



Reliability of systems, equipments and components

Part 2. Guide to the assessment of reliability

Fiabilité des systèmes, équipements et composants
Partie 2. Guide pour l'évaluation de la fiabilité

Zuverlässigkeit von Systemen, Einrichtungen und Bauteilen
Teil 2. Anleitung zur Auswertung der Zuverlässigkeit

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Foreword

During the period from 1971 to 1976 BSI published the series of Drafts for Development DD 10 to DD 16, 'Guide on the reliability of engineering equipment and parts'. These were prepared by Committee E/-/6, Engineering reliability, taking into account work carried out by IEC* Technical Committee 56, Reliability and maintainability, and represented the most up-to-date information at that time. The guide was issued in the Draft for Development series of publications to permit its use on a provisional basis so that information and experience of its practical application could be obtained. Those who used the guide were urged to write to BSI giving their experience in putting its principles into practice and giving constructive proposals for improvements. It was the original intention that a British Standard would be published when sufficient knowledge and reaction has been accumulated.

During 1977 the committee was redesignated QMS/2/3, Reliability systems, under the direction of the Quality, Management and Statistics Standards Committee. The committee was charged with the task of converting the Drafts for Development into a British Standard in the light of comments received, improved techniques and other factors such as new health, safety and product liability legislation. The change in the committee title and reporting structure was influenced by the need to extend the application of reliability to disciplines other than strictly engineering, e.g. building and construction, services, etc.

The first decision taken (by QMS/2) was to incorporate the reliability terminology (previously given in DD 12) in the revision of BS 4778, in order (a) to bring together the closely allied disciplines of quality assurance and reliability and (b) provide for a wider readership and appreciation of reliability terms and definitions.

A standardized systematic approach is essential to ensure that statements are consistent and unambiguous in all communications concerning reliability. As well as providing a basis for the preparation of reliability programmes, this British Standard can be used in preparing documents concerned with the specification of reliability or the reporting of reliability data or tests.

It is vital that reliability be considered in a manner similar to a characteristic of a system, equipment or item from its design conception to the end of its working life, and that the reliability experience of one generation of products, engineers or technology is used by the next. This requires that each factor that affects reliability at any stage in the life of an item be identified and considered in its relationship to the other factors. This procedure can be formalized into a reliability programme. It may be necessary to divide such a programme into two parts, one of which may be the concern of the designer and manufacturer, and the other the concern of the user. It is most important that consideration be given to the reliability of items or parts that are used by one manufacturer but produced by another, i.e. subcontractor evaluation and vendor appraisal (see BS 5179 and BS 5750).

Satisfactory reliability has traditionally been achieved by the use of codes of practice or methods of working that have been shown by experience to give good results for established products. However, with the increasing pace of change and performance expectations it is no longer sufficient to consider reliability in purely qualitative terms when its achievement often demands the specification of quantitative acceptance/rejection criteria also.

Accordingly the onus should be on all parties (i.e. the designer, supplier, purchaser or any other responsible authority) to specify quantitative requirements wherever possible.

*International Electrotechnical Commission.

The need to satisfy competing constraints, including stricter health and safety requirements, the advent of new materials and pressure to reduce costs has made people realize that reliability can and should be expressed in quantitative terms if it is to be given its proper weight. For example, an item or product may be required to achieve at least a certain useful life, not to exceed a certain failure rate or to have an extremely high availability. Quantitative statements relating to reliability should then be made which may be accompanied by stipulations that certain codes of practice be followed.

One aspect of quantitative reliability that requires stressing is that often reliability can be predicted even for items or projects of new design and high cost, such as a ship's propulsion system or a new kind of nuclear reactor pressure vessel, but only if there is sufficient knowledge of the processes leading to failure, including data based on statistical observations.

The designer's or user's confidence that his reliability requirement will be met is dependent not only on the adequacy of the data concerning failure, which should be obtained either from reliability testing, field data, or from a knowledge of the physical processes leading to failure, but also on the application, in the first place, of a disciplined approach to the specification and design of a product or service. It is essential not only that the basic data be adequate, but also that they be interpreted correctly, in the light of the environmental and other particular factors appropriate to the subject.

Although only three Parts of this standard have been planned at present, further Parts are envisaged in order

to give greater emphasis to certain aspects of reliability and to take account of new developments. The first three Parts may be summarized as follows.

Part 1. Guide to reliability programme management.

This Part discusses the essential features of a comprehensive reliability programme for the direction and control of the reliability of systems, equipments and components. Recommendations are given for the drafting of specifications for reliability achievement. The main considerations for the assessment of reliability are reviewed and this is followed by details of the means by which reliability data are transmitted from the point of collection and recording to the point of storage and/or usage.

Part 2. Guide to the assessment of reliability. This Part recommends general procedures for the assessment of reliability and contains guidance, for the reliability practitioner, on the quantitative and statistical aspects of reliability such as reliability modelling, the provision of data, and the concepts of redundancy and simulation.

Part 3. Examples of reliability practices. This Part will contain authentic practical examples illustrating the principles established in Parts 1 and 2 of BS 5760.

NOTE. This British Standard makes numerous cross-references to BS 4778 'Glossary of terms used in quality assurance (including reliability and maintainability terms)'. The glossary contains definitions relating to reliability concepts applicable to the guide and it is essential that these definitions and concepts be fully understood if this guide is to be interpreted correctly. For this reason it is recommended that BS 4778 be read in conjunction with BS 5760.

British Standard

Reliability of systems, equipments and components

Part 2. Guide to the assessment of reliability

Section one. General

1. Scope

This Part of BS 5760 provides guidance, for the practitioner, on the quantitative aspects of reliability.

Section one of this guide provides references to basic statistical terms and lists the symbols used in reliability. This amplifies the definitions given in BS 4778 and BS 5532 and may be passed over by users with a good knowledge of statistics.

Guidance is given on the procedures that should be followed in estimating failure and repair time distributions.

Section two explains basic reliability modelling, the provision of reliability data and system reliability which introduces the concepts of redundancy and simulation.

Section two concludes with reliability testing and Bayesian analysis.

Appendix A presents, in tabular form, properties of various statistical distributions used in the assessment of reliability.

2. References

The titles of the standards publications referred to in this standard are listed on the inside back cover.

In addition, the following texts, papers and articles have been considered during the preparation of this guide. The references are a selection only and do not purport to be an exhaustive list of recommended reading.

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- (4) Green, A E and Bourne, A J. *Reliability technology*, Wiley, 1972
- (5) Rau, J G. *Optimization and probability in systems engineering*, Van Nostrand Reinhold, 1970
- (6) Sandler, G H. *System reliability engineering*, Prentice Hall, 1958
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- (9) Jardine, A K S. *Maintenance, replacement and reliability*, Pitman, 1973
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- (12) Ireson, W G Ed. *Reliability handbook*, McGraw-Hill, 1966.
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- (14) Siegel, S. *Non-parametric statistics for the social sciences*, Wiley, 1972
- (15) Singh, C and Billinton, R. *Systems reliability, modelling and evaluation*, Hutchinson, 1977
- (16) Carter, A D S. *Reliability reviewed*, Proc. Instn. Mech. Engrs. Vol 193, No. 4, 1979
- (17) Finkelstein, J M and Schafer R E. *Improved goodness of fit tests*, Biometrika. Vol 58 pp 641 — 645, 1971
- (18) Gredenko, B V. *Mathematical methods of reliability theory*, Academic Press, New York, 1969
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3. Terminology and symbols

3.1 Terminology. For the purposes of this British Standard, the terminology given in BS 4778 and BS 5532 apply.

3.2 Symbols. (See also appendix A.)

a	Location parameter for the Extreme value distribution
A	Availability
$A(t)$	Availability function
b	Scale parameter for the Extreme value distribution
c	Shape parameter for the Gamma distribution
C	Number of censored items
D	Kolmogorov-Smirnov statistic
D_{i1}	Determinant formed by deleting the i th and first column of a matrix
$f(t)$	Failure probability density function (failure p.d.f.)
F_i	Number of failures in interval i
$F(t)$	Cumulative distribution function (c.d.f.)
$F(t^1)$	The transformed failure c.d.f.
$F^{-1}(t)$	The inverse function of the transformed c.d.f.

$g(t)$	Repair probability density function (repair p.d.f.)	δ	Variable in the log Extreme Value distribution
$G_{\alpha}(\beta; \nu, w)$	Inverse distribution function of probability α of the β distribution with parameters ν, w .	δt	Small finite increment of t
H	Cumulative hazard	η	Scale parameter for the Weibull distribution
$H(t)$	Cumulative hazard function	θ	Mean time between failures (m.t.b.f.)
I	Identity matrix		Mean value of the failure p.d.f. (First moment about the origin, i.e. $\theta = \mu_{i,0}$)
k	Parameter used in the Hyper-exponential distribution	$\bar{\theta}$	Column vector of mean time to first failure
$L, (S)$	Load (strength) or likelihood	$\hat{\theta}$	Current m.t.b.f.
$\bar{L}, \bar{S}$	Mean load or mean strength	$\bar{\theta}_i$	Mean passage time from initial state, i , to first failure
$L_{\max}$	Maximum likelihood	θ_s	Column vector with each element given by $\bar{\theta}_{si} = (\bar{\theta}_i)^2$
MR	Median rank	λ	Failure rate
MO_i	Mean order number of the i th failure of a total sample of N	λ_u	Upper limit of failure rate
N	Matrix defined as Q^{-1}	λ_L	Lower limit of failure rate
	Number of items in a population	μ	Repair rate
n	Number of items in a sample	$\mu_{a,r}$	The r th moment of $f(t)$ about a
P	Probability	ρ	Parameter used in the Maximum likelihood estimation
P_A	Availability matrix	σ	Standard deviation of a random variable or of a population
P_F	Failure probability	σ^2	Variance of a random variable or of a population. Variance is the second moment about the mean (i.e. $\sigma^2 = \mu_{2,0}$)
P_I	Initial state probability		
P_R	Reliability matrix	$\sigma_L^2, (\sigma_s^2)$	Variance of load (strength) distribution
P_a	Probability of acceptance	ϕ	Mean value of the repair p.d.f. ($g(t)$)
Pr	Proportion of open-circuit failures	χ^2	Chi-squared distribution
q	Probability of an open-circuit failure	ψ	Parameter used in the Maximum likelihood estimation
Q	Matrix defined as $I - P_R$	ω	Parameter used in the Maximum likelihood estimation
r	Number of failures		
R	Reliability		
$R(t)$	Reliability function		
s	Estimate of standard deviation		
	Probability of a short-circuit failure		
S	Number of surviving items		
	Stress		
	Mean stress		
t	Time		
t_{50}	Median		
T	Cumulative time		
U	Unit column vector		
	Unavailability		
V	Unit reliability in terms of open and short circuit failures ($V = 1 - q - s$)		
$\bar{x}$	Sample mean		
$z(t)$	Hazard rate (or instantaneous failure rate) function		
Z	Hazard		
α	Distribution parameter for the Birnbaum-Saunders distribution		
	Variable in the log Extreme Value distribution		
	Producer's risk		
β	Weibull shape parameter		
	Distribution parameter for the Birnbaum-Saunders distribution		
	Consumer's risk		
γ	Weibull (or exponential) location parameter		
Γ	Gamma function		

Section two. Reliability assessment

4. Some basic statistical terms in reliability

4.1 Distribution. For further reference see 2.17, 2.18, 1.30 to 1.48 of BS 5532 : 1978 and 5.3 of this guide. Reliability work is generally concerned with the distribution of times to failure or, in the case of repairable (maintained) systems, times between failures. Apart from one-shot devices such as ammunition, items are designed, manufactured and constructed to endure for a period of time (or a number of cycles or a distance run or some other appropriate variable).

Because of variations in the construction of a series of nominally identical items and further variability of treatment in use, times to or between failures exhibit variation. Frequency analysis shows some sections of the possible range of time to occur more frequently than others. For convenience of calculation, the frequency analysis is often fitted by curves described by mathematical functions called *probability density functions* (p.d.f.'s). A p.d.f. yields a curve for the probability that an item, taken at random from a large population of like items, will have a certain life or time between failures. A p.d.f. can be estimated from a random sample of times to failure of individual items by arranging the times in bands of equal width (see figure 1). This histogram is then smoothed to estimate the p.d.f.

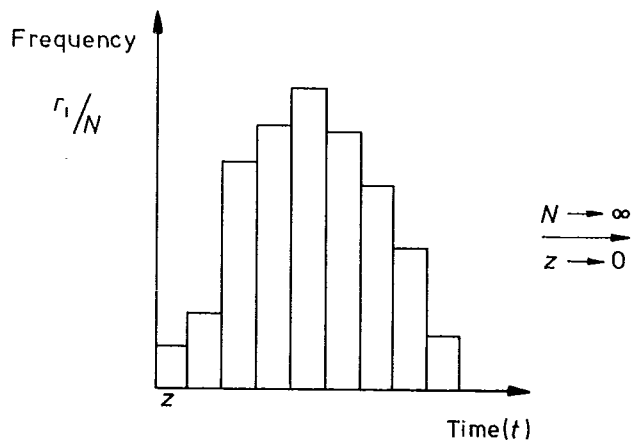


Figure 1. Histogram

Conceptually the p.d.f. is the theoretical shape of the curve as the sample becomes infinite and the time bands become infinitesimal. The total area under the curve is always unity.

$$f(t) = f(a, b, c \dots t)$$

where

a, b, c are constants called *parameters*.

The integral of the p.d.f. $f(t)$ is the proportion of an infinite population that fails before t . This is called the *cumulative distribution function* or c.d.f., $F(t)$.

Thus:

$$\text{Cumulative density function } F(t) = \int_0^t f(t) dt$$

$$\text{Reliability function } R(t) = 1 - F(t)$$

$$\text{Reliability of a one-shot device } R = (1 - p)$$

where

p is the proportion that fails to operate

4.2 Moments of a distribution. For further reference see 1.20 to 1.22, 2.36, 2.37 of BS 5532 : 1978. If the values of a p.d.f. are each multiplied by a power, r , of the distance from some fixed point a on the time scale thus:

$$\mu_{a,r} = \int_{-\infty}^{+\infty} (t - a)^r f(t) dt$$

then $\mu_{a,r}$ is the r th moment of $f(t)$ about a . Usually the moments are taken about the mean (central moments) or the origin. The mean is itself the first moment about the origin. The second moment about the mean is the

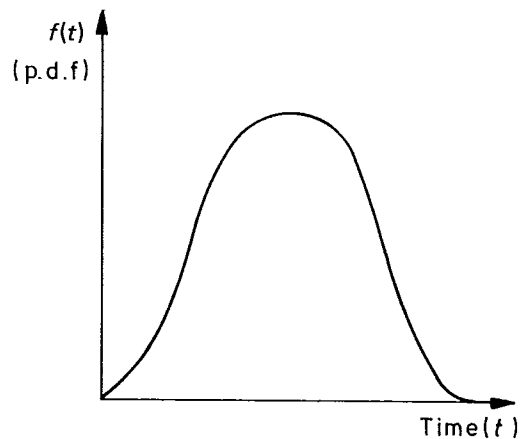


Figure 2. Probability density function

variance, and its positive square root is called the *standard deviation*.

$$\theta = \mu_{1,0} = \int_{-\infty}^{+\infty} t f(t) dt$$

$$\sigma^2 = \mu_{2,\theta} = \int_{-\infty}^{+\infty} (t - \theta)^2 f(t) dt$$

Dimensionless coefficients can be used to compare the shapes of different distributions, namely,

σ/θ is the coefficient of variation

$\mu_{3,\theta}/\sigma^3$ is the coefficient of skewness

$\mu_{4,\theta}/\sigma^4$ is the coefficient of kurtosis (peakedness)

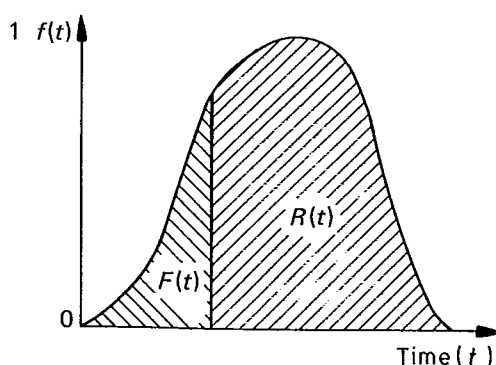
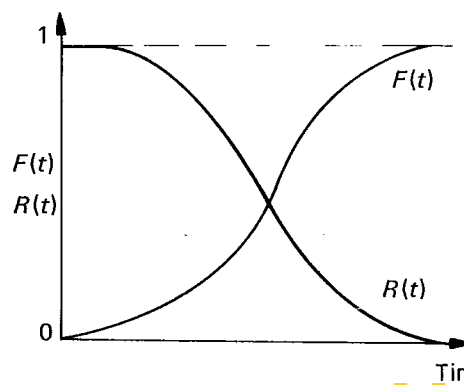
4.3 Hazard. Statistical hazard (used in its statistical context as distinct from its qualitative meaning, e.g. in the chemical industry) is a fundamental reliability concept. If n items start to operate at time $t_0 = 0$, and if r_1 of them fail and w_1 are withdrawn from service for other reasons before t_1 and if, after t_0 , r_2 fail and w_2 are withdrawn before t_2 , then the hazard $Z_{1,2}$ for the interval t_1 to t_2 is defined as the number failing divided by the number at risk. Strictly, the number at risk used in a calculation of $Z_{1,2}$ should be the average number of survivors for the interval, i.e.

$$s_{1,2} = \frac{1}{2} \left[(n - r_1 - w_1) + (n - r_2 - w_2) \right]$$

but a frequent convention is to use the number of survivors at the end of the previous interval, namely $s_1 = n - r_1 - w_1$ is the number at risk at the start of the interval (t_1 to t_2). The hazard in either case is given by

$$Z_{1,2} = (r_2 - r_1)/s$$

The mean hazard rate or observed failure rate for the interval t_1 to t_2 is given by $Z_{1,2}/(t_2 - t_1)$.

Figure 3. Relationship of $F(t)$ to $R(t)$ 

4.4 Hazard rate function. The hazard rate function usually denoted by $z(t)$, alternatives being $h(t)$ and $\lambda(t)$, is the instantaneous failure rate. It represents the conditional probability of failure in an interval t to $t + \delta t$ given survival to t , divided by δt as δt tends to zero, i.e.:

$$z(t) = \lim_{\delta t \rightarrow 0} \frac{F(t + \delta t) - F(t)}{[1 - F(t)] \delta t}$$

The limit of $[F(t + \delta t) - F(t)] / \delta t$ is simply $f(t)$ and $1 - F(t) = R(t)$, so:

$$z(t) = f(t)/R(t)$$

If $z(t)$ is a constant, usually denoted λ , then $f(t) = \lambda e^{-\lambda t}$ which is the exponential p.d.f.

Other useful relationships which stem from this definition are as follows:

$$\begin{aligned} z(t) &= \frac{-1}{R(t)} \frac{dR(t)}{dt} \\ &= -\frac{d}{dt} \ln R(t) \end{aligned}$$

i.e.

$$R(t) = \exp \left(- \int_0^t z(t) dt \right)$$

The Cumulative Hazard Function is defined as the integral of the hazard rate function, i.e.:

$$\begin{aligned} H(t) &= \int_0^t z(t) dt \\ &= -\ln R(t) \end{aligned}$$

$$R(t) = e^{-H(t)}$$

$H(t)$ is *not* a probability and can exceed unity.

4.5 Likelihood (For further reference see 9.49 of BS 5532 : 1978). If n independent and random times to failure of like items are observed to be $t_i, i = 1, 2, \dots, n$, then the likelihood that they all come from a p.d.f. $f(t, a)$, where a is a parameter of the distribution, is given by the product rule of probability, i.e.

The likelihood function of the sample, n

$$= L = \prod_{i=1}^n f(t_i)$$

The likelihood function is a joint density of the n variables and is a function of a . Likelihood is most often used to estimate distribution parameters (see BS 5532), so if all the parameters of L are known or assumed except a , then a likelihood function for a can be synthesized from the sample values t_i by calculating L for various trial values of a .

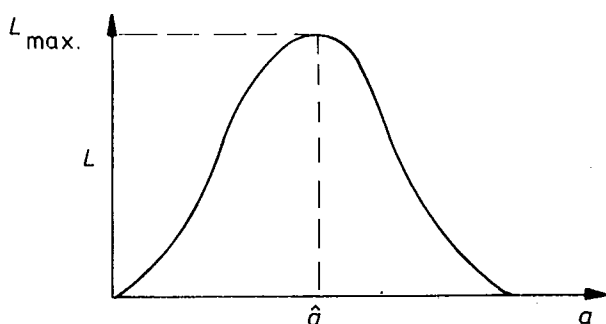


Figure 4. Likelihood function

The value $\hat{a}$ which corresponds to $L_{\max}$ is the maximum likelihood estimate (MLE) of a , given the other parameters.

More formally, the value of a which maximizes the likelihood function is found. Often the probability distribution function $f(t_i, a)$ will involve exponential terms, so it is convenient to work with $\ln L$. The logarithm of a function and the function itself will assume their maximum values together.

Therefore

$$L(a) \text{ is defined as: } \ln \prod_{i=1}^n f(t_i, a) = \sum_{i=1}^n \ln f(t_i, a)$$

and

$$\frac{\partial L}{\partial a} = 0 \text{ is solved for } a.$$

This value of $\hat{a}$ is the maximum likelihood estimate.

NOTE. Maximum likelihood estimators tend to be normally distributed as n increases.

4.6 Confidence limits. Frequently the purpose of a statistical technique is to obtain an estimate of some population or distribution parameter from sample data. In such cases it is not uncommon to require an indication of the uncertainty surrounding this estimate; in other words it is necessary to determine a confidence interval from the estimate. Clearly a 'best estimate' is desired, which is the term usually applied to an estimator that is both unbiased and most efficient. An estimator is said to be unbiased if its average, or mathematical expectation, is equal to the population parameter being estimated. If there are two or more unbiased estimators available, the one having the least variance (i.e. the one in which distribution clusters most closely about the population parameter being estimated) is called the most efficient. For further reference see 2.59 to 2.69 of BS 5532 : 1978.

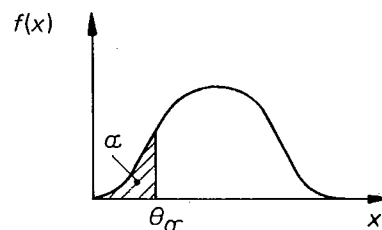


Figure 5. Single-sided confidence limits

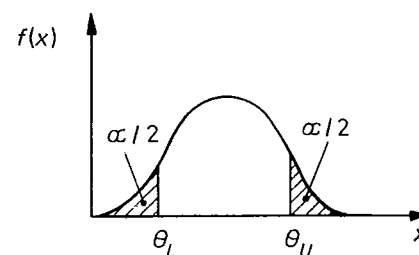


Figure 6. Two-sided confidence limits

The form of the p.d.f. for the parent population is known, but the parameters have to be estimated from the sample data. Although only one distributional form and one population parameter, θ , is considered here, the method can be readily extended to any number of unknown parameters and to other distributions.

From the sample data $(x_1, x_2, \dots, x_n)$, and using the best estimate, it is generally possible to find two limits θ_L (lower) and θ_U (upper) such that the true value of the unknown parameter, θ , lies in the interval $(\theta_L \leq \theta \leq \theta_U)$ with some pre-assigned probability $(0 < \alpha < 1)$. θ_L and θ_U are themselves functions of the sample data and the interval $(\theta_L : \theta_U)$ is a random interval. Each sample will produce its own limits and intervals and, by continuing the sampling process, a whole family of intervals would be generated; 100 $(1 - \alpha)$ % of these intervals are such that they contain the true value θ . This is often loosely interpreted as: 'There is 100 $(1 - \alpha)$ % confidence that the true θ lies between θ_L and θ_U limits derived from the sample'.

The interval between the confidence limits θ_L, θ_U , obtained from the sample data, is called the *two-sided confidence interval*.

$(1 - \alpha)$ is the confidence level, commonly used values being 0.99, 0.95, 0.90 and 0.50.

Two-sided limits are normally symmetrical with $\alpha/2$ in each 'tail', but the other divisions of the available α can be chosen if desired. In the limit, if α is confined to either the upper or the lower tail, this gives an upper or lower single-sided confidence interval.

4.7 Measures of central tendency, mean, median and mode.

Any likelihood function, including a p.d.f., has a mean, a median and a mode. In various applications all these measures of central tendency are used in reliability: they only coincide in special cases such as the normal distribution p.d.f.

For example, a frequently required estimate of central tendency is the 'best' estimate to assign to the i th failure out of a sample of n times to failure, the objective being to estimate the function $F(t)$ parameters from the sample data. In this case the likelihood function for $F(t_{i,n})$, the 'true' value of $F(t)$ for the i th failure out of a sample of n is of Beta form. The Beta is an inversion of the Binomial distribution, namely

$$F(t_i) = \left\{ F(t)^{i-1} [1 - F(t)]^{n-i} \right\} / \int_0^1 u^{i-1} (1-u)^{n-i} du \\ = \left\{ n! F(t)^{i-1} [1 - F(t)]^{n-i} \right\} / (i-1)! (n-i)! \\ \text{for } i \text{ integer}$$

By taking moments and numerical integration it can be shown that

$$\begin{aligned} \text{mean rank } \bar{L}_{F(t_{i,n})} &= i/(n+1) \\ \text{median rank } L_{0.5, F(t_{i,n})} &\approx (i-0.3)/(n+0.4) \\ \text{mode or maximum likelihood estimate (m.i.e.)} \\ \hat{L}_{F(t_{i,n})} &= (i-1)/(n-1) \end{aligned}$$

The meanings are as follows. Normally the true rank of a failure is unknown, but if N samples of size n are taken from the distribution of $F(t_i, n)$, then, as $N \rightarrow \infty$, the average of the values of $F(t_i)$ corresponding to the N times of the i th failure out of n would be the mean rank, i.e.

$$\lim_{N \rightarrow \infty} \sum_{j=1}^N F(t_{i,n})_j / N = i/(n+1)$$

Alternatively use an estimate that will be too high half the time and too low the other half of the time: the median rank.

In the same experiment 50 % of the $F(t_{i,n})$ would lie below the median rank value of $F(t_{i,n})$ and 50 % above.

The $\frac{i-0.3}{n+0.4}$ expression due to B  nard is a very good

approximation to the value obtained from the binomial expansion solution. Furthermore, in the same experiment the value that would appear most often would be the mode or maximum likelihood estimate $(i-1)/(n-1)$.

In distribution parameter estimation, these values are used in reverse as estimators of $F(t_{i,n})$ parameters. Different estimates can thus be obtained from the same sample data. Which measure to use is a matter of judgement, previous practice and circumstances. If confidence limits are required for these estimators of rank, the median is often used but, if not, the mean is more usual because it is easier to calculate. Maximum likelihood estimators are commonly used for parameter estimation.

4.8 Types of value for reliability characteristics.

For further reference, see 4.1 and BS 4778. Reliability or the characteristics of reliability (such as mean time to failure, moments of the distribution) can be known or estimated in several ways which result in different values, as follows,

(a) *Population values* are based upon complete data for an entire finite population. By their nature they cannot be known during the life of the items, and they are in any case degraded to sample values as soon as another item is put into service. Population values are simple to calculate, e.g. the reliability (to time t , if appropriate) is simply the number surviving (to time t) divided by the number in the population.

$$R = (n - r)/n$$

Any population, particularly a small one, can be considered as only a sample from an infinite population and, if this is done, it is appropriate to treat the population data as if they were sample data from which the larger population parameters are estimated.

(b) *Sample or observed values* are calculated from sample data. Such values are easy to calculate but may be biased as estimators of the true value. For example, the observed mean and variance of a sample of n items are:

$$\bar{x} = \sum_{i=1}^n x_i / n \text{ which is unbiased}$$

and

$$s^2 = \sum_{i=1}^n (x_i - \bar{x})^2 / n$$

which is biased (see 2.58 of BS 5532 : 1978).

(c) *Estimated and assessed values* are derived from data usually using statistical methods. These values are usually quoted as a measure of central tendency with confidence limits, i.e. the confidence interval between the limits contains the true value with confidence $(1 - \alpha)$ (see 4.6).

(d) *True values* are conceptual only and can never be known precisely from sample data. They are population values for an infinite population. They are commonly estimated from sample data by a measure of central tendency and confidence limits.

4.9 **Sample data: censoring.** It is often required to estimate values of $F(t)$ from a sample of times to failure, some of which may be incomplete, i.e. observation ceased before failure took place. It is not correct to ignore these incomplete lives: knowledge that an item operated for a

certain time without failure is useful and should not be discarded. Moreover, failures may occur by a mode which is not under investigation as well as by the mode that is being studied. The former are incomplete lives in the context of the estimation of $F(t)$ for the particular mode of failure. Data sets with incomplete lives as well as failure times are called *censored*. There are three types of censoring as follows.

Type I. Time terminated. The test is continued until all items have either run for a fixed time or failed prior to this time.

Type II. Failure terminated. The test is stopped as soon as r out of n of the items have failed.

Type III. Items are withdrawn at random from the test. This type of censoring, also called *progressive censoring*, is typical of data from actual service and of tests in which failures are replaced during testing by new or repaired items.

Types I and II are particular cases of type III.

NOTE. Withdrawing items depending on their condition after inspection is not censoring but preventive maintenance. In censoring, all items have an equal chance of failing up to the withdrawal time, and the selection of items to be censored is random.

A common method of dealing with censored data is first to arrange all the data in time order and assign an order number to each failure for the mode being studied. In general, the order numbers of failures following the first suspension will no longer be integers but will take fractional values to allow for the suspended or censored item. The general formula for the next increment, i.e. as soon as the first suspension is encountered is:

$$\text{New increment} = \frac{(n+1) - (\text{previous rank order number})}{1 + (\text{number of items beyond the present suspended item})}$$

This increment is added to the previous rank order number until another suspended item is reached when a new increment is calculated. In general, the next failure will have a rank order number equal to the preceding number plus the new increment. As a check, the last increment calculation added to the last rank order number should equal $n+1$. Median ranks and confidence limits can then be determined from the median rank tables using the sample size column and interpolating fractional order numbers.

5. Estimation of failure distributions for items

5.1 Introduction. The aim of this guide is to give only an outline of the procedure that should be followed in estimating failure and repair time distribution functions and to provide some warnings about common errors. Any procedure should consist of statistical analysis to sufficient depth to elicit the required information, or such information as can be obtained from the sometimes limited data.

NOTE. The limitations of the data should be recognized by calculating confidence limits as well as mean or median estimates, and it is advisable to plot the results on suitable graph paper to obtain a visual impression of the data limitations.

The exponential model (constant failure rate) is so simple to use that it is tempting to omit the procedures involved in distribution parameter estimation and simply take the observed failure rate. The collection of suitable data for full distribution analysis costs more than for observed failure rate calculation, and some justification

for this extra work is given below:

(a) The form of the distribution gives clues as to the nature of the failures. For example, an exponential distribution (demonstrated rather than assumed) may mean that an item is subject to several failure modes but that none of these is dominant. This might give confidence that the design or the maintenance schedule is well balanced. A hyper-exponential distribution might indicate the possibility of ineffective maintenance, and a Normal distribution could indicate the presence of a dominant wearout failure mode. A lognormal distribution of failures to identical parts would lead one to suspect fatigue as the cause.

(b) The prediction of reliability and reliability confidence limits will be enhanced by knowledge of the underlying distribution. This is essential knowledge for establishing economic maintenance and replacement intervals and for determining how many machines may be required to sustain a required rate of output. For example, it is possible to modify an equipment in such a way that, although its mean failure rate is reduced, the reliability over a vital mission time has actually fallen. If the development engineers fail to monitor the distribution as well as the mean, they could be misled.

5.2 Suggested outline procedure. More details of distributions are given in 5.3 and, for more detailed analysis of any particular step, standard reference books may be consulted. This outline is set out so that analysis may be taken to the desired or necessary depth with minimum wasted effort. See flow chart (figure 7).

Let the data consist of n times to failure x_i ($i = 1, 2, \dots, n$) and $N - n$ censored times t_j ($j = 1, 2, \dots, N - n$).

$$(1) \text{ Determine the sample mean } \bar{x} = \left(\sum_{i=1}^n x_i + \sum_{j=1}^{N-n} t_j \right) / n$$

(2) Estimate the standard deviation from the sample data but, if the sample is heavily censored, go straight to step (5) below.

$$s = \left[\sum_{i=1}^n (x_i - \bar{x})^2 \right]^{1/2} / (n-1)$$

(3) Estimate roughly the sample median value as the middle failure time or the average of the two middle failure times if the sample number, n , is even.

(4) If $\bar{x} \approx s$, it is likely that the hazard rate, $z(t)$ is constant, i.e. a negative exponential distribution. Constant $z(t)$ may indicate the need for deeper analysis of the causes of the failure because it frequently arises as the sum of several failure modes, none being dominant. Qualitative information will be needed for this analysis. Data from maintained systems also often has this form of distribution. The next most common distribution form is the Normal, evidence of which is when mean and median are approximately equal in the sample data.

(5) The next step is to place the complete and censored times (unknown finishes) in ascending order of time. Divide the time between the first and last failures into about 10 equal intervals. Then construct a table (see table 1) of the hazard, survivors, cumulative hazard and hazard rate. In the table, x_i represent individual times to failure and t_j are the times at which the data are censored.

Table 1. Survivors, hazard, hazard rate and cumulative hazard for a range of time intervals

Interval	<i>F</i> Failures	<i>C</i> Censored	<i>S</i> Survivors	<i>Z</i> Hazard	<i>z(t)</i> Hazard rate	<i>H</i> Cumulative hazard
0 to $x_1 = t_1$	0	C_1	$N - C_1 = S_1$	0	0	0
x_1 to t_2	F_1	C_2	$N - F_1 - C_2 = S_2$	$Z_1 = \frac{F_1}{(N - C_1)}$	$\frac{F_1}{(N - C_1)(t_2 - x_1)}$	$\frac{F_1}{(N - C_1)}$
t_2 to t_3	F_2	C_3	$S_2 - F_2 - C_3$	$Z_2 = \frac{F_2}{S_2}$	$\frac{F_2}{S_2(t_3 - t_2)}$	$H_1 + Z_2$
t_i to t_{i+1}	F_i	C_{i+1}	S_i	$\frac{F_i}{S_i}$	$\frac{Z_i}{(t_{i+1} - t_i)}$	$H_{i-1} + Z_i$

tion of hazard are made. If there are only a few actual failure times, they may be used as such. The interval hazard is taken to be the number of failures in the interval divided by the number at risk at its start. The first calculation is usually made at the time of the earliest failure so that $t_1 = x_1$.

Plot $z(t)$ against t . If $z(t)$ is increasing, the distribution is probably either Weibull or Normal. If $z(t)$ increases and then either levels off or falls, the Lognormal or Gamma distributions are possible. If $z(t)$ is falling, the distribution is Hyper-exponential, and either the true Hyper-exponential or Weibull models ($\beta < 1$) should be used.

(6) By plotting the cumulative hazard, H , versus time on log against log paper, it is possible to get a rough estimate of the Weibull parameters. The slope of the graph estimates β and the intercept at $H(t) = 1$ estimates η . In deciding whether β is greater or less than 1, in borderline cases regard should be paid to the number of data points involved. If $\beta \approx 1$, the Exponential distribution applies.

(7) If item failure results from the first or last failure of many components (e.g. the first blade to fail on a turbine constitutes turbine failure; the last engine to fail brings the aircraft down), then Extreme Value, log Extreme Value or Weibull distributions may give the best fit.

(8) When a distribution form is being chosen, it will usually be required to estimate the c.d.f. $F(t)$ using graphical means or other methods. It is strongly advised that confidence limits be calculated or plotted as well as central estimates.

(9) A number of statistical goodness of fit tests are applicable to discover which of a number of possible distributions gives the best model. Although directly plotting failures against time is a useful first step, it can give a false visual impression, and a graphical alternative (to judge the suitability of a model) is to draw confidence limits at (say) 5 % and 95 % about the distribution form and line which looks the best fit. Then see how well alternatives fit within these limits.

5.3 Statistical failure models. It is strongly emphasized that finding the most suitable distribution model and its parameters is as much a matter of judgement, experience and the actual failure mechanism as it is of the statistics and mathematics. For this reason, computer programs to find parameters may not be

satisfactory unless they produce graphical representations of trial distributions for the reliability engineer to consider the possible interactions.

NOTE. Appendix A contains an illustrated summary of the properties of the statistical distributions discussed in 5.3.

5.3.1 The Exponential distribution

$$f(t) = \frac{1}{\theta} e^{-t/\theta}$$

$$R(t) = e^{-t/\theta}$$

$$z(t) = \frac{1}{\theta}$$

$$\text{mean} = \theta$$

$$\text{variance} = \theta^2$$

The Exponential distribution is the most widely used distribution in life testing. This is because it is an appropriate model where failure of an item is due not to deterioration as a result of wear, but rather to random events. However, it is necessary to emphasize that the failure process has to be random and in accordance with the postulates of the Poisson process in that:

- (a) the process is time homogeneous and the occurrence of random events is independent of past occurrence;
- (b) the probability of exactly one random event in any time interval is proportional to the length of that time interval.

In other words, if the number of failures occurring over an interval of time is Poisson distributed, the time between failure is exponentially distributed.

Equally it can be seen that the occurrence of random failure with constant proportionality implies a constant hazard (or constant hazard rate).

IMPORTANT NOTE. It is unfortunate that, in many cases, the justification for the use of this model has been its simplicity and, in consequence, ease of mathematical manipulation, rather than a clear understanding of the processes that lead to it. In many cases it may be not unreasonable to assume that an item has a constant hazard function over a limited part of its life, but to assume a constant hazard function over its entire life, or (to a constant hazard function over any part of an item's life) without adequate justification, is bound to result in misleading or irrational conclusions.

A more appropriate application of the exponential distribution is as a time to failure model for complex systems, where the system consists of a large number of components in series, none of which individually

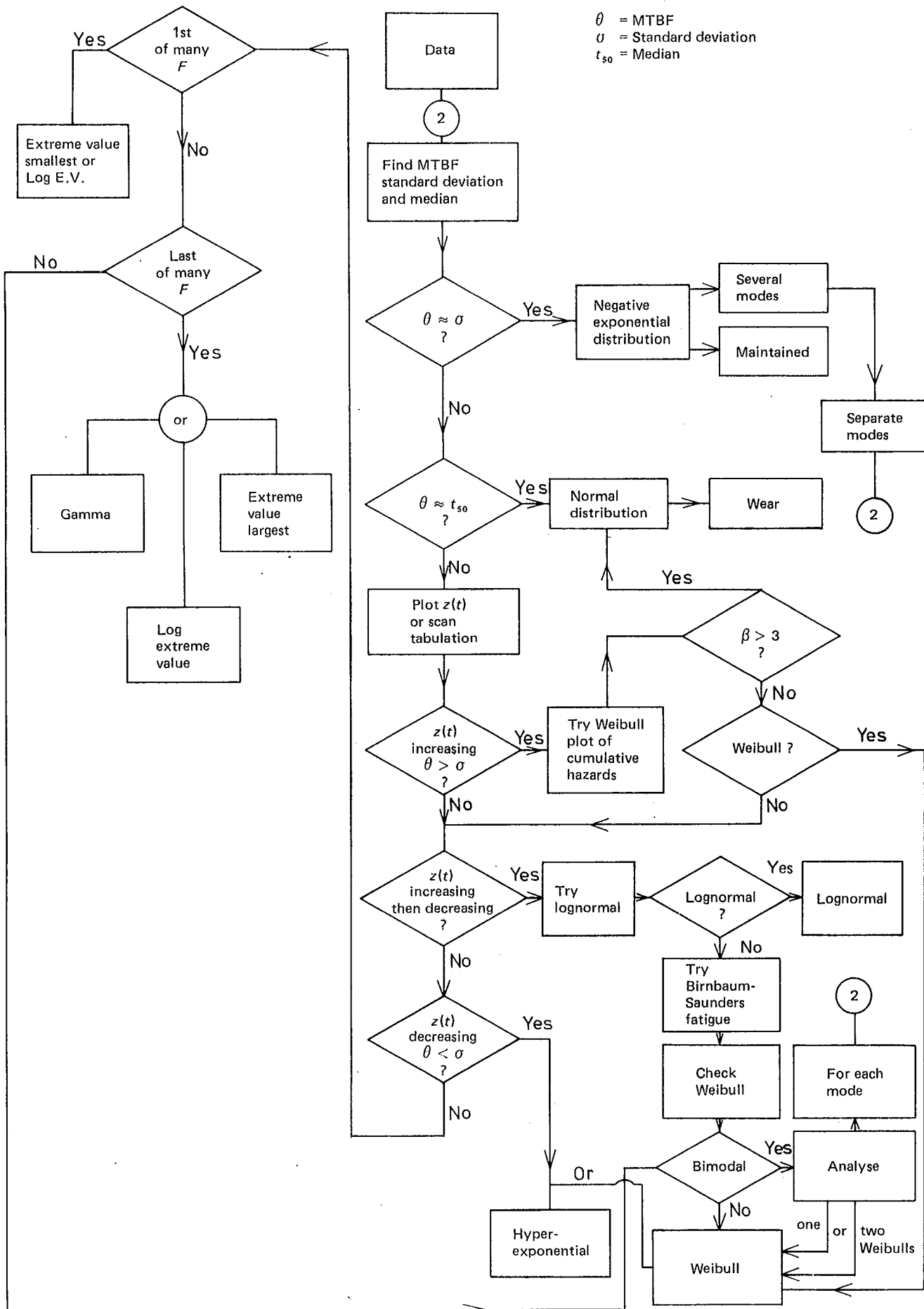


Figure 7. Analysis of failure data flow chart

contributes significantly to the total failure density. Equally, if in the system a particular component fails and is immediately replaced by another of the same type, the exponential distribution can be considered to be appropriate as a time between failure model. It is worth noting that this model is suitable irrespective of the failure distribution of any of the individual components. This parallels the situation in non-life statistics where the central limit theorem establishes the Normal distribution as an appropriate model in many applications.

However, to consider the exponential model as a suitable model for all systems irrespective of the system condition is as erroneous as the assumption of normality for all non-life situations. A further considerable constraint of the exponential is that it does not necessarily enjoy the robust properties of the Normal distribution.

A more general form of the Exponential distribution is the two parameter form where

$$z(t) = 0 \quad \text{when } 0 \leq t < \gamma$$

$$z(t) = \frac{1}{\theta} \quad \text{when } t \geq \gamma$$

which leads to:

$$f(t) = \frac{1}{\theta} e^{-\frac{(t-\gamma)}{\theta}}$$

$$R(t) = e^{-\frac{(t-\gamma)}{\theta}}$$

γ often being referred to as the *threshold* or *shift parameter*.

5.3.2 The Gamma distribution

Shape parameter $c > 0$

Scale parameter $\lambda > 0$

$$f(t) = \frac{\lambda^c t^{c-1} \exp(-\lambda t)}{\Gamma c}$$

$$R(t) = \int_t^\infty f(t) dt$$

where

Γc is the gamma function of c and is equal to

$$\int_0^\infty \exp(-u) u^{c-1} du$$

where

c is a positive integer, the distribution is known as the *Erlang distribution* and:

$$R(t) = \exp(-\lambda t) \sum_{i=0}^{c-1} \frac{[(\lambda t)^i]}{i!}$$

$$z(t) = f(t)/R(t)$$

$$\text{mean} = c/\lambda$$

$$\text{variance} = c/\lambda^2$$

if

- (1) $c > 1$ the distribution is unimodal;
- (2) as c becomes larger, $f(t)$ quickly tends to Normal;
- (3) $c = 1$, the distribution is exponential;
- (4) $c < 1 \approx$ Hyper-exponential (decreasing failure rate).

The Gamma distribution is a natural extension of the Exponential distribution in that it represents the c -fold convolution of an exponential distribution. In fact the Exponential distribution is a special case of the Gamma distribution.

A further important case of the Gamma distribution is:

The Chi Square distribution,

where

$$\lambda = 0.5 \text{ and } c = \frac{n}{2}$$

The Gamma distribution has the facility to model many reliability situations. Typically:

- (1) The time to the n th failure of a system, where the underlying distribution is exponential.
- (2) The time to failure of an item whose failure results from receiving precisely c shocks, the times between which are exponentially distributed.
- (3) The number of units in need of repair in an extensive and complex group of equipments.

The facility of the Gamma distribution to traverse the entire spectrum of shapes of other important life distributions lends it to the arbitrary fitting of data. However, such a practice should be avoided unless there are sound technical reasons to expect the data to be Gamma distributed, or adequate data exist to preclude the distributions that will give a more systematic interpretation of the data such as those of the Extreme Value family or Normal distributions.

5.3.3 The Weibull distribution

$$f(t) = \frac{\beta (t - \gamma)^{\beta-1}}{\eta^\beta} \exp \{ - [(t - \gamma)/\eta]^\beta \}$$

$$R(t) = \exp \{ - [(t - \gamma)/\eta]^\beta \}$$

$$z(t) = [\beta (t - \gamma)^{\beta-1}] / \eta^\beta$$

$$\text{mean} = \eta \Gamma(1 + 1/\beta) + \gamma$$

$$\text{variance} = \eta^2 \{ \Gamma(1 + 2/\beta) - [\Gamma(1 + 1/\beta)]^2 \}$$

N.B. Often the Weibull distribution is expressed using

$$\alpha = \eta^\beta \quad \text{or} \quad \delta = 1/\eta^\beta$$

The ability of the Weibull distribution to model failure situations, where non-constant hazard functions apply, makes this distribution one of the most generally useful distributions for analysing failure data.

If $\beta > 1$, this indicates an increasing $z(t)$ which may be symptomatic of wear out. Where $\beta = 2$, this means that $z(t)$ is linearly increasing with t and the resulting distribution is a special case of the Weibull known as the *Rayleigh*.

If $\beta = 1$, $z(t)$ is constant, which means that incidences of failure are random and in accordance with the postulates of the Poisson process, the resulting distribution is the Exponential (a further special case of the Weibull distribution).

If $\beta < 1$, this indicates a decreasing $z(t)$.

A considerable number of research studies have shown that the Weibull distribution can be applicable in such diverse areas as particle sizing, corrosion studies, sales analyses, droplet sizes in sprays, surface finishes as well as an extensive number of reliability studies for electronic, electromechanical and mechanical components or equipments. The theoretical justification of the Weibull distribution in these areas owes much to the extreme value theory (see 5.3.4) which demonstrates it as one of the asymptotic distributions of smallest extremes.

A further considerable factor that enhances the importance of the Weibull distribution is that many phenomena that, as yet, cannot be theoretically shown to be Weibull

distributed, can be *empirically* shown to be Weibull distributed. The versatility of this distribution in itself suggests that it can form the basis of a formally structured approach to analysing data. Initial Weibull plots will, with experience, suggest what other special graph papers might be tried.

5.3.4 Extreme value theory. Failure of a system or item may be dependent on its weakest link, or its largest flaw, i.e. the smallest or largest element. Unfortunately in many cases the true distribution of the elements may not be known, nor can it be evaluated. However the distribution of the smallest or largest element will in general be dependent on the sample size of the element and the nature of the element's initial distribution. If the sample size is suitably large, and if it is possible to make some limited assumptions concerning the initial distribution, it can be shown that, for an extensive number of initial distributions, the asymptotic distribution of both the largest and smallest element will take one of the following forms:

- (1) type I, which is the Extreme Value (or Gumbel) distribution;
- (2) type II, which is the log Extreme Value (or Rosin Rammler) distribution;
- (3) type III, which is the Weibull distribution.

5.3.5 Extreme Value distribution

NOTE. The distribution of the smallest extreme is given. To obtain the distribution of the largest extreme put $(a - t)$ for $(t - a)$.

Range $-\infty \leq t \leq +\infty$

Location parameter a

Scale parameter $b > 0$

$$f(t) = \frac{1}{b} \exp \left[(t - a)/b \right] \exp \left\{ -\exp \left[(t - a)/b \right] \right\}$$

$$R(t) = \exp \left\{ -\exp \left[(t - a)/b \right] \right\}$$

$$z(t) = \frac{1}{b} \exp \left[(t - a)/b \right]$$

$$\text{mean, } \theta = a - 0.57721b$$

$$\text{variance, } \sigma^2 = b^2 \pi^2/6$$

(where 0.57721 is Euler's constant).

The Extreme Value distribution is another distribution of wide applicability and, as such, arises in the study of floods, aeronautics, metrology, breaking strengths, etc.

Its importance in reliability is that it is an appropriate model in which the hazard function is exponential in form. It can be demonstrated that this distribution is the limiting model of the asymptotic distribution of independent maximum values as n approaches infinity where the initial distribution's right-hand tail is unbounded and is of the exponential type, i.e. the initial cumulative distribution function approaches unity at least as rapidly as the exponential distribution. Since the Normal, Lognormal and restricted cases of the Gamma and Weibull distributions are all unbounded to the right and are of the exponential type, this leads to a wide range of initial distributions. Thus the Extreme Value distribution for maximum values serves as a suitable model for failure times of a system in many situations in which the system can be considered as having n parallel elements and it can be assumed that n is large and the component part failure times come from the same exponential type distribution and are independent.

Equally the Extreme Value distribution for minimum values will often be found to be appropriate as a model for failure times of a system that can be considered as having n elements in series and where the failure distribution of the components are reasonably uniform and similar.

5.3.6 The log Extreme Value distribution

$$f(t) = \left\{ \frac{\delta \alpha^\delta}{t^\delta + 1} \right\} \exp \left[- \left(\frac{\alpha}{t} \right)^\delta \right]$$

$$R(t) = 1 - \exp \left[- (\alpha/t)^\delta \right]$$

$$z(t) = f(t)/R(t) \quad (\text{No simplification})$$

The log Extreme Value distribution stands in the same relationship to the Extreme Value distribution as does the Lognormal distribution to the Normal distribution in that the logarithms of log Extreme Value distributed variates are Extreme Value distributed.

Whilst not having the wide applicability in reliability as some other distributions, the log Extreme Value distribution is appropriate where the failure of a component is dependent on its strength which deteriorates logarithmically with time, e.g. in a separately bladed turbine the strength of the weakest blade fixing. If this strength decreases logarithmically with time, e.g. failure due to fatigue, the failure distribution for the turbine rotor (or wheel) will be of a log Extreme Value form.

5.3.7 The Normal distribution

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[- \frac{(t - \theta)^2}{2\sigma^2} \right]$$

$$R(t) = \int_t^\infty f(t) dt$$

$$z(t) = f(t)/R(t)$$

$$\theta = \text{mean}$$

$$\sigma^2 = \text{The variance}$$

The integrals are tabulated for reference. The Normal distribution is unquestionably the most important of all statistical distributions in that it can be considered to be an appropriate distribution model in 80 % or more cases of distributional analyses in a non-life situation.

The theoretical justification for this is based upon the central limit theorem which, simply stated, says that the distribution of n independent observations from any distribution or even up to n different distributions, with a finite mean and variance, approaches a normal distribution as n becomes increasingly large, which leads to the conclusion that the normal distribution is the model for the sum of n positive variates under very general conditions. While the Normal distribution is often applicable to non-life statistics, it is not so relevant when considering life statistics where it appears to have the restricted role of explaining wear-out failures. It should be noted that the function exists to $-\infty$ and so may have significant value at $t = 0$. Since negative time is usually meaningless in reliability, great care should be used in manipulating this model unless $\theta > 3\sigma$, when the tail area below $t = 0$ can be considered negligible. Although the Normal distribution has only the limited applicability to life data, it may sometimes be considered in cases other than those where failure is due to wear.

This distribution, especially with modification, has been used, with varying levels of success, to explain failures other than wear. For instance, consider a flywheel of a motor which is assembled from two parts of the same type. It is important that these parts are of nearly the