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Guide to

Data analysis and quality control using cusum techniques —

**Part 3: Cusum methods for
process/quality control by
measurement**

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Foreword

It was originally intended that this British Standard should form part of an overall revision of British Standards relating to statistical quality control chart techniques (BS 1313 and BS 2564). However, cusum methods have found applications in fields beyond that of quality control in manufacturing (e.g. management, government administration, medical science, marketing, commerce, etc.) and comments on the draft standard revealed interest in a general standard on cusum methods. Part 1 of this standard is an attempt to meet this demand, and is intended as an introduction to the principles of cusum procedures, and charting.

Part 2 details decision rules, significance tests for retrospective analysis of accumulated data, and contains a variety of applications of cusum charting, some concerned mainly with data presentation, some with monitoring in quality or budgetary control, and others with retrospective analysis.

Part 3 deals with applications of cusum methods to process and quality control by measurement.

The techniques cover control of both average level and variability, and may be implemented using charts, tabulations or computers. They are especially appropriate when location of change points, and estimation of the corrections required to bring the process on to target, are desirable features of the control system.

Part 4¹⁾ deals with cusums for counted data.

Acknowledgement is made to the authors of papers in the statistical journals whose work has been drawn upon, to the contributors of examples of applications, to Mr P L Goldsmith for useful discussions, to ICI Fibres (Pontypool) for provision of computing facilities, and to Dr Derek Bissell (a representative of the Institute of Statisticians) who was primarily responsible for the development work on this standard, carried out on behalf of BSI committee QMS/1/4.

NOTE BS 1313 has been superseded by BS 5701 and BS 2564 is currently being revised and will be superseded by BS 5702. A new standard BS 5700¹⁾ gives general guidance on the applicability of all the various control chart methods referred to in BS 5701 to BS 5703.

A British Standard does not purport to include all the necessary provisions of a contract. Users of British Standards are responsible for their correct application.

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Summary of pages

This document comprises a front cover, an inside front cover, pages i to iv, pages 1 to 62, an inside back cover and a back cover.

This standard has been updated (see copyright date) and may have had amendments incorporated. This will be indicated in the amendment table on the inside front cover.

¹⁾ In course of preparation.

0 Introduction

0.1 Basis of cusum chart. To avoid repetition of material in the earlier Parts, it is necessary to assume that the reader is familiar with the basis of cusum charting as presented in BS 5703-1, and with the concepts underlying decision rules as set out in BS 5703-2. However, for convenience a brief recapitulation of the essential features of cusum charts and decision rules is given.

The cusum chart is a highly informative means of graphical presentation of data which can be ordered into a logical sequence. For process or quality control purposes, this sequence will generally correspond to the order of observation on a time scale (but examples of other bases for ordering are given in BS 5703-2).

The essential principle is that a *target* value, T , is subtracted from each observation. This target value is usually a constant (examples where the target value is not constant are given in BS 5703-2). In this Part, the target value(s) are related to some desirable feature(s) of the process when it is running in a satisfactory manner.

The cumulative sum of the deviations from target is formed, and this *cusum*, C , is plotted against the serial number of the observations.

Unlike conventional plotting (where the average level of the values in the series corresponds to height above the serial number axis), the cusum method of plotting results in the representation of average by the local *slope* of the chart. When the average corresponds to the target value, the path of the cusum lies roughly parallel to the sequence axis. When the local average of the series is greater than the target value, the cusum slopes upwards (away from the sequence axis); conversely, when the local average is less than the target value (negative), the cusum slopes downwards. The greater the discrepancy between the local average and the target value, the steeper the slope of the cusum path.

The result of plotting in the cusum mode is that changes in average level over different subdivisions of the sequence of observations are clearly indicated by changes in slope of the chart. The local averages in such subdivisions can be readily calculated, either from the numerical values of the cusum (from which the chart is plotted) or directly, albeit with some small loss in accuracy in some cases, from the chart itself.

0.2 Simple example of cusum chart. The above principles are best appreciated from a simple example. The calculations and plotting procedure will, at this point, be developed without mathematical symbolism; appropriate algebraic formulae are presented later.

It is supposed that the data given in Table 1 have been obtained, over a time sequence in the order shown, and that a target value of 15 is appropriate.

For a conventional chart, as shown in Figure 1, the observed values are plotted against their corresponding observation numbers. There is some indication that the last dozen values appear to be clustered around a different mean level from the first 20 or so.

Plotting in the cusum mode gives a much clearer display than the conventional chart. The cusum (column 4 of Table 1) is plotted against the observation number, on conventional axes, using the y (“vertical”) axis for the cusum and the x (“horizontal”) axis for the observation number, as shown in Figure 2.

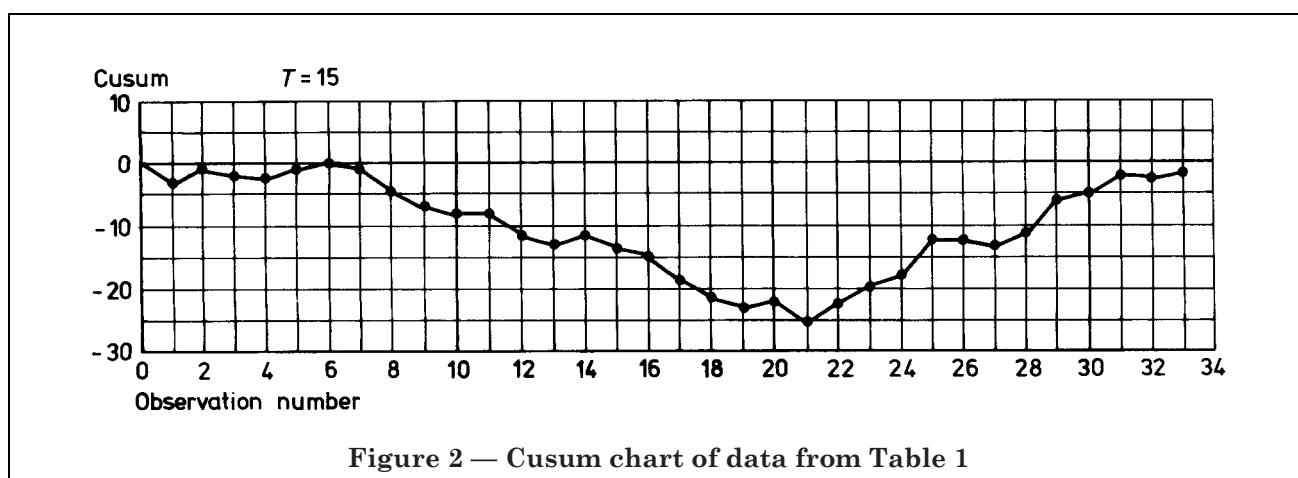
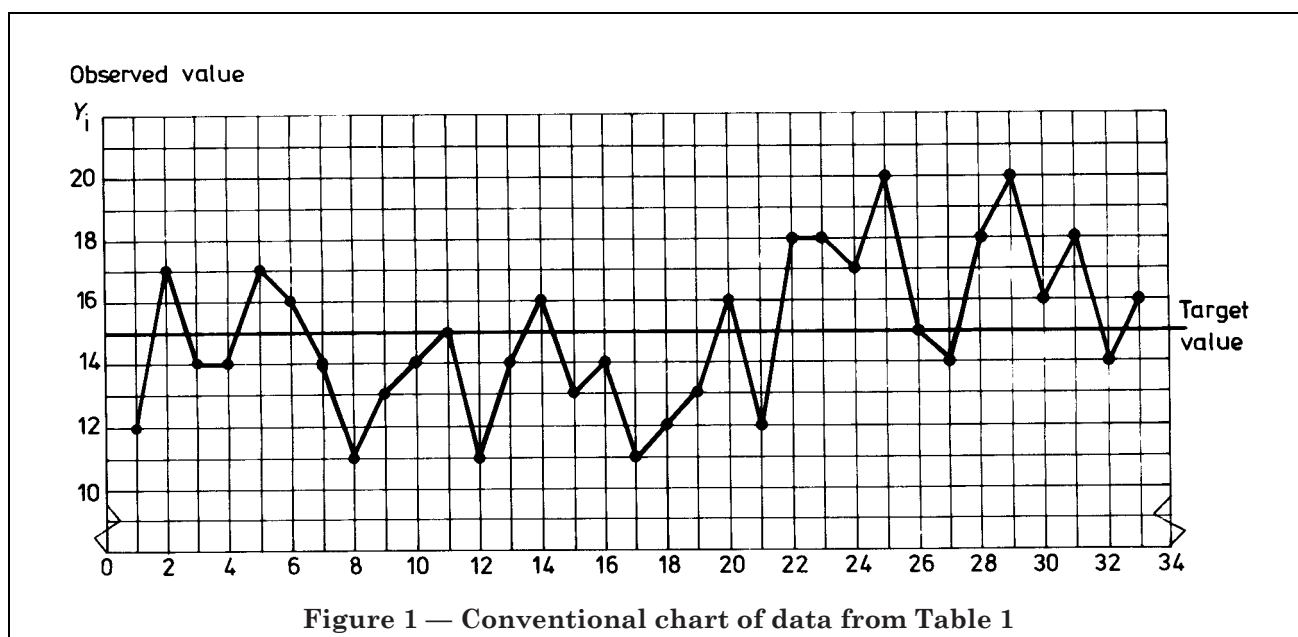
The chart clearly separates into three segments. From observations numbered 1 to 7 (inclusive) the cusum path is generally parallel to the observation number axis, i.e., the path is, in conventional terms, roughly horizontal. From observations 8 to 21 inclusive the path is recognizably downward (despite local irregularities such as at observations 14, 20). From observations 22 to 33, the path is upward (again with local irregularities which do not blur the overall pattern).

Thus it could be inferred that:

- a) observations 1 to 7 constitute a sample from a “population” whose mean is at or near the target value (15);
- b) observations 8 to 21 appear to have been sampled from a population whose mean is below 15;
- c) observations 22 onward appear to come from a population whose mean is greater than 15.

Table 1 — Data for cusum plotting

Observation number	Observed values	Deviation from target (15)	Cumulative deviations	Successive differences (see 0.6)
1	12	− 3	− 3	5
2	17	+ 2	− 1	− 3
3	14	− 1	− 2	0
4	14	− 1	− 3	3
5	17	+ 2	− 1	− 1
6	16	+ 1	0	− 2
7	14	− 1	− 1	− 3
8	11	− 4	− 5	2
9	13	− 2	− 7	1
10	14	− 1	− 8	1
11	15	0	− 8	− 4
12	11	− 4	− 12	3
13	14	− 1	− 13	2
14	16	+ 1	− 12	− 3
15	13	− 2	− 14	1
16	14	− 1	− 15	− 3
17	11	− 4	− 19	1
18	12	− 3	− 22	1
19	13	− 2	− 24	3
20	16	+ 1	− 23	− 4
21	12	− 3	− 26	6
22	18	+ 3	− 23	0
23	18	+ 3	− 20	− 1
24	17	+ 2	− 18	3
25	20	+ 5	− 13	− 5
26	15	0	− 13	− 1
27	14	− 1	− 14	4
28	18	+ 3	− 11	2
29	20	+ 5	− 6	− 4
30	16	+ 1	− 5	2
31	18	+ 3	− 2	− 4
32	14	− 1	− 3	2
33	16	+ 1	− 2	



There are now a number of questions that might be asked.

- In the light of the underlying variability (as indicated, for example, by the minor irregularities in the cusum path) can it be concluded that the changes in slope represent real shifts in average rather than merely lucky or unlucky runs of samples from a stable population?
- If the changes are real, how should the data be used to estimate local averages?
- To what extent might the inferences or estimates be affected by the choice of the target value or the cusum scale factor? Thus Figure 3 and Figure 4 show the same series plotted firstly with the same cusum scale but with a target of 12; and secondly with a target of 15 but a compressed cusum scale.

In both Figure 3 and Figure 4, the change in slope around observations number 8 is less apparent, though the change around number 21 is still visible, but it is less easy to “pinpoint” in Figure 4. Thus the choice of target value and scale factor need careful thought, to avoid either the suppression of useful information or conversely the exaggeration of spurious effects. It is also clear from Figure 3 that use of an inappropriate target value may result in the chart running off the upper or lower edge of the graph paper (although this problem may also be overcome by replotting from a new zero at any point in the sequence; (see clause 15 of BS 5703-1:1980).

0.3 Monitoring or retrospective analysis. Questions b) and c) are covered in detail in BS 5703-1, and question a) in BS 5703-2, where two distinct situations are recognized.

1) The object of the checking is to monitor the behaviour of a series of observations against some specified or standard value. This situation is the one that applies especially to process and quality control. Basic decision rules are given in clause 4 of BS 5703-2:1980 and are expanded in this Part.

2) The object is to examine historical data retrospectively, so as to detect any differences between segments. No formal standard or specification need exist. The methods described in clause 5 of BS 5703-2:1980 may apply in the field of process or quality control, especially in checking data from feasibility or process capability studies, in order to detect any anomalous behaviour of the system under observation. This use of cusums is described in clause 10 of this Part.

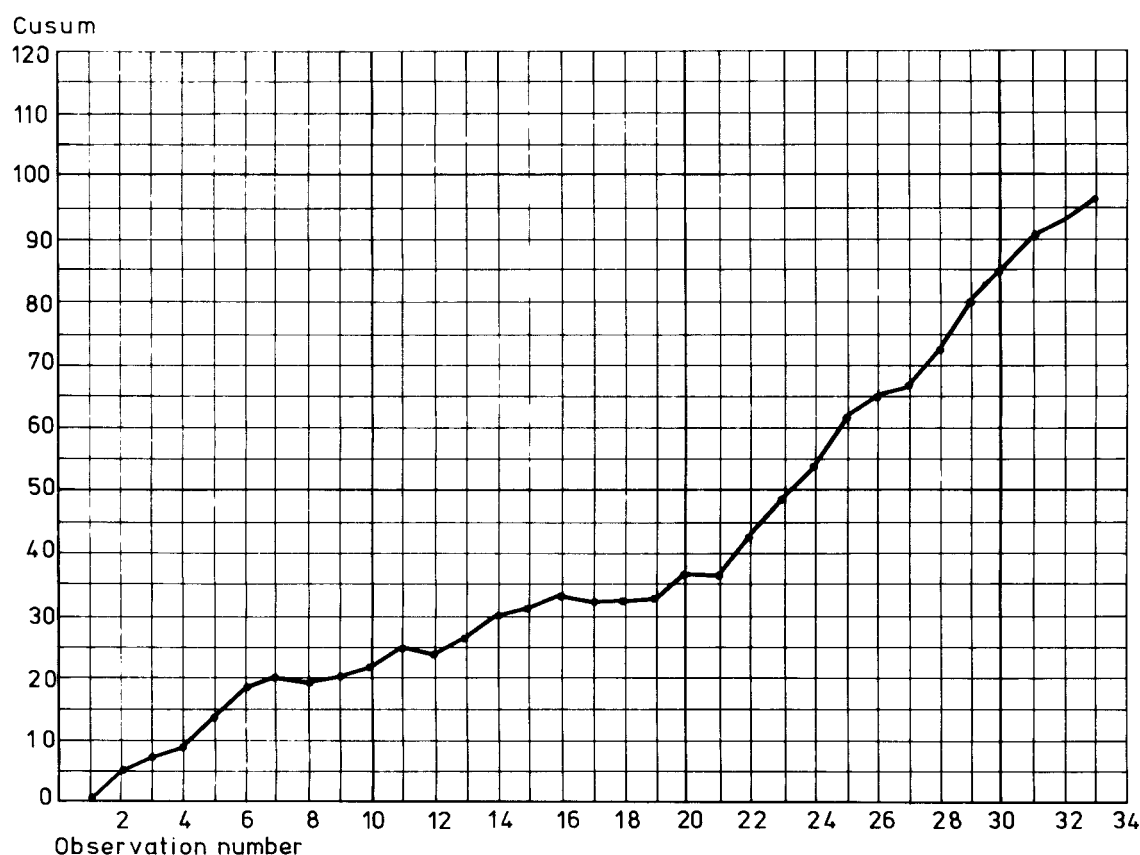


Figure 3 — Cusum chart of data from Table 1, with target value 12

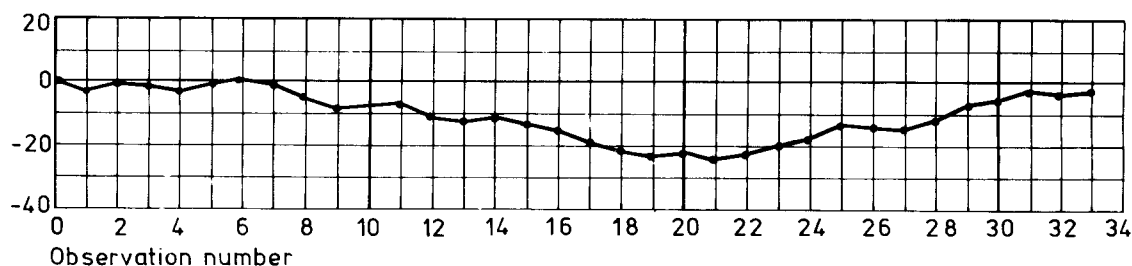


Figure 4 — Cusum chart of data from Table 1, with target value 15 but compressed cusum scale

0.4 Basic decision rules. The significance of a change in the average level of a plotted variable, appearing as a change of slope on the cusum chart, depends on the magnitude and persistence of the change. Thus a large change in slope (indicating a gross shift in average level) should rapidly yield a signal, whereas a smaller change would need to persist over a longer sequence of observations before it could be taken as real evidence of a shift in average.

The simplest cusum decision rule is embodied in a truncated V-mask. It comprises a datum point, indicated by A in Figure 5, which may usefully be identified by a small notch in the vertical section BC. Two sloping arms run from B to D and C to E, but may be extended indefinitely beyond D or E if required. The vertical half-distances AB, AC are termed the *decision interval*, and the lines BD, CE, the *decision lines*. The slope of these decision lines, measured in scale units per sample plotting interval, corresponds to the reference value introduced in clause 7 of BS 5703-2:1980 and in clause 8 of this Part.

The construction is detailed in Figure 5. The distances $5\sigma_e$, $10\sigma_e$ may be transferred from the cusum scale if desired, or calculated from the standard error, σ_e , and a scale factor if preferred (see clause 8 of BS 5703-1:1980).

The mask is used by placing the datum, A, over any point on the chart; this will often be the most recent point plotted, or the last point in some segment of particular interest. The AF axis is laid parallel to the sample number axis of the chart. If the path of the cusum wanders outside the sloping arms (or their extensions beyond D or E), then a significant departure from the *target* value is signalled. However, if the entire cusum path remains inside the arms, no significant shift is indicated.

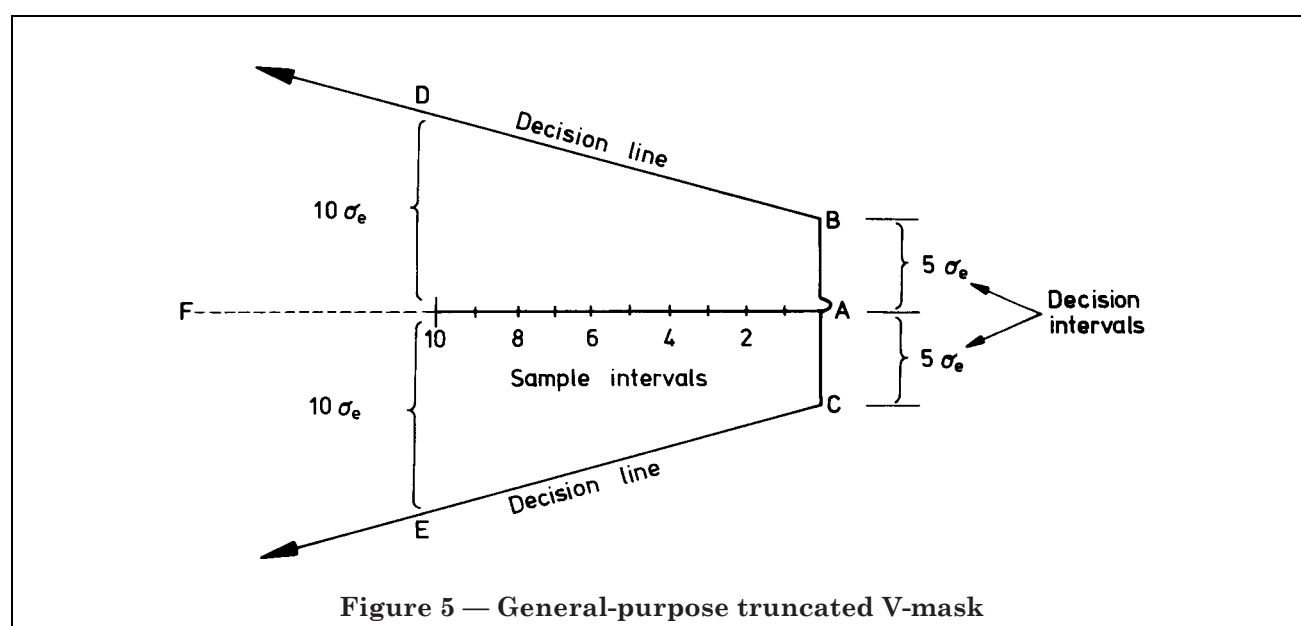


Figure 6 and Figure 7 show the truncated mask applied at two points to the cusum chart of the data from 0.2 (Table 1, Figure 1 and Figure 2). The value of σ_e is assumed to be 2.0. Applying the mask at observation 16, the cusum remains within the sloping arms, and the segment from samples 8 to 16 does not differ significantly from the target value of 15. However, on applying the mask at sample 18, the cusum path is seen to touch the *upper* decision line, indicating that by the time the 18th value is obtained, sufficient evidence has accumulated to signal a *downward* shift from the target value.

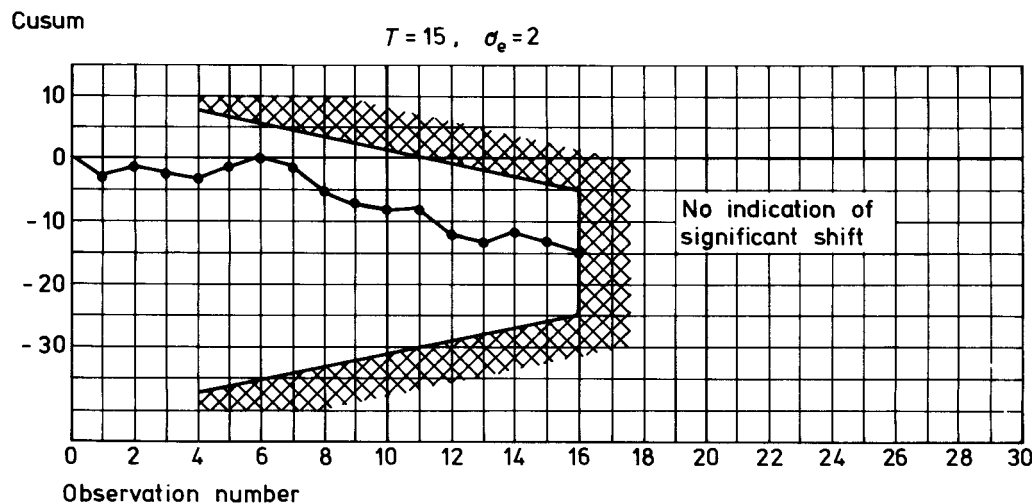


Figure 6 — Truncated V-mask applied to cusum chart: no indication of shift

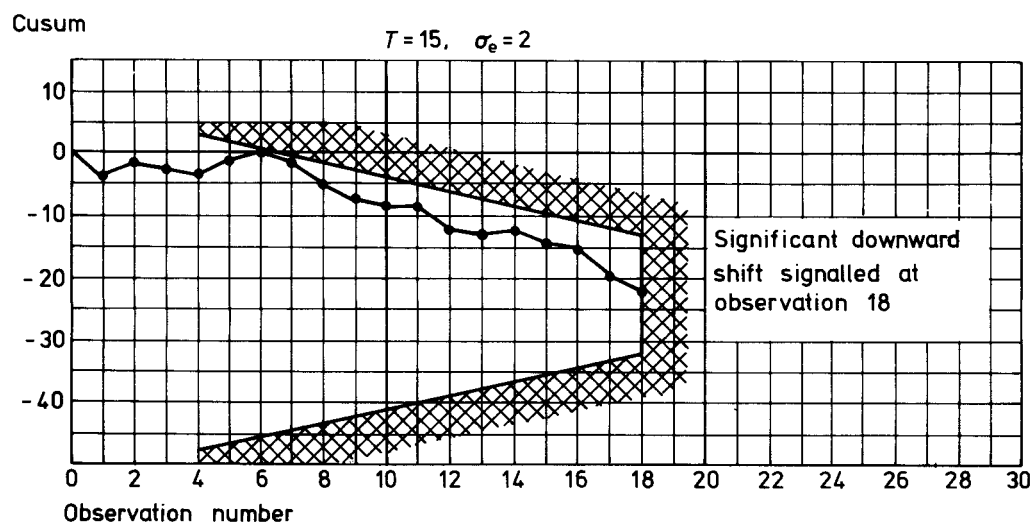


Figure 7 — Truncated V-mask applied to cusum chart: indication of shift

Several points are worth noting here.

- The upper decision line is *touched* at sample points 6 and 7. Touching or crossing the decision line are deemed to be violations yielding signals.
- The point at which the decision line is violated is on the extended arm of the mask, i.e. more than ten samples behind the datum. The inference of change in mean is none the less valid.
- If the mask were applied at any point before 18, no decision would be signalled. Applied at any point between 18 and 21 (the end of the downward segment) a significant downward shift is indicated. Repeated (even unnecessary) use of the mask does not invalidate or modify the signals obtained.
- The signal occurs at sample 18. At this point, the evidence is sufficient to support a conclusion that the mean has changed, but the change is likely to have set in at some earlier stage; inspection of the chart suggests sample 7 or 8 as the first of a sequence in which the average is displaced from the target value.

e) In a control situation, appropriate action would be indicated after plotting sample value 18 and obtaining a signal from application of the V-mask. An estimate of the new mean level may be useful in invoking the correct adjustment, and use of the method described in 13.4 of BS 5703-1:1980 yields:

$$C_{18} = -22, \quad C_6 = 0;$$

$$\bar{x} = 15 + \frac{-22 - 0}{18 - 6} = 13.16$$

If action is delayed or ineffective, this estimate may be up-dated at a later stage if required.

f) Similar use of the V-mask later in this series would reveal a violation of the *lower* arm, indicating an *upward* shift, by the 29th sample point.

g) The present procedure is not appropriate for testing the *difference* in means between segments 8 to 21 and 22 onward; it is designed for testing against the target value only.

0.5 Practical points about cusum masks. Before considering alternative forms of the preceding decision rule, some essentially practical points regarding the form and construction of masks are considered. These may be constructed from card or even from paper, but a mask on transparent material (acrylic or acetate sheet) is more useful. A slope protractor may then be incorporated on the same sheet of material, rendering estimation of local averages even simpler (see clause 13 of BS 5703-1:1980).

When constructed from transparent material, actual removal of the truncated-V section is unnecessary, and a decision is signalled if the cusum path crosses a decision line.

If a non-transparent material is used, either of the two following forms may be adopted:

- Cut out the truncated-V section, leaving the remainder. When using the mask, a decision is signalled if the cusum path disappears behind the mask.
- Remove and use the truncated-V section, discarding the remainder. A decision is then signalled if the cusum path appears outside the mask; as long as it remains hidden by the mask, no signal is generated.

Selection of either of these forms is purely a matter of personal preference, but the transparent mask has the advantage that the entire cusum path remains visible, whether or not a decision is signalled or impending.

In cases where one-sided control is appropriate, a half-mask may be used if preferred. When monitoring against an upward shift, the lower portion is required, and, for a downward shift, the upper half of the mask is needed. However, a full mask may still be preferred on grounds of simplicity, and one may choose to ignore any shifts in the irrelevant direction. In many cases, evidence of such a change is of interest in directing attention to possible reasons for good process performance; a signal may then be observed, though the action generated will obviously differ from that triggered by an adverse shift.

0.6 Scaling the chart. The relationship between the cusum and sample number scales will determine the appearance of the chart. Too large a cusum scale will produce a chart with a saw-tooth appearance on which small changes will be difficult to detect, while large shifts in average will quickly send the cusum path off the top or bottom of the graph sheet. Too small a scale will effectively damp out minor variations, but may also suppress real changes in mean level.

The most widely accepted convention is that a distance on the vertical (cusum) scale equal to one sample interval should represent approximately $2\sigma_e$ units of the plotted variable. This convention is most easily applied as follows.

Decide on a convenient interval for the sample number axis (the *i*-scale). This may be, for example, 5 mm (or 0.2 in) for desk use, or 1 cm, 2 cm (or 0.5 in, 1 in) for wall display. Mark off this interval on the cusum scale, and let it represent about $2\sigma_e$ units of the values to be plotted. If $2\sigma_e$ is an inconvenient value, an alternative between $1.5\sigma_e$ and $3\sigma_e$ may be preferred for ease of plotting.

Further details of plotting conventions are given in clause 8 of BS 5703-1:1980.

To establish the scale factor (and to implement the decision rules) an estimate of σ_e , the standard error of the plotted points, is required. This subject is considered in clause 7 and Appendix B of BS 5703-1:1980, and in clause 10 of this Part. The simplest estimate is obtained by forming the successive differences between the values of a suitable sequence of observations, calculating the average of these differences and taking eight-ninths of the average as an estimate of σ_e . In summation notation, for a sequence of observations y_j , $j = 1$ to m :

$$\hat{\sigma}_e = \frac{8}{9(m-1)} \sum_{j=1}^{m-1} |y_j - y_{j+1}|$$

the summation extending over the $m - 1$ differences between the m values (the symbol $| \ |$ indicates “absolute difference”, i.e. the difference without regard to sign).

For the data given in Table 1, the total of the successive absolute differences is 80, and for 33 values (32 differences) the mean successive differences is thus 2.5. Hence σ_e is estimated as $8/9 \times 2.5 = 2.2$.

Figure 2 shows the data plotted with ≈ 0.4 cm intervals between the abscissa of the sample numbers. On the vertical (cusum) scale, ≈ 0.4 cm should therefore represent about 2×2.22 , or 4.4, units of the variable. As this is not convenient for plotting, it has been rounded to 5.0, well within the recommended range of $1.5 \sigma_e$ to $3 \sigma_e$ (3.3 to 6.6).

1 Scope

This Part of the guide provides decision procedures, methods of process and quality control, schemes for means, control of short term variation, tabulation methods, computer applications and analysis of feasibility (process capability) data using cusum techniques.

2 References

The titles of the publications referred to in this standard are listed on the inside back cover.

3 Symbols and terminology

General. To avoid an unduly lengthy document, it is necessary to assume some familiarity with basic statistical concepts and terminology. In this Part, a similar knowledge of the language of quality control will be assumed. The essential requirements are an understanding of the following:

- $\bar{x}$ arithmetic mean of a set of data;
- μ arithmetic mean or expected value of the population (real or imaginary) from which the data is a sample;
- s sample estimate of the standard deviation;
- σ standard deviation of the population;
- Σ summation, in particular as used in the following:

$$\begin{aligned}\bar{x} &= \frac{1}{n} \sum_{i=1}^n x_i \\ s &= \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2} \quad \text{or} \\ s &= \sqrt{\frac{1}{n-1} \left\{ \sum x^2 - \frac{(\sum x)^2}{n} \right\}}\end{aligned}$$

using the abbreviated summation notation, where the range of x is assumed to extend over $i = 1, \dots, n$.

σ_e is used to denote the standard error of a sample statistic, the conventional measure of variation of a variable calculated from a sample; the standard error of the mean is the most familiar example.

Some concepts from elementary probability and distribution theory (independence, correlation) are used in this standard. Where decision rules are discussed, the distribution of the observed variable is specified.

An account of this basic material may be found in most elementary statistical text books. The only other requirement of the reader is an appreciation of the terms used in plotting graphs on two axes. The convention adopted is that the sequence in which the data occur is referred to the x-axis (abscissa, or conventional “horizontal” scale), and the cumulative sum is referred to the y-axis (ordinate, or “vertical” scale).

Careful thought has been given to the notation adopted for cusum parameters. In the statistical journals, an informal convention has evolved using the symbols H (or h), K (or k), and θ (theta). There seems to be no reason to depart from the use of H and K , but θ has been adopted as a standard symbol (for another purpose) in BS 5532.

In cusum terms, the required symbol represents an angle; ϕ (phi) has therefore been adopted, leading to f to represent its tangent.

The upper case forms H , K , F are used in conjunction with charts referred to an observed variable, and the lower case (h , k , f) to a standardized variable, e.g. one having a standard normal distribution $\mu = 0$, $\sigma = 1$.

3.1 Symbols and terms

Symbol	Term	Definition
a, A	scale factor	$A = a\sigma_e$, and is the number of units of the plotted variable, measured on the cusum axis, corresponding to one sample interval on the sequence axis of the cusum chart.
AQL	acceptable quality level	The limiting value of the process average regarded as acceptable for process control purposes. NOTE This definition is compatible with those given in BS 4778 and BS 5532, but extends its use to cover measurable characteristics in a manner consistent with current industrial practice.
C	cusum	The cumulative sum of deviations of a variable from a target value.
f, F	reference shift	The difference between target and reference values for a cusum tabulation. $F = f\sigma_e$ corresponds to the slope of the limb of a V- or truncated mask.
h, H	decision interval	The cumulative sum of deviations from a reference value required to yield a signal. $H = h\sigma_e$ corresponds to the half-height at the datum line of a truncated V-mask.
i, j, r		Subscripts used to identify values of a variable, or of a cusum.
K	reference value	The value from which differences are calculated for a cusum tabulation, and the qualifying value for initiating a cusum sequence. $K = T + F$ for monitoring against upward shift, and $K = T - F$ for monitoring against downward shift.
m	segment length	The number of sample intervals in a cusum sequence for which a test of change in level is carried out.
T	target value	At a given quality level, the target or standard value from which departure in average level requires to be detected.
x, y, Y	variables	Symbols denoting a random variable.
$\bar{x}, \bar{y}, \bar{Y}$	mean values	Arithmetic mean of a set of random variables x, y, Y .
R, w	sample range	The difference between the largest and smallest items in a sample.
RQL	rejectable quality level	In the quality control sense, a level of quality regarded as unsatisfactory, which should be detected quickly by a quality control procedure.
RL	run length	At a given quality level, the number of samples taken up to the point at which a signal occurs.

Symbol	Term	Definition
ARL or L	average run length	At a given quality level, the average number of samples taken up to the point at which a signal occurs. This is generally related to a stated quality level and the symbol L then carries an appropriate subscript, as for example L_a or L_r for average run length at AQL or RQL respectively.
s	sample standard deviation	The estimate, $s = \sqrt{\frac{1}{n-1} \sum (x - \bar{x})^2}$ of standard deviation obtained from a sample of n items.
n	sample size	The number of individual items comprising a sample.
σ	population standard deviation	A known or assumed value of standard deviation.
σ_e	standard error of a simple statistic	Measure of variation of a sample statistic. For sample means in samples of n , often assumed to be equal to $\sigma/\sqrt{n}$.
μ_a	acceptable process mean level	A value of the process mean level regarded as providing acceptable quality.
μ_r	rejectable process mean level	A value of the process mean level regarded as “rejectable” in the sense that detection of any shift from μ_a to μ_r should yield a control signal.

Subscripts

a	The variable at acceptable quality level.	} Although a and r have other uses in this standard, e.g. a as a scale factor, r as a summation indicator, the context will indicate the appropriate usage.
r	The variable at rejectable quality level.	

4 Cusum decision procedures

4.1 Purpose of decision rules

4.1.1 As detailed in clauses 4 and 5 of BS 5703-2:1980, decision rules are necessary where some action is to be taken on the evidence of a cusum (or other) chart or tabulation. This Part is concerned with process or quality control applications of cusum methods, and the methods described in clause 4 of BS 5703-2:1980 are therefore relevant (rather than those of clause 5 of BS 5703-2:1980 which deal with retrospective analysis).

4.1.2 Objective rules are required so that decisions do not differ according to the bias or whim of the individual assessing the data. The risks of erroneous decisions need to be considered, and are most readily assessed using average run length (ARL) criteria, as described in 4.6 of BS 5703-2:1980, and in 4.3 of this Part. These are:

- a) the risk of deciding that the process requires adjustment or investigation (i.e. “action”) when in fact it is running satisfactorily;
- b) the risk of failing to detect an out-of-control situation, at least for some time after its onset.

4.2 Principles of decision rules

4.2.1 The essential features of a cusum chart relevant to the detection of changes in the underlying average level of the monitored variable are the steepness of the cusum slope (slope corresponds to average level) and the number of samples over which this slope persists. When the process is operating at or near its target level, there will inevitably be occasions when the cusum path apparently diverges from the horizontal. The decision rules are required to identify segments where the steepness and persistence of the slope (away from the horizontal) are significant of a real change in average, rather than manifestations of sampling variation.

4.2.2 The simplest form of decision rule is embodied in a truncated V-mask used in conjunction with a cusum chart. The construction of the mask is based on the *standard error* of the plotted variable, its distribution and on the required average run length properties.

Methods of estimating the standard error, σ_e are detailed in clause 7 and Appendix B of BS 5703-1:1980 and briefly recapitulated in 0.6 of this Part. Further consideration of methods of representing the underlying variation is incorporated in clause 10 of this Part.

The way in which the distribution of the variable affects the decision parameters is considered in clause 7. Some general indications are given in Appendix C of BS 5703-2:1980.

A basic decision procedure for a normally distributed variable with ARL properties suitable for many process and quality control applications, is provided in 4.2 of BS 5703-2:1980. Useful variants of this procedure are described in 4.3 of this Part. The basic scheme is described in 0.4.

4.2.3 The essential features of the mask are shown in Figure 5. They are:

- a datum point, A, which is placed over any point of interest on the chart; often the most recent point plotted;
- the vertical half-distances AB, AC, known as decision intervals;
- the sloping decision lines BD, CE; an out-of-control signal is indicated if the cusum path crosses or touches either of these lines;
- for ease of alignment on the chart, the horizontal line AF may be useful; this line (representing the zero slope of the cusum when the process is running at its target level) should be parallel to the sample axis of the chart.

4.2.4 The method of using the mask is shown in Figure 6 and Figure 7, using the data and cusum chart given in clause 0. Applying the mask at observation 16 (as shown in Figure 6) the cusum remains within the sloping arms, and no signal is generated, i.e. the process is assumed to remain in control. When the mask is applied at observation 18 (as shown in Figure 7), the cusum path touches the upper decision line of the mask, and a signal occurs, indicating the need for correction or investigation. Similar use of the mask at observation 29 would result in the cusum path crossing the lower decision line indicating an upward shift in average level and the need for correction or investigation.

4.2.5 When a signal occurs, an estimate of the mean level of the process (since the onset of the change) may be useful in deciding what corrective action is required. The local average level is obtained as the average slope of the cusum, in chart units per sample interval, plus the target value, i.e.:

$$\bar{x}_{i+1,j} = \tau + \frac{C_j - C_i}{j - i}$$

where

C_j, C_i are the values of the cusum at sample numbers j and i .

Thus for the segment $i = 6, j = 21$, giving the average value from samples 7 to 21 inclusive:

$$\bar{x}_{7,21} = 15 + \frac{(-26) - (0)}{21 - 6} = 13.27$$

[In 4.2.3 f) of BS 5703-2:1980, the average is calculated over samples 7 to 18 inclusive.]

4.2.6 Where a tabulation is preferred to a chart (for the reasons detailed in 7.2.5 of BS 5703-2:1980, the procedure is modified as follows:

- In place of $C = \sum_{r=1}^i (x_r - \tau)$, the cumulative sum is tabulated as:

$$S_1 = \sum_{r=1}^i (x_r - K_1) \quad \text{for detection of a possible increase in mean, or}$$

$$S_2 = \sum_{r=1}^i (x_r - K_2) \quad \text{for detection of a possible decrease in mean,}$$

where

$$K_1 = T + f\sigma_e$$

$$K_2 = T - f\sigma_e$$

The parameter f is the slope, in standardized units per sample interval, of the limbs of the mask. For the decision rule considered in this clause, $f = 0.5$.

b) Any negative value of S_1 or positive value of S_2 is ignored, the cusum being re-set to zero.

c) A signal occurs if S_1 or S_2 exceeds the decision interval $H = h\sigma_e$. In this case, $h = 5$.

4.2.7 The tabulation procedure, applied to the data given in Table 1, is detailed in Table 2. The procedure yields identical indications to that of the chart and mask.

4.3 Run length properties of the decision rule

4.3.1 The average run length properties of the basic cusum decision rule for a normally distributed variable, with decision interval $H = 5\sigma_e$ and reference shift $F = 0.5\sigma_e$, are described in 4.6 of BS 5703-2:1980. In Figure 8, the ARL curve of this scheme is reproduced, and compared with that for conventional control charts either with 1 in 1 000 action lines, or with action lines combined with 1 in 40 warning lines.

These ARL curves refer to one-sided control at a target value, i.e. monitoring against either an upward departure from target, or a downward shift, but not both. The effect of monitoring simultaneously for upward and downward shifts from the same target is considered in 5.5.

4.3.2 Even where appreciable shifts in average level of the process occur, response of the control procedures is by no means immediate. The meaning of ARL in such cases is the average, number of samples taken from the point of occurrence of the change until its detection by a signal from the chart. This is a most important point, in that, when a signal occurs, it may be necessary to look for the cause of the trouble several samples back from the one at which the decision is made.

It is here that the cusum is especially useful, in that the slope of the chart (or the pattern of contributions to the cusum where a tabulation is used) provides a guide to the change-point, as well as to the magnitude of the shift.

4.3.3 The cusum procedure (whether plotted or tabulated) therefore offers three advantages over conventional chart methods.

- a) Over a wide range of possible changes in mean level, detection is more rapid, sometimes by a factor of three or more as compared with action and warning charts, or as much as five when compared with action line only charts. This speed of response is achieved without risk of more frequent “false alarms”, as indicated by the ARL for zero shift (i.e. the ARL at the target level).
- b) When a change is signalled, the cusum “corner” provides an estimate of the point of change. This may be a valuable aid in locating the cause of the trouble.
- c) The local average level, which can be readily calculated from the cusum chart or tabulation, offers a basis for estimating the magnitude of the required process adjustment.

These advantages are gained at the expense of a minor increase in computation (the cumulation of deviations from target or reference values) and the somewhat unusual representation of average level by slope (rather than level above a baseline).

4.3.4 The ARL curves for the cusum procedure and control chart (i) cross at about $2.7\sigma_e$ shift, and those for the cusum procedure and control chart (ii) at about $2\sigma_e$ (see Figure 8). Alternative cusum decision rules which extend the advantage of shorter ARL to larger shifts, with little loss of response of small or moderate shifts, are given in 6.2 to 6.3.

Table 2 — Cusum calculations for tabulation procedure
 $T = 15, \sigma_e = 2.0$

 Upper reference value $K_1 = T + (0.5 \times 2.0) = 16$

 Lower reference value $K_2 = T - (0.5 \times 2.0) = 14$

 Decision interval $H = 5.0 \times 2.0 = 10$

i	x_i	Upper cusum		Lower cusum	
		$x_i - 16$	$S_1 = \Sigma (x_i - 16)$	$x_i - 14$	$S_2 = \Sigma (x_i - 14)$
1	12	-4	< 0	-2	-2
2	17	1	1	3	> 0
3	14	-2	< 0	0	—
4	14	-2	—	0	—
5	17	1	1	3	—
6	16	0	1	2	—
7	14	-2	< 0	0	—
8	11	-5	—	-3	-3
9	13	-3	—	-1	-4
10	14	-2	—	0	-4
11	15	-1	—	1	-3
12	11	-5	—	-3	-6
13	14	-2	—	0	-6
14	16	0	—	2	-4
15	13	-3	—	-1	-5
16	14	-2	—	0	-5
17	11	-5	—	-3	-8
18	12	-4	—	-2	-10 ^a
19	13	-3	—	-1	-11
20	16	0	—	2	-9
21	12	-4	—	-2	-11
22	18	2	2	4	-7
23	18	2	4	4	-3
24	17	1	5	3	0
25	20	4	9	6	—
26	15	-1	8	1	—
27	14	-2	6	0	—
28	18	2	8	4	—
29	20	4	12 ^a	6	—
30	16	0	12	2	—
31	18	2	14	4	—
32	14	-2	12	0	—
33	16	0	12	2	—

^a Cusum signals.

5 Scope of methods for process and quality control

5.1 Definition of “process”

5.1.1 For the valid application of *control* procedures, a process should conform to the basic requirement of continuity. A practical definition of continuity for the purposes of this clause is that inferences based on the recent history should provide a sound basis for adjustment of the process, so that its later behaviour satisfies defined objectives. A technical understanding of the process is thus required to permit translation of signals from the control procedure into effective adjustment. In many circumstances, the facility for extracting local averages, and locating approximate points of change, are important advantages of cusum methods.

5.1.2 There are, however, many circumstances where a signal may instigate investigation, segregation or other action, rather than direct process adjustment. Such applications monitor rather than control the process, but both types of application involve similar assumptions about the process, and they are therefore considered together in this clause.

5.1.3 In both types of application, there should be an element of process continuity. Truly continuous processes, such as paper-making, plastics extrusion, etc., are obvious examples but streams of discrete articles produced under steady conditions are equally suitable for continuous control methods. Even where conditions vary from one batch to another, it may be reasonable to use continuous methods, either to assess whether *average* batch quality changes over time, or where (for example) changes in raw material properties are compensated by process adjustments, to ensure that these adjustments achieve the desired results in terms of a steady level of output quality.

5.1.4 With cusum techniques (or indeed conventional control charts), decision rules are based on the assumption that successive values of the observations, on which control is based, are independent. Methods of testing the validity of this assumption are detailed in **10.5**. However, some situations require special care. Examples are processes where thermostatic or other control devices produce systematic variations in process conditions; the “topping-up of raw materials by addition to a reservoir only partly depleted; and the oscillatory effect produced where data values are obtained as successive differences between cumulative readings on an instrument. In the latter case (as may arise in monitoring fuel consumption), an error on one gauge reading may lead to errors in the two data values to which it contributes, these errors being opposite in sense and sign.

5.2 Process parameters

5.2.1 In process and quality control, while there may be a wide range of properties or parameters required to be monitored, these fall generally into two types: parameters representing location or general magnitude and parameters representing variation.

Firstly, and generally of major importance, are parameters representing location or general magnitude of the variable(s) concerned. Such parameters may be various kinds of average (arithmetic means, medians, indices), proportions, rates (of occurrence or consumption, production, etc.). The object of monitoring is to detect any shift in general location or magnitude, so that appropriate control action or investigation may be initiated.

5.2.2 In order to assess the significance of apparent shifts in location, it is necessary to have some measure of the typical variation or “noise” that occurs under normal conditions. In a few situations involving counting of defects, defectives or incidents, the average level of these occurrences may itself provide a measure of their variation using the binomial or Poisson distributions. These applications are the subject of BS 5703-4²⁾, while for the cases forming the material of this Part, a separate measure of variation is required. This, in most cases, is provided by ranges or standard deviations in samples of two or more items. Occasionally differences between successive values of the control variable may provide the only measure of variation, as for example when the sample statistic used for control is a single-valued observation (such as a rate of use of materials or the result of a chemical analysis) rather than an average of several observations.

5.2.3 The decision rules for assessing shifts in location are based on the level of variation in the process. It is therefore wise to monitor this variation so that, if it becomes consistently larger or smaller, the decision rules for detecting shifts in location may be modified accordingly. This is important for ensuring that targets or specifications continue to be met if variation increases, but also so that appropriate economies may be effected if variation decreases (either fortuitously or by deliberate process improvements).

In addition to this indirect interest in the process variation, situations commonly occur when control or monitoring of variation is of direct concern.

As described in **5.6**, when specifications require control of the proportions of individual values exceeding, or falling short of, some critical value, the location and dispersion aspects achieve equal importance.

²⁾ In course of preparation.

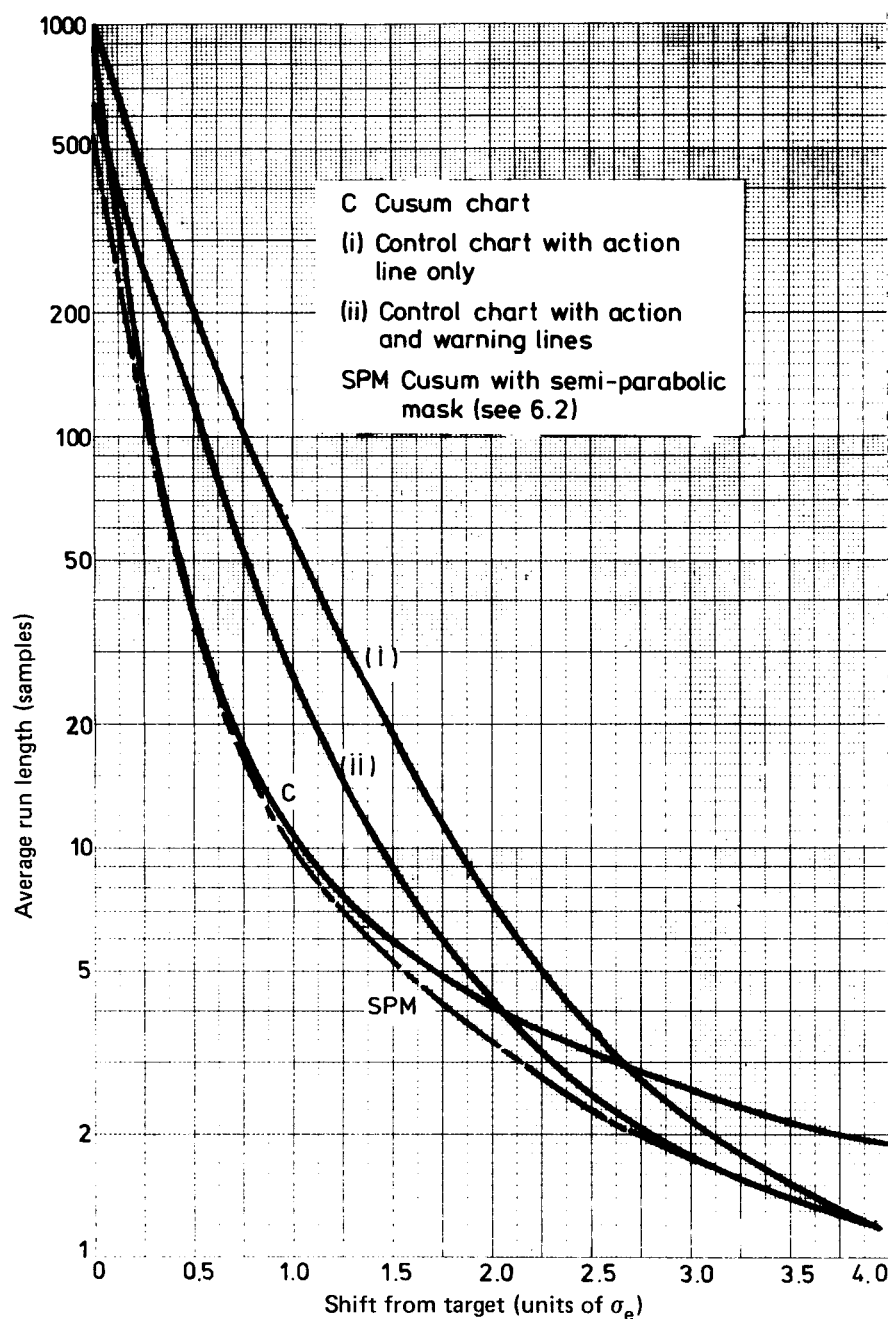


Figure 8 — Average run length data for cusum and control charts

5.2.4 A wide variety of possible location and dispersion parameters exists, but it is impracticable to consider all of them in detail in this standard. The parameters for which detailed procedures are presented are as follows.

- Average (arithmetic mean) values of measurements obtained from sequences of samples all of the same size.
- Single values measuring some feature of a process whose average value (over time) is of concern. Such cases will include measurements of consumption, production, chemical composition, productivity, etc. It is assumed that each value in the sequence of observations is of equal importance.
- Ranges (differences between largest and smallest values) in samples of equal size.