

Power law model — Goodness-of-fit tests and estimation methods

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National foreword

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**Modèle de loi en puissance –
Test d'adéquation et méthodes d'estimation
des paramètres**

**Power law model –
Goodness-of-fit tests and estimation methods**



Commission Electrotechnique Internationale
International Electrotechnical Commission
Международная Электротехническая Комиссия

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INTRODUCTION

This International Standard describes the power law model and gives step-by-step directions for its use. There are various models for describing the reliability of repaired items, the power law model being one of the most widely used. This standard provides procedures to estimate the parameters of the power law model and to test the goodness-of-fit of the power law model to data; to provide confidence intervals for the failure intensity and prediction intervals for the length of time to future failures. An input is required consisting of a data set of times at which relevant failures occurred, or were observed, for a repaired item or a set of copies of the same item, and the time observation of the item was terminated, if different from the time of final failure. All output results correspond to the item type under consideration.

Some of the procedures may require computer programs, but these are not unduly complex. This standard presents algorithms from which computer programs should be easy to construct.

POWER LAW MODEL – GOODNESS-OF-FIT TESTS AND ESTIMATION METHODS

1 Scope

This International Standard specifies procedures to estimate the parameters of the power law model, to provide confidence intervals for the failure intensity, to provide prediction intervals for the times to future failures, and to test the goodness-of-fit of the power law model to data from repaired items.

2 Normative references

The following normative documents contain provisions which, through reference in this text, constitute provisions of this International Standard. For dated references, subsequent amendments to, or revisions of, any of these publications do not apply. However, parties to agreements based on this International Standard are encouraged to investigate the possibility of applying the most recent editions of the normative documents indicated below. For undated references, the latest edition of the normative document referred to applies. Members of IEC and ISO maintain registers of currently valid International Standards.

IEC 60050(191):1990, *International Electrotechnical Vocabulary (IEV) – Chapter 191: Dependability and quality of service*

3 Definitions

For the purposes of this International Standard, the terms and definitions of IEC 60050(191) apply.

4 Symbols and abbreviations

For the purposes of this International Standard, the following symbols and abbreviations apply:

β	shape parameter of the power law model
$\hat{\beta}$	estimated shape parameter of the power law model
β_{LB}, β_{UB}	lower, upper confidence limits for β
C^2	Cramer-von-Mises goodness-of-fit test statistic
$C^2_{1-\gamma}(M)$	Critical value for the Cramer-von-Mises goodness-of-fit test statistic at γ level of significance
χ^2	Chi-square goodness of fit test statistic
$\chi^2_{\gamma}(\nu)$	γ th fractile of the χ^2 distribution with ν degrees of freedom
d	number of intervals for groups of failures
$E[N(t)]$	expected accumulated number of failures up to time t
$E[t_j]$	expected accumulated operating time to j th failure

$\hat{E}[N[t(i)]]$	estimated expected accumulated number of failures up to $t(i)$
$\hat{E}[t_j]$	estimated expected accumulated operating time to j th failure
$F_\gamma(\nu_1, \nu_2)$	γ th fractile for the F distribution with (ν_1, ν_2) degrees of freedom
i	general purpose indicator
j	general purpose indicator
k	number of items
L, U	multipliers used in calculation of confidence intervals for failure intensity
λ	scale parameter of the power law model
$\hat{\lambda}$	estimated scale parameter of the power law model
M	parameter for Cramer-von-Mises statistical test
N	number of relevant failures
N_j	number of failures for j th item
$N(t)$	accumulated number of failures up to time t
$N[t(i)]$	accumulated number of failures up to time $t(i)$
R	difference between the order number of future (predicted) failure and order number of last (observed) failure
T	accumulated relevant operating time
T^*	total accumulated relevant operating time for time terminated test
T_j	total accumulated relevant operating time for j th item
T_{RL}, T_{RU}	lower, upper prediction limits for the length of time to the R th future failure
$\hat{T}_{N+1}$	estimated median time to $(N+1)$ th failure
t_i	accumulated relevant operating time to the i th failure
t_{ij}	i th failure time for j th item
t_N	total accumulated relevant operating time for failure terminated test
t_{Nj}	total accumulated relevant operating time for to N th failure of j th item
$t(i-1), t(i)$	endpoints of i th interval of operating time for grouped failures
$z(t)$	failure intensity at time t
$\hat{z}(t)$	estimated failure intensity at time t
z_{LB}, z_{UB}	lower, upper confidence limits for failure intensity

5 Power law model

The statistical procedures for the power law model use the relevant failure and time data from the test or field studies.

The basic equations for the power law model are given in this clause. Background information on the model is given in annex A and examples of its application are given in annex B.

The expected accumulated number of failures up to test time t is given by

$$E[N(t)] = \lambda t^\beta \text{ with } \lambda > 0, \beta > 0, t > 0$$

where

λ is the scale parameter;

β is the shape parameter ($0 < \beta < 1$ corresponds to a decreasing failure intensity; $\beta = 1$ corresponds to a constant failure intensity; $\beta > 1$ corresponds to an increasing failure intensity).

The failure intensity at time t is given by

$$z(t) = \frac{d}{dt} E[N(t)] = \lambda \beta t^{\beta-1} \text{ with } t > 0$$

Thus the parameters λ and β both affect the failure intensity in a given time.

Methods are given in 7.2 for maximum likelihood estimation of the parameters of λ and β . Subclause 7.3 gives goodness-of-fit tests for the model and 7.4 and 7.5 give confidence interval procedures. Subclause 7.6 gives prediction interval procedures and 7.7 gives tests for the equality of the shape parameters.

The model is simple to evaluate. However when $\beta < 1$, theoretically $z(0) = \infty$ (i.e. $z(t)$ tends to infinity as t tends to zero) and $z(\infty) = 0$ (i.e. $z(t)$ tends to zero as t tends to infinity); but this theoretical limitation does not generally affect its practical use.

6 Data requirements

6.1 Case 1 – Time data for every relevant failure for one or more copies from the same population

The normal evaluation methods assume the observed times to be exact times of failure of a single repaired item or a set of copies of the same repaired item. The figures below illustrate how the failure times are calculated for three general cases.

6.1.1 Case 1a) – One repaired item

For one repaired item observed from age 0 to age T , the relevant failure time, t_i , is the elapsed operating time (that is, excluding repair and other down times) until the occurrence of the i th failure as shown in Figure 1.

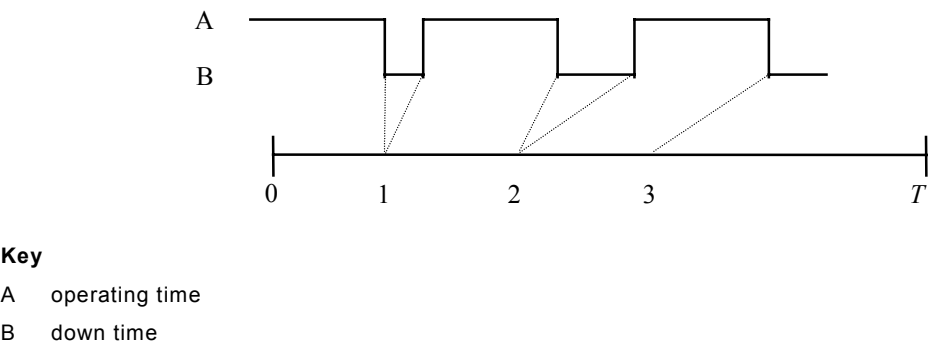


Figure 1 – One repaired item

Time terminated data are observed to T^* , which is not a failure time, and failure terminated data are observed to t_N , which is the time of the N th failure. Time terminated and failure terminated data use slightly different formulae.

6.1.2 Case 1b) – Multiple copies of repaired item observed for same length of time

It is assumed there are k items, which all represent the same population. That is, they are nominally identical copies operating under the same conditions (e.g. environment and load). When all copies are observed to time T^* , which is not a failure time (i.e. time terminated data), then the failure time data are combined by superimposing failure times ($t_i, i = 1, 2, \dots, N$) for all k systems on the same time line as shown in Figure 2.

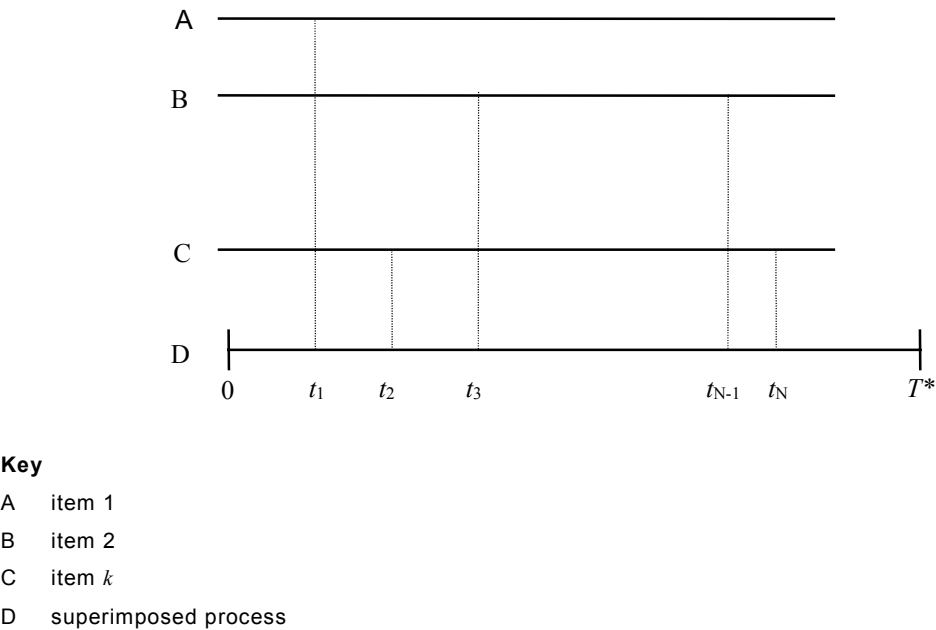
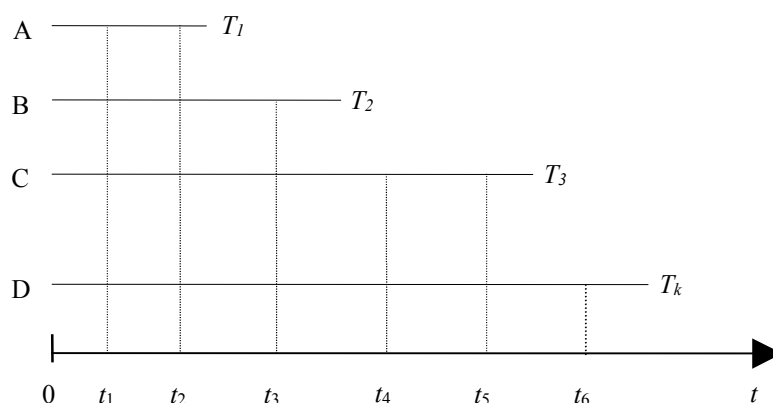


Figure 2 – Multiple copies of repaired item observed for same length of time

6.1.3 Case 1c) – Multiple copies of repaired item observed for different lengths of time

When all copies do not operate for the same period of time, then the time at which observation of the j th copy is terminated T_j ($j = 1, 2, \dots, k$), where $T_1 < T_2 < \dots < T_k$, is noted. The failure data are combined by superimposing all the failures times for all k copies on the same time line as shown in Figure 3. The operating times to failure are t_i , $i = 1, 2, \dots, N$, where N = the total number of failures observed accumulated over the k copies.



Key

- A item 1
- B item 2
- C item 3
- D item k
- t time

Figure 3 – Multiple copies of repaired item observed for different lengths of time

NOTE If each item is a software system then the repair action should be done to the other systems which did not fail at that time.

6.2 Case 2 – Time data for groups of relevant failures for one or more copies from the same population

This alternative method is used when there is at least one copy of an item and the data consists of known time intervals, each containing a known number of failures.

The observation period is over the interval $(0, T)$ and is partitioned into d intervals at times $0 < t(1) < t(2) < \dots < t(d)$. The i th interval is the time period between $t(i-1)$ and $t(i)$, $i = 1, 2, \dots, d$, $t(0) = 0$, $t(d) = T$. It is important to note that the interval lengths and the numbers of failures per interval need not be the same.

6.3 Case 3 – Time data for every relevant failure for more than one repaired item from different populations

It is assumed there are k items which do not represent the same population and are to be compared. It should be noted that if each item is to be considered individually then it is appropriate to use case 1a) in 6.1.1.

If direct comparisons of the systems are to be made then as an extension of 6.1 the following notation is used:

t_{ij} denotes the i th failure time for the process corresponding to the j th item;

N_j denotes the number of failures observed for the j th item;

t_{N_j} is the time of the N th failure for the j th item;

where $i = 0, 1, 2, \dots, N_j$ and $j = 1, 2, \dots, k$.

7 Statistical estimation and test procedures

7.1 Overview

In case 1 – time data for every relevant failure – the formulae given for failure terminated data assume one repaired item, that is $k=1$. All output results correspond to that item. The formulae given for time terminated data assume k copies of the item observed for the same length of time. If there is only one repaired item then $k=1$. The point estimation procedures for all the aforementioned cases are given in 7.2.1. The appropriate procedures for the case when all copies are observed for different lengths of time are given in 7.2.2. Procedures for the case of time data for groups of relevant failures are given in 7.2.3.

An appropriate goodness-of-fit test, as described in 7.3 shall be performed after the parameter estimation procedures of 7.2. Note that these tests, and the procedures given in 7.4 to 7.7 for constructing interval estimates and carrying out statistical tests, distinguish only between the cases of time data for every relevant failure (i.e. all instances of case 1 data – 1a), 1b) and 1c)) and time data for groups of relevant failures (i.e. case 2).

NOTE The inference procedures that follow provide approximate estimates in some circumstances and so caution is required if they are to be applied to very small samples of data.

7.2 Point estimation

7.2.1 Case 1a) and 1b) – Time data for every relevant failure

This method applies only when the time of failure has been logged for every failure as described in 6.1.1 and 6.1.2.

Step 1: Calculate the summation:

$$S_1 = \sum_{j=1}^N \ln \left(\frac{T^*}{t_j} \right) \quad (\text{time terminated})$$

$$S_2 = \sum_{j=1}^N \ln \left(\frac{t_N}{t_j} \right) \quad (\text{failure terminated})$$

Step 2: Calculate the (unbiased) estimate of the shape parameter β from the formula:

$$\hat{\beta} = \frac{N-1}{S_1} \quad (\text{time terminated})$$

$$\hat{\beta} = \frac{N-2}{S_2} \quad (\text{failure terminated})$$

Step 3: Calculate the estimate of the scale parameter λ from the formula:

$$\hat{\lambda} = \frac{N}{k(T^*)^{\hat{\beta}}} \quad (\text{time terminated})$$

$$\hat{\lambda} = \frac{N}{(t_N)^{\hat{\beta}}} \quad (\text{failure terminated})$$

Step 4: Calculate the estimate of the failure intensity $z(t)$, for any time $t > 0$, from the formula:

$$\hat{z}(t) = \hat{\lambda} \hat{\beta} t^{\hat{\beta}-1}$$

NOTE $\hat{z}(t)$ estimates the current failure intensity for t over the range represented by the data. "Extrapolated" estimates for a future t may be obtained similarly, but should be used with the usual caution associated with extrapolation.

Step 5: Given N observed failures the last of which occurred at t_N , the median time to the $(N+1)$ th failure can be estimated from the formula:

$$\hat{T}_{N+1} = t_N \exp \left[\frac{\frac{-1}{0,5 \frac{1}{N+1} - 1}}{\hat{N}\hat{\beta} / (N-1)} \right] \quad (\text{time termination})$$

$$\hat{T}_{N+1} = t_N \exp \left[\frac{\frac{-1}{0,5 \frac{1}{N+1} - 1}}{\hat{N}\hat{\beta} / (N-2)} \right] \quad (\text{failure termination})$$