jasinski⁵⁵. Their work laid the foundation for the tangent and reduced (double) modulus theories of buckling. Some further refinements were introduced more recently by Shanley⁹⁵. By simply replacing the elastic modulus E by the tangent modulus E_t in the Euler buckling load, eq. (6), the critical buckling load of an ideal inelastic column is obtained. Recognition of an elastic process in the unloading fibers of a buckling column leads to the double modulus theory^{83,102}. Both these theories yield very similar results: they appear to apply reasonably well also to slender reinforced concrete columns¹⁶.

A procedure for determining the inelastic buckling of geometrically imperfect columns stems from the classical theory of von Kármán⁵⁹. The approach consists of the following stages. First, the elasto-plastic mechanical properties of a material are carefully defined, *i.e.*, its stress-strain relation is established. Then, a compatible deformed column shape corresponding to a given axial force is determined. For exam-

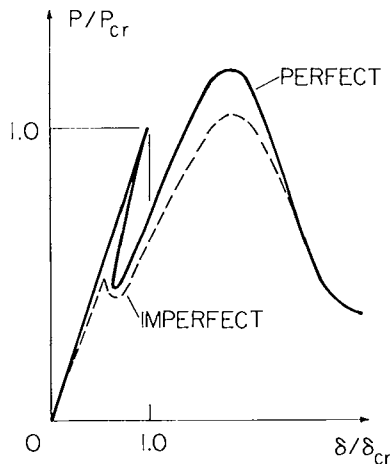


Fig. 20 Path of Equilibrium of an Oval Cylinder⁵²

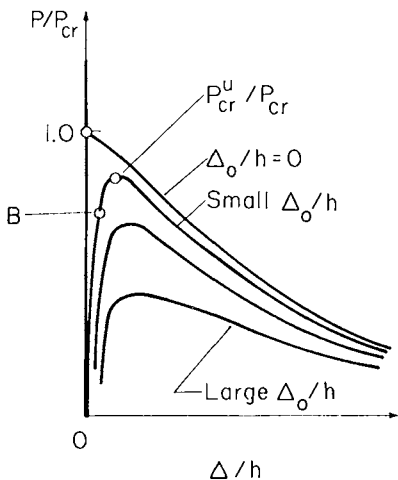


Fig. 21 Buckling Behavior of Eccentrically Loaded Elasto-Plastic Struts

ple, for a small initial ratio Δ_0/h , such a solution could lead to point *B* in Figure 21. By making a large number of such equilibrium solutions for different bent configurations of an elasto-plastic column, an entire path of equilibrium for an initially assumed ratio of Δ_0/h can be established. The maximum of this path of equilibrium is the ultimate capacity P_{cr}'' of a column, corresponding to the assumed Δ_0/h . This very laborious process has been simplified by, among others, Chwalla²⁰, and by Westergaard and Osgood¹¹². It was applied with notable success to slender reinforced concrete columns by Broms and Viest¹⁶.

A representative diagram from Ref. 16 is shown in Figure 22. These analytically derived curves for the ultimate capacity of reinforced concrete pin-ended columns are in good agreement with experimental results. It should be noted that the specimens analyzed in Ref. 16 were symmetrically reinforced; caution must be exercised when extrapolating these results to other situations.

A rational theory of inelastic plates was developed by Ilyushin⁵⁴ and Stowell⁹⁷, with appropriate modifications it can be applied to reinforced concrete plates. According to this theory, the material is assumed to be inelastic in the loaded direction, and a tangent modulus is assumed in the definition of its stiffness. In the unloaded, orthogonal direction, the plate is assumed to remain elastic. Then, for uniaxial loading, the plate equation analogous to eq. (14c) is

$$D \left[\frac{E_t}{E} \frac{\partial^4 w}{\partial x^4} + 2\sqrt{E_t/E} \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] - N_x \frac{\partial^2 w}{\partial x^2} = 0 \quad (32)$$

Equation (32) is of the type developed earlier by Huber⁵⁰ for orthotropic rectangular plates, and its solution can be readily obtained. Successful attempts to apply such solutions to the buckling analysis of reinforced concrete plates have been made recently by Swartz and Rosebraugh⁹⁸, who compared them with their experimental results. Good correlation was also achieved by assuming isotropic plate behavior based on the use of a tangent modulus in both directions. Thus, the refinements introduced by eq. (32) do not appear necessary. It seems that similar conclusions hold for reinforced concrete shells, although here analytical and experimental evidence appears to be rather limited.

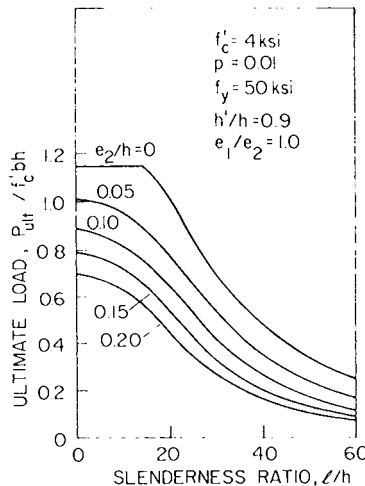


Fig. 22 Graphs of Ultimate Buckling Load for Reinforced Concrete Hinged Columns¹⁶

Techniques similar to those employed by Broms and Viest in the analysis of slender reinforced concrete columns¹⁶ might be applied in the analysis of inelastic plates and shells with initial imperfections. However, the problem is vastly more complicated, and few solutions are available. Recently, progress in the analysis of inelastic shells was made with the aid of computers⁶⁹. The shell (or plate) is idealized either as an assembly of inelastic elements with material properties that vary depending on the stress level, or as an assembly of layered elements that represent concrete and reinforcement, and that may possess inelastic material properties. The latter approach has been applied successfully to several types of shell surfaces, including hyperbolic paraboloids and surfaces of revolution⁶⁹. So far, geometric nonlinearity has been restricted to small strains and large rotations. As yet, conclusions which would permit the use of a simplified analytical procedure are lacking.

CREEP BUCKLING

In the discussion of inelastic buckling, instantaneous behavior with no time effect was assumed. However, shrinkage and creep due to sustained constant load are known to cause additional deformations. These long-term deformations are influenced by many factors, including exposure, quality and age of concrete at the time of loading, amount of reinforcement, etc. These factors are recognized in codes⁴ and, based on experimental evidence, empirical magnification factors are introduced to account for these phenomena¹⁵. The current state-of-the-art is assembled in a recent publication⁵. Virtually all of the available data, however, deal with the long-term deflection of beams. Only a limited amount of information is available on long-time load effects on columns¹⁶ and even less on plates and shells. The work that is available has often been directed to the study of the effects of long-term load on short reinforced concrete columns, typical of the situations encountered in buildings¹⁰⁸. Nevertheless, such data as are available provide a good basis for the understanding of the behavior of slender reinforced concrete compression elements. Conclusions are summarized in the following.

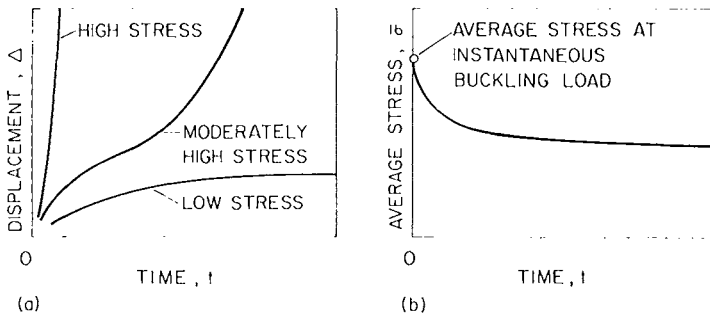


Fig. 23 Creep Buckling of Struts: (a) Effect of Stress Level on Transverse Deflection, (b) Buckling Capacity Variation with Time (adapted from Ref. 87)

Long-term deflections due to shrinkage and creep exceed those that occur at the initial application of load, concrete creeping under sustained stress. As a result, the stress in the compressive steel reinforcement increases progressively, thereby decreasing somewhat the compressive stress in the concrete. The stress in the tension steel—if it exists—increases only very slightly, because the effective lever arm is reduced⁸². This behavior is ever changing with time, and is very complex. Qualitatively, it can be visualized as shown in Figure 23^{87,94}. For a low axial stress, the initial lateral deflection Δ_0 , caused by the accidental eccentricity of the applied load or by end moments, increases but tends to an asymptote (Figure 23a). The ability of reinforced concrete elements to reach equilibrium in this manner can be attributed to the presence of compressive reinforcement, a fact clearly recognized by codes⁴. For improperly designed compression elements, the displacements continue to grow without bound. However, even a correctly designed slender element can exhibit different load-carrying capacities depending on the length of time a load is sustained. This kind of response is illustrated in Figure 23b. A column shows its greatest strength for an initial load of a virgin specimen. For a sustained load, the longer the period of load application, the lower the collapse load. The extent to which the load-carrying capacity of a column is reduced with time has not been definitely established¹⁶.

Methods for the investigation of creep buckling of columns were developed largely in response to the needs of the aircraft industry, and were mainly concerned with the behavior of elements subjected to high temperatures^{46,51,107}. The usual methodology consists of idealizing the mechanical properties of materials by means of a Maxwell and/or Kelvin model⁷⁵. In this manner, the problem is treated as linearly viscoelastic, although solutions based on nonlinear viscoelastic or viscoplastic behavior are also available⁶². Fortunately, solutions based on the concepts of linear viscoelasticity appear to be satisfactory for reinforced concrete¹¹³.

Early work on the analysis of effects of creep on reinforced concrete elements can be traced to the investigations of Dischinger²⁵; it was elaborated upon more recently by Kóllár and Dulácska⁶⁶. Refined approaches for creep analysis of reinforced concrete based on hereditary concepts with the use of the Boltzmann-Volterra superposition principle have been advanced by Arutyunyan¹¹. It appears that a sufficiently accurate approach consists of the introduction of a creep modulus of elasticity E_{cr} , approximately one-third the value of the modulus of elasticity E , into the critical load formulae. This conclusion has been further justified by a series of careful tests¹⁰⁶ performed on micro-concrete models by Vandepitte *et al.* Nevertheless, this approach involves significant simplification and hence requires the application of somewhat larger factors of safety.

DYNAMIC BUCKLING

In some applications, reinforced concrete columns, plates or shells may be subjected to rapidly varying or transient loads. If such loads are applied relatively slowly, *i.e.*, if their frequency is of the order of magnitude of the natural frequency of the structure, no special consideration need be given. Wind and seismic loads fall within this category, since the structural response for such loads is sufficiently slow not to affect the material properties (although the hysteretic behavior should be included in the analysis). In the rare instances of application of a blast load of very short duration measured in microseconds—load frequency being significantly smaller than the natural frequency of the structure—a dramatic change in element behavior takes place. The mechanical properties of materials are significantly increased and, due to the inertia effect, the critical buckling load may be increased severalfold. A qualitative illustration of this behavior is given in Figure 24¹⁰⁷. It appears that the effects of dynamic loading of plates and shells are qualitatively similar.

METHODS OF ANALYSIS

From the preceding discussion of the nature of the buckling phenomenon in shells it is clear that, unless analytical or experimental evidence establishes beyond any doubt that a shell of given geometry, with given support conditions, and subjected to given load, experiences no reduction in the critical load in the postbuckling range, the analysis of the problem must be based on a large deflection theory and that, in addition, the effects of possible geometric imperfections have to be taken into account.

The matter is further complicated by the material characteristics of reinforced concrete: cracking, creep, the presence of reinforcement, and the inelasticity of materials. Thus, the analysis becomes a truly formidable task and it is not surprising that only a very few solutions have been obtained by "classical" methods. This is true even if the large deflections and the geometric imperfections are the only effects taken into account, the reinforced concrete material characteristics being neglected. Some of these solutions of great elegance, shedding new light and leading to a new understanding, have been discussed in preceding paragraphs.

The modern era of shell analysis, including buckling analysis, began with the advent of the electronic computer, which made possible a rational approach to an almost intractable problem. Thus, to speak of modern shell buckling analysis is to speak of numerical analysis based on the use of computers. In the following, such numerical techniques are briefly reviewed. For a detailed study of these questions, reference should be made to the review by Schnobrich⁹¹ in this volume.

Numerical Integration of shell equations.

This is a classical numerical integration procedure adapted to the computer. The technique is applicable to those shell problems which can be reduced to an ordinary differential equation: primarily the problems of shells of revolution. The problem is formulated as an initial value problem posed in terms of a system of first order differential equations. This system is then integrated numerically using an algorithm such as the Runge-Kutta method²¹, usually in conjunction with a matrix transfer method⁵⁶. The Runge-Kutta algorithm is self starting; the value of the function at the i -th step is obtained from its value at the $(i-1)$ -th step. Refinements through the use of the value of the function and its derivatives at several previous steps are possible. Accuracy can be improved by iterating at every step. Because of the round-off errors due to the exponential form of the solution it may be necessary to subdivide the problem into several subproblems of suitable range; this increases the amount of numerical work, and the required computer time.

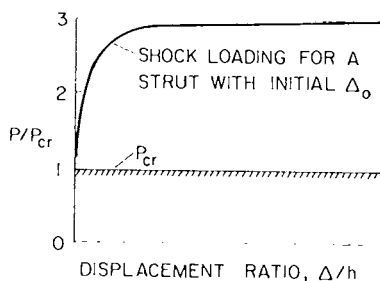


Fig. 24 Increase in Strut Buckling Capacity under Shock Loading¹⁰⁷



Finite Difference Procedures

The finite difference procedures in classical form are not well suited to the computer because of the need to treat each problem separately—it does not seem practical to set up a general computer program capable of treating all the different boundary conditions without modification.

A form which is well suited to the computer is the finite difference energy method. First proposed by MacNeal in 1953⁷³, the method can be classified as a numerical variational procedure. All quantities which enter into the calculation are discretized through the introduction of suitable finite difference expressions. The total strain energy of the system, as well as the potential of the work done by the external forces, is obtained by summation over the surface of the shell. Next, a system of algebraic equations is obtained by extremizing the expression for the total potential energy of the system written in terms of nodal displacements, with respect to these displacements. The method is applicable to nonlinear as well as linear problems. An established computer program, STAGS^{7,8}, of great power and generality, is based on this procedure.

Finite Element Method

This is a computer-oriented method that can be fairly said to have revolutionized the analysis of structures. It is particularly well suited to creation of general computer programs based on the displacement method of structural mechanics. It is a discretization procedure in which the structure is divided into a finite number of component elements connected at a finite number of nodes. The elements are carefully designed to possess properties such that the conditions of equilibrium and compatibility are satisfied at the nodes, and the conditions of continuity are satisfied (in general) along the joint lines between the neighboring elements. Literature of the finite element method is replete with various types of finite elements ready for use in a given case¹¹⁶. The elements include: flat, curved, triangular, rectangular, quadrilateral and other elements, with a varying number of nodal points.

From the point of view of the analysis of buckling of shells, recent important progress includes the development of shell elements capable of handling large deflections, geometric imperfections, cracking, creep, inelastic material properties, and steel reinforcement. Scordelis⁶⁹ developed a layered element which he used in the analysis of hyperbolic paraboloid shells through the elastic and inelastic stages of behavior. The analysis is capable of predicting shell behavior up to failure; surface of origin and extent of cracks, as well as their progression with load, can be predicted. Therefore, the procedure is suitable for the analysis of buckling of reinforced concrete shells. Another systematic line of attack on the problem of prediction of the behavior of concrete shells through the inelastic stage and up to failure is being followed by Dutch researchers⁴².

Experimental Analysis

Experimental analysis of buckling is generally recognized as a valid approach to the problem of establishing the path of equilibrium of a shell structure^{2,77}. In general, if the material characteristics of the prototype are to be successfully simulated, the model has to be made of concrete (and hence relatively large), or of microconcrete. A number of difficulties arise in that it is not possible to scale down the cement component, and the amount of water in microconcrete is relatively greater than in the standard mix. In general, the concrete and reinforcement should be simulated by materials with the same stress-strain characteristics through the inelastic range, a difficult undertaking. Nevertheless, until the recent progress in the finite element and other computer methods of analysis of shells, microconcrete model analysis appears to have been the only reasonably reliable technique for studying the response of concrete shells throughout the whole range of behavior, from elastic, through inelastic to failure.

However, it should be noted that because of the great inherent strength of shells, care has to be taken to design an experiment using a concrete shell model in such a way that the failure is unequivocally due to a loss of stability. For a detailed review of experimental analysis of buckling of concrete shells, reference should be made to the paper by Billington and Harris¹⁴ in this volume.

CURRENT PRACTICE IN BUCKLING ANALYSIS

In the following, the recommendations for the analysis of buckling of concrete shells by two leading professional organizations are reviewed. This is done with the expectation that such recommendations reflect current practice. The organizations are: the American Concrete Institute (ACI) and the International Association for Shell and Spatial Structures (IASS).

American Concrete Institute

In 1959, ACI Committee 334 on concrete shells was formed. Working under the chairmanship of Anton Tedesco, the committee published in 1964 a report² that was divided into two parts. *Criteria* covered general design considerations. *Commentary* discussed data of general interest to shell designers. In the *Criteria*, the discussion of stability was limited to the sentence (paragraph 3):

When investigating shells for stability, consideration shall be given to the possible reduction in the value of the buckling load caused by large deflections, creep effects, and the deviation between the actual and theoretical shell surface.

This brief paragraph was intended to "red flag" the problem, directing the designer's attention to the uncertainties of buckling analysis of concrete shells. The list of the possible buckling load reducing effects was by no means complete. As an example, no mention was made of the nonlinear characteristics of concrete and steel materials, or cracking of concrete. No attempt was made to provide a formula or a procedure for the calculation of buckling loads within this portion of the report.

In contrast to the intentional terseness of the *Criteria*, in the *Commentary* the discussion of stability occupies approximately 30% of the text. The problem of buckling of shells is discussed qualitatively. The relative importance of postbuckling behavior for different types of shells is brought in, and the significant reduction in the buckling load when both principal in-plane stresses are compressive is mentioned. Several methods of estimating buckling loads are listed, including energy, equilibrium and compatibility, and experimental methods, but no details are given. Some general suggestions are made regarding acceptable procedures in obtaining an approximate estimate of the buckling load. The report assumes that information on buckling of ideal shells for cylinders and spheres may be used as the starting point of the analysis. For *synclastic shells* it is stated that "...an estimate of the critical stress may sometimes be obtained by comparing the given shell to a cylinder or a sphere." For *anticlastic shells* "...it is often possible to use the linear buckling theory." The report recommends that the *combined apparent safety factor* F calculated from the formula

$$F = \frac{F_1 F_2}{F_1 + F_2} \quad (33)$$

be not less than 5. In this formula, F_1 and F_2 are the apparent safety factors calculated for an ideal shell in the direction of the principal radii of curvature, using an analogous sphere (or cylinder) in each direction. The rather large value of the combined apparent factor of safety F was intended to allow for the uncertainties of the various effects not included in the calculation. The effect of creep was allowed for as follows:

Assume a reduced value of E or, knowing the principal membrane forces at any point of a given shell, determine the tangent modulus of elasticity. Divide the tangent modulus of elasticity by a multiplier for long-term deflections. The multiplier shall be not less than 2. The use of a reduced E accomplishes the same purpose as a buckling load reduction factor.

Finally, the report mentions that a shell with two layers of reinforcement has greater buckling load than a shell with only one layer. In part III of the report, a selected bibliography, six references to buckling of shells are listed.

In general, the report reflected the concern with and the imperfect state of knowledge of buckling of shells at the time of its publication in 1964, and relied on the expertise of designers and their ability to make use of existing literature in the field.

The report of ACI Committee 334 had a significant influence in the sense that it was the source for several national recommendations for the design of concrete shells. In the United States, it served as the source for what became in time Chapter 19 of ACI Standard 318-77⁴, and the accompanying *Commentary*, ACI 318R-77⁴. The paragraph pertaining to buckling of concrete shells in the Standard is essentially the same as the paragraph in the *Criteria* part of the ACI Committee 334 Report cited above. However, the corresponding Commentary was cut significantly, being reduced to but four lines in paragraph 19.2.4, where the designer is referred to another publication for theoretical and practical guidance in the evaluation of the effects of creep. The list of references pertaining to the problem of stability of shells was also pared down in the ACI 318R-77 to only one reference, that an old one dating back to 1952, and not included in the list of references in the ACI Committee 334 Report.

It seems fair to say that the problem of stability of concrete shells in the *Commentary* to the ACI Standard was cut to the point where almost no information of use to a shell designer survived. It is of interest to note that in accordance with the standard procedures of the American Concrete Institute, the text of the ACI Committee 334 Report⁴ is currently under review and that this text will presumably become the basis for a review of Chapter 19 of the *ACI Standard* and *Commentary* in time for the 1983 edition.

International Association for Shell and Spatial Structures.

A Working Group on Recommendations for Concrete Shells and Folded Plates was formed in 1969, and it developed a text intended to serve as a recommended practice in the design, analysis and construction of concrete shells used in buildings (shells such as cooling towers, etc., are excluded). The final text of the IASS Recommendations was approved in 1976, and publication was completed in 1979⁷⁷. The question of stability of reinforced concrete shells is discussed in paragraph 3.5, and is based largely on Kóllar and Dulácska⁶⁶.

After a qualitative discussion of the problem, and having pointed out the importance of postbuckling behavior, recommendations are given for the calculation of buckling loads as follows:

- (1) Compute the initial buckling load as predicted by the linear theory for an ideal shell.
- (2) Ascertain if the problem falls within the category of those for which postbuckling behavior does not govern; this is done on the basis of established analytical or experimental results which show conclusively that the shell does not experience a reduction in the buckling load in the postbuckling range. If this is the case, modify the buckling load for geometrical imperfections and materials effects as outlined in paragraph (4) below.
- (3) If the problem does not fall within the scope of paragraph (2) above, the initial linear buckling load is presumed to decrease in the postbuckling range. If this is

the case, modify for large deflections as follows:

- Compute (or obtain from experiments) the deflection w'_0 of the shell, the deflection $w_0 = w'_0$, and the deflection-to-thickness ratio w_0/h .
- Obtain reduced upper critical load p_{cr}^u from the graph in Figure 25, or from similar graphs or solutions which might be available in the literature for shells of given geometry and load. If no solution is available, use the lowest curve in Figure 25, *i.e.*, the curve applicable to an axially compressed cylinder and a spherical cap.

After the modification of the critical load has been completed, proceed to paragraph (4) below.

- (4) Modification for geometric imperfections (shape deviation). Calculate, or assume, the shape imperfection w''_0 . Compute total deflection, $w_0 = w'_0 + w''_0$, and the ratio w_0/h . Obtain the reduced value of the upper critical load p_{cr}^u from Figure 25 as previously explained in paragraph (3).
- (5) Modifications for material properties.
 - Effect of creep: Calculate reduced value of the creep-reduced modulus of elasticity of concrete.

$$E_{cr} = \frac{E_c}{1 + C_u} \quad C_u = 4 - 2 \log f_c \quad (34)$$

where E_c is the modulus of elasticity of concrete and f_c is the strength of concrete at the time of loading measured in N/mm^2 . Then reduce the previously calculated value of the critical load p_{cr}^u in the same proportion as the reduction of the value E_c .

- Effect of reinforcement and cracking: Compute the quantity:

$$\frac{E_s}{E_{cr}} \cdot \frac{A_s}{A_c} \quad (35)$$

where E_s is the modulus of elasticity of steel, A_c is the area of concrete, and A_s is the area of steel in one direction only. Obtain from Figure 26a the value of coefficient ψ . Next, for this value of ψ , and the calculated value of w_0/h obtain from Figure 26b the value of $p_{cr, reinf}^u$.

- Effect of inelasticity of concrete: Compute the critical load p_{cr}^{plast} from the approximate interaction formula:

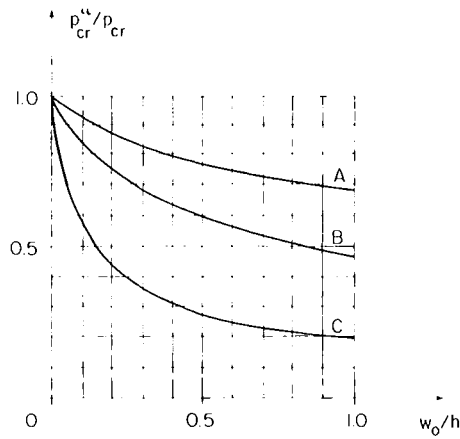
$$\left(\frac{p_{cr}^{plast}}{p_{plast}} \right)^2 + \left(\frac{p_{cr}^{plast}}{p_{cr, reinf}^u} \right) = 1 \quad (36)$$

where p_{plast} is the load that a shell with a computable eccentricity w'_0 of the compressive in-plane forces can carry, if buckling is neglected. The IASS Recommendation suggests that in the calculation of p_{plast} , the thickness of shells thinner than 8 cm be reduced by 1 cm, to allow for the possible inaccuracies of construction resulting in a shell thinner than designed.

The IASS Recommendation provides that the factor of safety $F.S.$

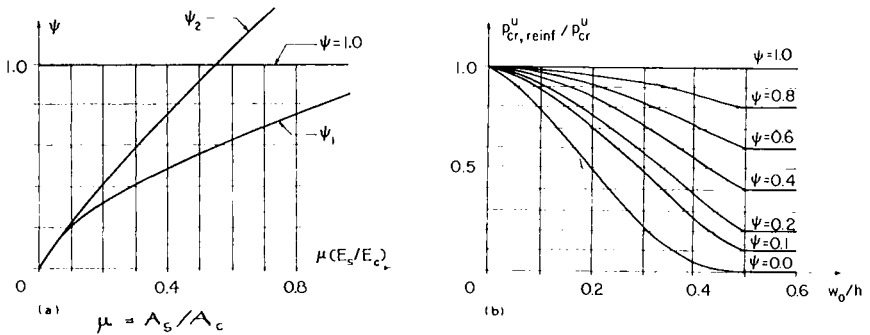
$$F.S. = \frac{p_{cr}^{plast}}{p} \quad (37)$$

where p is the actual pressure, be not less than 1.75 for shells that meet the conditions of paragraph (2) (postbuckling behavior does not govern), and not less than 3.5 for shells of paragraph (3) (postbuckling behavior governs).



- A. Long cylinder compressed in ring direction
- B. Short cylinder compressed in ring direction
- C. Axially Compressed cylinder and sphere under radial pressure

Fig. 25 Reduction in Critical Load Due to Postbuckling Behavior and Initial Imperfections



- Ψ - coeff. for uncracked concrete without reinforcement
- Ψ_1 - coeff. for cracked concrete with one middle layer of reinforcement
- Ψ_2 - coeff. for cracked concrete with reinforcement in two layers near the surfaces

Fig. 26 Modification of Critical Load Due to Presence of Reinforcement and Cracking
(a) Values of Coefficient Ψ , (b) Reduction of Buckling Capacity⁷⁷

It is seen that the IASS Recommendations attempt to provide in a rational manner for a number of effects (although not necessarily all) which might influence buckling in a concrete shell. Throughout, a conservative approach is taken. The effects of large displacements, geometric imperfections, creep, reinforcement and cracking are all cumulative; similarly, the effect of inelasticity of concrete is calculated from a formula which includes the other effects. Another example of the extent to which the IASS Recommendations approach is conservative can be seen in the suggestion that if no solution of the buckling problem can be found in the literature for a given shell, the very conservative low value of the critical load applicable to cylinders in compression, or spherical caps under radial pressure, be used. If all of the various cumulative effects take place, the critical load may be as little as 1% to 2% of the linear buckling load; the allowable load is even less, since a factor of safety is yet to be applied. A more detailed discussion of the load levels to be used in the design of reinforced concrete shells of different geometries can be found in the companion papers in this volume.

The discussion of the buckling provisions contained in the ACI publications, and in the IASS Recommendations, applies, presumably, to shells of any geometry. However, some specialized types of shell structures, such as hyperboloid cooling towers, may present a special problem. These are discussed in appropriate publications, and in the paper by Abel and Gould¹ in this volume.

CONCLUSIONS

This review of the problems of buckling of concrete shells was begun by considering the reasons why the problem is growing in importance. It appears that the tendency of modern architects, and modern shell designers, is to provide for ever-larger spans, with ever-larger radii of curvature-to-shell thickness ratios. Whereas the thickness of traditional shells was dictated primarily by construction considerations—cover to and ease of placement of reinforcement—the spans of modern shells are such that in-plane stresses have become significant compared to critical buckling stresses.

Qualitatively, the shell's path of equilibrium shows that in many cases postbuckling behavior is decisive in the evaluation of buckling loads. This appears to be invariably true in the case of shells such that both principal in-plane stresses are compressive. Thus, a large displacements (geometrically nonlinear) analysis of such shells is imperative. Further, such an analysis must take into account the effect of geometric imperfections — the difference between the shape of the ideal shell and the shape of the shell as built.

In addition, for reinforced concrete shells a number of other effects have an important influence on the magnitude of the buckling load. These include: creep of concrete under load, cracking of concrete, inelastic properties of concrete and reinforcement, and finally the presence of reinforcement.

Thus, the analysis of buckling of reinforced concrete shells can be expected to present formidable difficulties. This is indeed the case. In the era preceding the development of computers, only very few solutions were obtained, even when the problem was reduced to just the large displacements and geometric imperfections. Such solutions as were obtained were limited to a few problems of shells of simple geometry such as circular cylindrical tubes and panels and spherical caps. Nevertheless, these solutions, many of surpassing elegance, provided significant insight into the problem of shells in general.

In particular, after a long search, these solutions provided the answer to the long-perplexing problem of the lack of agreement between the buckling loads predicted by the linear theory, and the behavior of shells as observed in models. In fact, it was the experimental work that spurred the development of the theory—the discrepancy was

simply too great, both qualitatively and quantitatively, to remain unexplained. The answer laid in the geometrically nonlinear theories, and in the studies of geometric imperfections, which finally resulted in the complete qualitative explanation of the phenomenon.

It remains for the computer-based solutions to complete the task. At this stage, the work is still in progress, but it appears that the problem has been, in principle, solved. Using the powerful technique of the finite difference energy method, and above all of the finite element method, solutions have been obtained for the whole range of shell behavior based on a nonlinear theory, and including the various characteristics of reinforced concrete: imperfections of shape, inelastic properties of materials (both concrete and reinforcement), creep, and cracking.

The need at this time is to extend the hard-gained understanding of concrete shell buckling to a greater number of shell geometries important in practical applications and to provide shell designers and analysts with reliable design aids, easily understood and used. The future will tell whether this is an attainable goal.

The progress in the area of shell buckling analysis alone will not be a quarantor that better shells will be built, for if any building—shells included—is to be considered an improvement on the past, it must be not just stronger, stiffer, more stable, it must also be more functional and more economical and, above all, it must be a source of delight to all users, those who occupy the space within as well as those who view the structure from without. Only when all the Vitruvian precepts—commodity, firmness, delight—are met can we say that a better shell has been built, a work of art as much as of science.

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Stability of Cylindrical Reinforced Concrete Shells

By Paul Seide

Synopsis: A state-of-the-art review of the stability of cylindrical shells is presented. The copious theoretical results available in the literature are discussed and compared with available experimental results. Reasons for the well-known discrepancies between theory and experiment are indicated and various design formulas which take these discrepancies into account are given.

Most of the available experimental data are for linearly elastic metallic or plastic specimens. The behavior of these can differ markedly from the behavior of reinforced concrete structures which are subject to cracking and material nonlinearity. The applicability of linearly elastic shell data to the buckling analysis of concrete shells is discussed briefly on the basis of a comparison of the few test results for reinforced concrete shells with those for elastic shells.

Keywords: buckling; cylindrical shells; reinforced concrete; shells (structural forms); stability.

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INTRODUCTION

Circular cylindrical shells (Fig. 1) find many uses as concrete structures. Complete cylinders are exposed to hydrostatic load as submerged transportation tunnels, deep mine shafts, and offshore oil production structures. Open or incomplete circular cylindrical shells are used as roof structures under dead load, snow, and wind. Circular bearing walls for dome roofs are an example of axially loaded cylindrical shells. The design of these shell structures in many cases depends on their buckling capacity.

A review of the existing knowledge of buckling of reinforced concrete shells was made by Bradshaw in 1963 (7) and by Ramaswamy in 1968 (40). Note was taken of the lack of information concerned with concrete shells as against that available for plastic and metal shells. Reinforced concrete has a nonlinear stress-strain curve and is subject to creep and tension cracking. Thus its buckling properties may differ. Some 16 years later the available information has improved somewhat. In addition to a better understanding of elastic buckling, there are a few test results for concrete shells which have served as the basis for empirical modifications of the methods for linearly elastic shells. The present paper seeks to review the current state of knowledge of buckling of linearly elastic cylindrical shells and the applicability of the known results to reinforced concrete structures.

HISTORICAL REVIEW

Progress in the analysis of the buckling of cylindrical shells illustrates the close relationship between theory and experiment needed to attain adequate design data and an understanding of the steps needed to improve current design methods. For some 200 years after the first paper on structural stability by Euler in 1744 (19), buckling was considered to be a closed subject so far as the underlying theory was concerned. To find the critical load of a structure it was deemed necessary only to assume that the structure deviated infinitesimally from its pre-buckling equilibrium position. The critical load was then determined as the load at which the structure remained in equilibrium in the perturbed position. The theoretical buckling loads for the many problems solved by this method were in reasonable agreement with experimental results for columns and plates. It was also found that buckling load could be exceeded by large amounts for supported plates. In 1928, however, experimental results for

axially compressed thin cylindrical shells were published by Robertson (41). These results indicated a great discrepancy between theory and experiment since the measured buckling loads were 40% to 60% of theoretical values. In addition, the buckling load was the maximum or collapse load of the cylindrical shell. Later results (4, 8, 28 and 35, for example) indicated that even more discrepancy between theory and experiment could occur and that the results for ostensibly identical specimens yielded a large scatter band. A compilation of much of the test data available for axially compressed shells as of 1960 is shown in Fig. 2. The constant coefficient derived theoretically differs from experimental values by factors ranging from 1.2 to almost 9 over the range shown.

Such a discrepancy was too much to accept blindly so that explanations were sought. First attempts were made by Flügge (20) who investigated plasticity induced by end restraint in the cylinders and by Donnell (17) who theorized that initial imperfections caused failure to occur when the stress calculated by a nonlinear shell theory reached the material yield stress anywhere in the cylinder. The amplitude of the initial derivations from the cylindrical shape required to produce the observed reductions in the failure load were of the order of many shell thicknesses. This theory was considered unsatisfactory, however, since imperfections of such a magnitude should have been observed in at least some of the test specimens but were not (16, 29).

Subsequent investigations homed in on a more plausible explanation by concentrating on the nonlinear theory of the postbuckling behavior of cylindrical shells. First attempts investigated a simple structural model consisting of a column supported at its center by a nonlinear spring (30). It was shown that for such structures the critical load was the maximum possible load for equilibrium in a slightly deflected position and that more than one equilibrium position could exist at loads below the critical value. Initial deviations from the perfect form of the structure were found to lead to significant decreases in its maximum load-carrying capacity. The postbuckling behavior of a perfect cylinder under axial compression was shortly thereafter shown to have characteristics similar to the column with a nonlinear support (31). Interestingly enough, the role of initial imperfections was not to be fully recognized for some years yet. An energy criterion for buckling was proposed by Tsien (48) and stimulated a great deal of discussion before it was put aside. Numerous calculations of the postbuckling load-deflection curve for long cylinders in axial compression were made in an attempt to obtain the so-called lower buckling load until it was found (26) that more and more accurate calculations approached a value of zero (Fig. 3).

The paper that can be said to have had the most influence on an understanding of the buckling process was written by Koiter in 1945 (33). Since the original study was written as a doctoral

dissertation and in the Dutch language, it was not readily accessible to workers in the field until a condensed English version appeared in 1963 (34). The study indicated that the postbuckling behavior of a structure determines its sensitivity to initial imperfections and indicated a method for determining the initial slope of the postbuckling load-deflection curve. An asymptotic calculation of the effect of imperfections in the shape of the initial buckle pattern was made which indicated that imperfection amplitudes much less than the thickness of the shell were sufficient to produce the collapse load reductions observed in test (Fig. 4). Similar results had appeared in 1960 (18), but did not have the impact of Koiter's paper since the calculations involved were much more difficult. The investigations of (1) and (45) indicate, however, that some of the discrepancy between theory experiment may be due to differences between assumed and actual support conditions. This is especially true in the case of buckling of cylindrical shells under external pressure where the critical pressures for clamped shells may be as much as 50% greater than that for simply supported shells in some cases. It has also been found that the absence of circumferential edge restraint can decrease the buckling loads of axially compressed cylinders by as much as 50%. This type of support condition is rare, however. Edge conditions are difficult to ascertain precisely so that the effects of this source of error are usually ignored in comparisons of theory and experiment. Much of the recent work on buckling of shells has been based on an alternative version of Koiter's work by Budiansky and Hutchinson (11) and has concentrated on calculation of imperfection-sensitivity parameters and reduction factors for stiffened and unstiffened cylindrical shells under various loading conditions. Some results are shown in Fig. 5 in the form of values of a sensitivity parameter b as a function of shell geometry for external pressure and torsion. The reduction factors for various values of b are shown in Fig. 6. At the same time it has been recognized that the accurate calculation of the buckling loads of cylindrical shells generally requires the use of a nonlinear shell theory. The results of computer codes which incorporate nonlinear effects are described in (32) and (36). Computer codes such as ANSYS, MARC, and STAGS are also available (39). These codes permit the calculation of bifurcation loads of perfect cylinders and collapse loads of imperfect cylinders with known imperfections or of perfect cylinders which fail by snap-through action.

With imperfections pinpointed as the primary load reducing agent in cylindrical shells, careful tests on nearly perfect specimens have succeeded in yielding buckling loads very close to the theoretical value for perfect shells. Experimental results for cylinders with carefully controlled imperfections have also verified theoretical findings (46,47). Thus, the circle has been completed.

DESIGN FORMULAS

Surveys of the state-of-the-art of the buckling of cylindrical shells have appeared at various times. A very complete discussion as of the late 1950's is given by Gerard and Becker (5,21,22). An exhaustive compilation of results in the literature is available in (15). Recommended design criteria are given in (50) and to a lesser extent in (27) and (40). The discussion given here is based largely on the work of (42) and (50).

Despite our enhanced theoretical understanding of the factors affecting buckling loads of cylindrical shells, design formulas are, for the most part, semi-empirical. Theory has been utilized to establish the form of the various buckling criteria. These are modified, however, by a correction factor, often called a knockdown factor, which is obtained experimentally. A perusal of the literature will soon disclose the wide variety of correction factors, each based upon somewhat different philosophical approaches, which have been recommended. Some are based upon averages of data, others on a statistical analysis of the data, still others on a lower bound to the data. The use of lower bound data is based upon the premise that when imperfections are unknown, the designer should plan for the worst that can be expected to be encountered. It should be mentioned that such lower bound curves become harder to establish as the art of making and testing specimen shells improves. With modern computing techniques it may be possible to calculate accurate buckling loads for cylinders with known imperfections. It then becomes possible to improve current design practices if the distribution and magnitude of expected imperfections can be established and if calculated and measured buckling loads can be shown to be in reasonable agreement. Initial attempts toward this end have been made by Singer, Arbocz, and Babcock (2,3,44) but the method is as yet in its infancy.

The results given here are lower bound data for complete cylinders, i.e., cylinders which are continuous in the circumferential direction. Theoretical and experimental results for circular cylindrical panels are almost identical to those for complete cylinders if the included angle is sufficiently large and if all edges are assumed to be supported with tangential edge restraint. The effect of free straight edges on the buckling loads of open cylindrical shells have been investigated (12,13,14,49) and has been found to be significant. Only a few calculations which are insufficient for general design synthesis have been made, however.

Uniform Axial Compression

The theoretical classical buckling stress for an axially compressed circular cylindrical shell of moderate length is given by

$$\begin{aligned}\sigma_{cr} &= \frac{1}{\sqrt{3(1-\nu^2)}} E \frac{t}{R} \\ &= 0.605 E \frac{t}{R} \quad (\nu = 0.3) \quad (1)\end{aligned}$$

The experimental values of Fig. 2 exhibit considerable scatter but indicate that the discrepancy between theory and experiment tends to increase as the radius-thickness ratio of the shell increases. This phenomenon is considered to be due to the decreased bending stiffness of the relatively thinner shells which renders them more susceptible to imperfections during construction. The correction to Eq. (1) is usually represented by a factor by which the right side of the equation is multiplied. The most conservative design formula is the lower bound representation given in (41) and (49) as

$$C = 1 - 0.9 \left(1 - e^{-\frac{1}{16} \sqrt{\frac{R}{t}}} \right) \quad (2)$$

which covers the range of data for cylinder dimensions of

$$0.5 < \frac{L}{R} < 5$$

$$100 < \frac{R}{t} < 3000$$

A comparison of the lower bound curve and experiment is shown in Fig. 2.

Pure Bending

The theoretical value of the critical maximum stress of cylinders in pure bending has been shown in numerous papers to differ only slightly from the critical uniform axial compressive stress. Indeed, it is only for very rapidly varying circumferential distributions of axial stress that the maximum critical value differs significantly from the value for uniform axial stress. The stress must vary significantly over the small distance equal to the buckle wave length for axial compression before the effect becomes apparent. Nevertheless, experimental values of critical maximum bending stress are significantly higher than those for uniform axial compression. The results of various investigations are shown in Fig. 7. The lower bound correction factor for pure bending is given by

$$C = 1 - 0.73 \left(1 - e^{-\frac{1}{16} \sqrt{\frac{R}{t}}} \right) \quad (3)$$

Shown also is the lower bound curve for uniform axial compression which is considerably lower.

An explanation for the difference between correction factors for pure bending and uniform compression suggested in (42) is that the circumferential distribution of imperfections for a shell is apt to be non-uniform and there is a lesser probability of the coincidence of the largest imperfection and the maximum stress. This hypothesis has been untested to date, however, since there are no results for the variation of the critical bending moment with location of the maximum bending stress with respect to a reference cylinder generator.

It is also possible that the effect of initial imperfections is less for pure bending than under uniform axial compression. Although the buckle patterns of cylinders in pure bending and axial compression are similar, the postbuckling load-deflection curves obtained experimentally in (42) seem to differ. No theoretical postbuckling and imperfection sensitivity studies have been made, however.

External Pressure

The effect of initial imperfections on the critical external pressure of cylindrical shells has been shown to be much less than for uniform axial compression (10). This theoretical result has been verified experimentally since the difference between theory and experiment are not so drastic as for axial compression. The theoretical critical pressure for moderately long simply supported cylinders is

$$p = \frac{\sqrt{6} \pi}{9(1-\nu^2)^{\frac{3}{4}}} \frac{E}{\frac{L}{R} \left(\frac{R}{t}\right)^{\frac{5}{2}}} \quad (v = 0.3) \quad (4)$$

$$\approx \frac{0.92E}{\frac{L}{R} \left(\frac{R}{t}\right)^{\frac{5}{2}}}$$

The correction factor by which the right side of Eq. (3) is multiplied is recommended in (42) as

$$C = 0.75 \quad (5)$$

for all values of R/t . For long cylinders which buckle into an oval shape ($L/R > 3.35 \sqrt{R/t}$) the critical pressure is given by

$$p = \frac{1}{4(1-\nu^2)} E \left(\frac{t}{R}\right)^3$$

$$\approx 0.27 E \left(\frac{t}{R}\right)^3 \quad (v = 0.3) \quad (6)$$

with a recommended correction factor of 0.90. A comparison of the recommended design curve and experimental results for moderate length cylinders is shown in Fig. 8.

Torsion

For cylindrical shells under end torque, which subjects the cylinder to uniform shear stress, the theoretical buckling stress has been found to be given by

$$\tau = \frac{0.75 E}{\left(\frac{R}{t}\right)^{\frac{5}{4}} \left(\frac{L}{R}\right)^{\frac{1}{2}}} \quad (7)$$

for moderately long cylinders and

$$\tau = 0.25 E \left(\frac{t}{R}\right)^{\frac{3}{2}} \quad (8)$$

for long cylinders which buckle with two circumferential waves ($L/R > 8.62\sqrt{R/t}$).

The imperfection sensitivity of cylindrical shells in torsion is less severe than for axial compression (9). A correction factor of 0.67 for moderately long cylinders is recommended, however, in (50) on the basis of the comparison of theory and experiment in Fig. 9.

Interaction Formulas

Theoretical and experimental results are available for various combinations of the above loads. A recent study including the effects of initial imperfections and the nonlinear prebuckling shape is given in (6). A conservative approximation to the interaction curve is the assumption that the sum of the stress ratios, calculated as the applied stress component divided by the corresponding corrected buckling stress under that stress component alone, is equal to unity. Another method suggested (43) would retain the general shape of the theoretical interaction relation. Some of the available experimental data are shown in Figs. 10 and 11.

Modification for Nonlinear Stress-Strain Relations

The stress-strain relation for reinforced concrete is nonlinear from the onset of loading rather than linear as for the materials for which theoretical and most experimental results are available. Although theoretical solutions for plastic buckling of cylindrical shells based upon the theory of (22) are available, the few experimental results available do not appear to verify them. Griggs (24) finds good agreement between experimental results for reinforced mortar and linearly elastic plastic cylindrical panels in axial compression provided tension cracking does not occur and if the linearly elastic Young's modulus E in the design formulas is replaced by the appropriate tangent modulus E_t of the reinforced

concrete. The radius-thickness ratio of the panels is 200. Haynes and Nordby (25) similarly state that theoretical plastic buckling expressions do not appear to apply to concrete structures. They do not attempt to correlate their data for buckling of complete cylinders under external pressure with any material property such as the tangent modulus but use the data to define an empirical plasticity reduction factor. The cylinders used in their tests have radius-thickness ratios ranging from 8 to 16. Their maximum circumferential buckling-stress equation for moderately long cylinders under external pressure is expressed as

$$\sigma = 583 \left(\frac{t}{R} \right)^{\frac{3}{2}} \frac{R}{L} f'_c \eta$$

where f'_c is the concrete compressive strength and η is the plasticity reduction factor shown in Fig. 12. The initial Young's modulus of the concrete has been taken as

$$E_i = 667 f'_c$$

and Poisson's ratio as 0.18.

IASS Recommendations

The recent IASS approach to the design of reinforced concrete shells for buckling (37) differs from that used in the aerospace industry. Rather than use definite correction factors such as are given by Eqs. (2), (3) and (5), curves of theoretical reduction factors as a function of the imperfection amplitude are given. The designer is told to use an imperfection amplitude equal to the sum of the maximum elastic deflection and the maximum anticipated geometric imperfection. It is difficult to compare this design recommendation with experimental results since the buckling load of the cylinder will vary with the confidence of the designer in the construction methods to be used. Presumably any tendencies toward unconservatism in the buckling analysis will be compensated by the compounding of corrections for creep, cracking and plasticity which are defined by various curves and equations.

CONCLUSIONS

The state-of-the-art of buckling of circular cylindrical shells has been reviewed briefly and some design criteria which are used in the aerospace industry are given. These results which are primarily for buckling of linearly elastic cylindrical shells appear to be applicable to reinforced concrete cylinders provided the appropriate material tangent moduli are used in the design formulas and provided tension cracking does not occur. An approach recommended by the IASS will yield conservative results, in general, due to the compounding of material nonlinearity effects. There is, however, a definite need for additional experimental and theoretical results for reinforced concrete cylinders under various types of loading and geometrical properties.