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Compendium of curves and approximating algebraic functions

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COMPENDIUM OF CURVES AND APPROXIMATING ALGEBRAIC FUNCTIONS

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COMPENDIUM OF CURVES AND APPROXIMATING ALGEBRAIC FUNCTIONS

1. GLOSSARY

No "Notation and Units" Section is given in this Data Item – which makes use only of standard mathematical symbols for variables and constants. However, some of the mathematical terms used may need explanation, as follows.

Asymptote: A straight or curved line that is closely approached by a plane curve so that the perpendicular distance between them decreases steadily to zero as the independent variable governing the curve approaches a critical value or tends to infinity.

Dependent Variable: The variable in a mathematical function whose value is entirely *dependent* on the chosen values of the independent variable(s) and on the function itself. (It is customary, whenever practical, for the dependent variable to appear on the left-hand-side of an equation and as the vertical axis (ordinate) on a simple graph.)

Discontinuity: A point on a curve where the function is not continuous, *i.e.* its value takes a sudden jump.

Extrapolation: The process of estimating the value of a function *beyond* the range of values already known, by the extension of its curve.

Even Function: A function, $f(x)$, is *even* if $f(x) = f(-x)$.

Independent Variable: The variable(s) in a mathematical function whose values are considered to be selected *independently*, *i.e.* without reference to the function or any other variable(s). (It is customary, whenever practicable, for independent variables to appear on the right-hand-side of an equation and as the horizontal axis (abscissa) of a simple graph.)

Interpolation: The process of estimating the value of a function at a point *within* the range of values already known.

Odd function: A function, $f(x)$, is *odd* if, $f(x) = -f(-x)$.

Overfitting: This is the term used when a multiple-term mathematical function fitted to a set of data tends to follow the irregularities resulting from possibly inaccurate individual points rather than the overall trends of the data. (See also Reference 3.)

Singularity: A point on an algebraic function where it is not differentiable although it is differentiable in the immediate neighbourhood of that point, *e.g.* the function $f(x) = 1/x$ on the real axis is infinity at the singularity, $x = 0$.

Underfitting: This is the term used when a multiple-term mathematical function fitted to a set of data has too few terms to enable it to follow satisfactorily the overall trends of the data. (See also Reference 3.)

2. INTRODUCTION

This Data Item presents families of curve shapes and algebraic functions as an aid to the effective representation of experimental, numerical and graphical data by the selection of suitable such functions. Three categories of functions are presented:

Section 3 and Table 7.1: General multi-term functions,

Section 4 and Table 7.2: Logic functions,

Section 5 and Table 7.2: Single-term functions.

In many cases more than one type of function can represent a curve or data-set satisfactorily. In such cases it can be beneficial to compare the best-fitting curves produced by different functions before deciding which gives the best overall representation.

2.1 Applications

No distinctions are made here between the various applications of such algebraic functions, *i.e.*

- (i) representation of experimental results,
- (ii) representation of output from calculation procedures,
- (iii) representation of pre-existing curves from graphical representations of lines, “carpets” or “lattices”.

ESDU 06025 (Reference 3) gives a range of examples of Applications (i) and (ii) using the least-squares method of fitting and exploring the strengths and weaknesses of the functions presented here in Section 3 and Table 7.1.

ESDU 81009 (Reference 1) provides an example of Application (iii) to simple lines and carpets, see Appendix B of Reference 1.

2.2 Implementation

When fitting an algebraic function to a set of points plotted on a graph it is helpful to start by drawing a smooth “freehand” curve to follow the general trends of the points without being too influenced by any irregularities or scatter*. In essence this should be a smoothing process rather than one of interpolation, which produces curves constrained to pass through all the points individually (see Reference 3). Having drawn a representative curve through the set of data points, a suitable algebraic function is chosen that is capable of reproducing the curve’s salient features, see Sections 3, 4 and 5 and Tables 7.1, 7.2 and 7.3. The use of the “least-squares” procedure in fitting the chosen function to the data is described in ESDU 06025 (Reference 3).

The use of statistical techniques to represent the distribution of “scatter” about any function is introduced in ESDU 91017 (Reference 2).

* This discussion is presented in terms of Applications (i) and (ii), rather than Application (iii) where the pre-existing curve(s) could correspond to a “zero-scatter” case.

3. GENERAL MULTI-TERM FUNCTIONS SUITABLE FOR CURVE APPROXIMATION

This section presents a few types of multi-term function that can be applied to a very wide range of curves. These functions are general in nature and can be modified to suit specific characteristics of a curve, for example by the omission of one or more terms.

Although the functions listed can be applied to an extensive range of engineering applications, the list is not exhaustive and other functions might be contrived to suit a particular set of data. However, in general, the use of over-complicated functions should be avoided as they can introduce extraneous features of their own, such as waves or discontinuities, that are not inherent in the actual data.

3.1 Polynomials in Positive Integer Powers of the Independent Variable (Table 7.1, Sketches 7.1 to 7.4)

The **general form**, including odd and even powers of x , is

$$f(x) = C_1 + C_2x + C_3x^2 + \dots + C_px^{(p-1)} \quad (3.1)$$

where $C_1, C_2, \dots$ are coefficients of the independent variable x .

The **symmetrical form**, with only even powers of x , is

$$f(x) = C_1 + C_2x^2 + C_3x^4 + C_4x^6 + \dots + C_px^{(2p-2)}. \quad (3.2)$$

The **antisymmetrical form**, with only odd powers of x , is

$$f(x) = C_1x + C_2x^3 + C_3x^5 + C_4x^7 + \dots + C_px^{(2p-1)}. \quad (3.3)$$

(Other sequences, with larger differences in the exponents of x can be derived.)

If the curve to be fitted is known to be symmetrical, such that $f(-x) = f(x)$, then Equation (3.2) should be used. If the curve is antisymmetrical, such that $f(-x) = -f(x)$ then Equation (3.3) is appropriate. In all other cases the general form of Equation (3.1) should be used.

These polynomials are easy to evaluate and are widely used for relatively simple curves. However, they cannot adequately reproduce asymptotic behaviour, such as a levelling out to a constant value, or singularities such as a rapid rise to infinity at some critical point in the data range. They also have a tendency to diverge abruptly from the general trends of the data beyond the ends of the data range, and therefore **cannot safely be extrapolated**. Another disadvantage is that if too many terms are employed they can be prone to problems of overfitting and to numerical instabilities in the computation of the coefficients. These features are discussed more fully in Reference 3; that Data Item also contains examples of the use of polynomial functions.

3.2 Legendre Polynomials (Table 7.1, Sketches 7.1 to 7.4)

These take the form

$$f(x) = C_1P_1(x) + C_2P_2(x) + C_3P_3(x) + \dots + C_pP_p(x) \quad (3.4)$$

where $P_1 = 1; P_2 = x; P_3 = \frac{(3x^2-1)}{2}; P_4 = \frac{(5x^2-3)x}{2}$ etc.

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They may be employed in fitting to simple curves in place of polynomials in powers of x . Legendre polynomials are less likely than polynomials in powers of x to give rise to numerical instabilities such as overflows and loss of significant digits. However, they also are unable to reproduce asymptotic behaviour and cannot safely be extrapolated due to a tendency to diverge abruptly beyond the range of data. They are also prone to the phenomenon of overfitting if too many terms are used. The features of Legendre polynomials are discussed further in Appendix B of ESDU 06025 (Reference 3).

3.3 Rational Function (Table 7.1, Sketches 7.1 to 7.12)

This is formed as the ratio of two polynomial functions. In its simplest form both polynomials could be in powers of x , the numerator having, say, p terms and the denominator having q terms, thus

$$f(x) = \frac{C_1 + C_2x + C_3x^2 + \dots + C_px^{(p-1)}}{1 + C_{(p+1)}x + C_{(p+2)}x^2 + \dots + C_{(p+q-1)}x^{(q-1)}}. \quad (3.5)$$

The rational function is extremely versatile and can produce a wide range of curves including those obtained from power series or Legendre polynomials. Unlike the polynomial functions, the rational function generates curves that show no tendency to diverge abruptly beyond the ends of the data range and which therefore can be extrapolated to a limited extent. It is also less prone to overfitting than polynomials having an equal number of terms.

The most useful feature of the rational function is its ability to reproduce asymptotic behaviour in a curve, which the polynomial functions are unable to do. It may also be used in cases where the data suggest singularities, such as when the curve tends to infinity at some point within the data range. This behaviour can be reproduced by zero values in the denominator of a rational function but cannot be obtained from a polynomial in powers of x . However, it is possible for a rational function to produce unwanted singularities within the data range, in which case the function should be rejected. This emphasises the need to check the plotted curve before a function is accepted as a fit to the data. The characteristics of the rational function are discussed further in Reference 3; that Data Item also contains examples of the use of rational functions.

3.4 Peak Function (Table 7.1, Sketches 7.13 to 7.15)

The peak function can take the form

$$F_p(x) = \frac{C}{1 + B\left(\frac{x-K}{\Delta x}\right)^2}. \quad (3.6)$$

This function produces a peak of height C (if the coefficient C is positive) or a trough of depth C (if the coefficient is negative), at a specified point along the x -axis. The parameter K defines the value of x at which the peak occurs. The parameter B controls the sharpness of the peak in terms of the interval Δx . The parameters K , B and Δx must be prescribed so as to suit the position and shape of the peak.

Peak functions may be used in isolation but are best employed in conjunction with other functions which can follow the background trends of the data, for example

$$f(x) = C_1 + C_2x + C_3x^2 + \dots + C_px^{(p-1)} + \frac{C_{(p+1)}}{1 + B_1\left(\frac{x-K_1}{\Delta x}\right)^2} + \frac{C_{(p+2)}}{1 + B_2\left(\frac{x-K_2}{\Delta x}\right)^2} + \dots \quad (3.7)$$

Properties of the peak function are discussed further in Reference 3; that Data Item also contains examples of the application of the peak function.

3.5 Trigonometric Functions (Table 7.1, Sketches 7.16 to 7.23)

Trigonometric functions are useful in representing curves which are periodic in nature such that $f(x) = f(x + 2L)$, where $2L$ is the period, or wavelength, over which the curve repeats itself. Such a curve can usually be approximated by a series of cosine and sine functions, several types of which are introduced in Sections 3.5.1 to 3.5.5 and illustrated in Sketches 7.16 to 7.24. For convenience they may be expressed in terms of an angle $\theta = \pi x/L$, so that θ varies between 0 and 2π as x varies from 0 to $2L$.

3.5.1 General form, including odd and even harmonics in both cosines and sines (Table 7.1, Sketch 7.16)

$$f(x) = C_1 + C_2 \cos \theta + C_3 \cos 2\theta + C_4 \cos 3\theta + \dots + C_p \cos(p-1)\theta \\ + C_{(p+1)} \sin \theta + C_{(p+2)} \sin 2\theta + \dots + C_{(p+q)} \sin q\theta. \quad (3.8)$$

3.5.2 Cosine terms only (Table 7.1, Sketch 7.17)

If a curve is seen to possess certain degrees of symmetry, some of the terms in the general series (*i.e.* Equation (3.8)) may be omitted. For example, if the curve is seen to be even, *i.e.* symmetrical about $x = 0$ such that $f(-x) = f(x)$, then only the cosine terms should be included, plus the initial constant term C_1 , to give

$$f(x) = C_1 + C_2 \cos \theta + C_3 \cos 2\theta + C_4 \cos 3\theta + \dots + C_p \cos(p-1)\theta. \quad (3.9)$$

3.5.3 Sine terms only (Table 7.1, Sketches 7.18 and 7.23)

If a curve is seen to be odd, *i.e.* is antisymmetrical about $x = 0$, such that $f(-x) = -f(x)$, then only the sine terms should be included, as follows,

$$f(x) = C_1 \sin \theta + C_2 \sin 2\theta + C_3 \sin 3\theta + \dots + C_q \sin q\theta. \quad (3.10)$$

3.5.4 Odd cosine terms only (Table 7.1, Sketch 7.19)

Sometimes further degrees of symmetry may be observed in a curve, in which case certain other terms may be omitted. For example, a curve which is even and also antisymmetrical about $x = L/2$ may be approximated by a cosine series containing only the odd harmonics, as follows,

$$f(x) = C_1 \cos \theta + C_2 \cos 3\theta + C_3 \cos 5\theta + \dots + C_p \cos(2p-1)\theta. \quad (3.11)$$

3.5.5 Odd sine terms only (Table 7.1, Sketches 7.20 to 7.22)

In a similar manner, if a curve is odd and also symmetrical about $x = L/2$ then only the odd sine terms are necessary, as follows,

$$f(x) = C_1 \sin \theta + C_2 \sin 3\theta + C_3 \sin 5\theta + \dots + C_q \sin (2q - 1)\theta. \quad (3.12)$$

4. LOGIC FUNCTIONS (TABLE 7.2)

Instead of a mathematical variable, the argument of a logic function consists of a conditional expression, the outcome of which determines the value taken by the function. A logic function takes the value unity if the condition is true, and zero if it is false.

Four general types of logic function are represented in Equations (4.1) to (4.4) (by convention enclosed in square brackets). These are illustrated in Table 7.2, Sketches 7.24 to 7.27 respectively to show how their values change according to the truth of the conditional expression.

$$\begin{aligned} y &= [x > x_1] \\ &= 0 \text{ for } x \leq x_1 \\ &= 1 \text{ for } x > x_1. \end{aligned} \quad (4.1)$$

$$\begin{aligned} y &= [x < x_1] \\ &= 1 \text{ for } x < x_1 \\ &= 0 \text{ for } x \geq x_1. \end{aligned} \quad (4.2)$$

$$\begin{aligned} y &= [x > x_1 \text{ AND } x < x_2] \\ &= 1 \text{ for } x_1 < x < x_2 \\ &= 0 \text{ for all other } x. \end{aligned} \quad (4.3)$$

$$\begin{aligned} y &= [x < x_1 \text{ OR } x > x_2] \\ &= 0 \text{ for } x_1 < x < x_2 \\ &= 1 \text{ for all other } x. \end{aligned} \quad (4.4)$$

Any logic function can be employed as a multiplier of other mathematical functions, when it serves the purpose of a switch to turn them on or off at specific values of the independent variable x . Examples of this are given in Sections 4.1 and 4.2.