

Mobilities and impedances of structures

Part II: Compendium of point mobilities of infinite structures

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MOBILITIES AND IMPEDANCES OF STRUCTURES PART II: COMPENDIUM OF POINT MOBILITIES OF INFINITE STRUCTURES

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MOBILITIES AND IMPEDANCES OF STRUCTURES PART II: COMPENDIUM OF POINT MOBILITIES OF INFINITE STRUCTURES

1. NOTATION

		<i>Units</i>	
		<i>SI</i>	<i>British</i>
A	ratio of plate bending stiffness to beam bending stiffness, Equation (5.24)		
a	length of plate in x -direction,	m	in
B	bending stiffness of beam	N m ²	lbf in ²
b	length of plate in y -direction, width of beam	m	in
c	damping constant (coefficient) of isolator	N s/m	lbf s/in
c_b	bending wave velocity	m/s	in/s
c_l	longitudinal wave velocity in bar ($\sqrt{E/\rho}$)	m/s	in/s
c_l^*	complex longitudinal wave velocity in bar ($\sqrt{E^*/\rho}$)	m/s	in/s
c_l'	longitudinal wave velocity in plate ($\sqrt{E/\{\rho(1-\nu^2)\}}$)	m/s	in/s
c_s	shear wave or torsional wave velocity in bar ($\sqrt{G/\rho}$)	m/s	in/s
D	bending stiffness of isotropic plate ($Eh^3/\{12(1-\nu^2)\}$)	N m	lbf in
D_x	bending stiffness in x -direction	N m	lbf in
D_y	bending stiffness in y -direction	N m	lbf in
D_{xy}	torsional stiffness of plate about x - and y -axes	N m	lbf in
d	diameter of circular bar	m	in
E	Young's modulus	Pa	lbf/in ²
E^*	complex Young's modulus ($E(1+i\eta)$)	Pa	lbf/in ²
E_x	Young's modulus in x -direction	Pa	lbf/in ²
E_y	Young's modulus in y -direction	Pa	lbf/in ²
E_1	Young's modulus of upper face plate of honeycomb	Pa	lbf/in ²
E_2	Young's modulus of lower face plate of honeycomb	Pa	lbf/in ²
F	force ($= F e^{i\omega t}$)	N	lbf

f	frequency	Hz	Hz
f_r	ring frequency, Equation (4.27)	Hz	Hz
G	shear modulus	Pa	lbf/in ²
G^*	complex shear modulus	Pa	lbf/in ²
G_s	shear modulus for thick plates	Pa	lbf/in ²
G_{xy}	shear modulus in xy plane	Pa	lbf/in ²
GQ	torsional stiffness (Section 5.4.2)	N m ²	lbf in ²
H	height of stiffeners	m	in
h	thickness of beam, plate or cylinder	m	in
h_b	height of stiffening beam	m	in
h_c	core height of honeycomb	m	in
h_{em}	equivalent thickness based on equal mass	m	in
h_{eb}	equivalent thickness based on equal bending stiffness	m	in
h_1	upper face plate thickness of honeycomb	m	in
h_2	lower face plate thickness of honeycomb	m	in
I	second moment of area	m ⁴	in ⁴
$\mathbf{j}$	unit vector in y -direction		
J	polar moment of area of cross-section	m ⁴	in ⁴
K	elliptical integral of first kind, Equation (4.23)		
K_1	numerical constant in torsion of non-circular components		
k	stiffness of isolator	N/m	lbf/in
k_b	wavenumber for bar or beam	m ⁻¹	in ⁻¹
k_{bc}	coupled wavenumber in the beam	m ⁻¹	in ⁻¹
k_d	dilatational wavenumber	m ⁻¹	in ⁻¹
k_l	wavenumber for longitudinal motion	m ⁻¹	in ⁻¹
k_p	bending wavenumber for plate	m ⁻¹	in ⁻¹
k_{px}	bending wavenumber component for plate in x -direction	m ⁻¹	in ⁻¹
k_{py}	bending wavenumber component for plate in y -direction	m ⁻¹	in ⁻¹

k_{rc}	coupled wavenumber in ring	m^{-1}	in^{-1}
k_{rc}^*	complex coupled wavenumber in ring	m^{-1}	in^{-1}
k_s	shear wavenumber	m^{-1}	in^{-1}
k_T	dimensionless wavenumber ($\delta\omega/c_s$)	m^{-1}	in^{-1}
l	length of bar, honeycomb width, half circumference of ring	m	in
m	mass of beam, plate or cylinder	kg	*
$n_d(f)$	modal density	modes/Hz	modes/Hz
$n_d(\omega)$	modal density	modes/(rad/s)	modes/(rad/s)
n_{db}	modal density for bending waves	modes/Hz	modes/Hz
n_{dd}	modal density for dilatational waves	modes/Hz	modes/Hz
n_{ds}	modal density for shear waves	modes/Hz	modes/Hz
P	power	W	lbf in/s
Q	torsional constant	m^4	in^4
R	mean radius of ring or cylinder	m	in
r	radius of gyration of beam cross-section about neutral axis	m	in
S	surface area of plate	m^2	m^{-1}
s	spacing between ribs or corrugations	m	in
s_1	k_b/k_p		
T	moment applied to rod, beam or plate	N m	lbf in
t	thickness of stiffeners	m	in
t_1	web thickness of honeycomb	m	in
v	velocity	m/s	in/s
x, y, z	coordinates		
Y	mobility	s/kg	*
Y_c	mobility of cylinder	s/kg	*
Y_e	mobility of engine or machine	s/kg	*
Y_f	mobility of foundations	s/kg	*
Y_I	mobility of isolator	s/kg	*

Y_r	mobility of ring	s/kg	*
Y_s	mobility in shear	s/kg	*
Y_∞	mobility of infinite structure, force excitation	s/kg	*
$Y_{m\infty}$	mobility of infinite structure, moment excitation	s kg ⁻¹ m ⁻²	*
$Y_{\Delta\omega}$	point mobility, averaged over space and frequency band $\Delta\omega$	s/kg	*
Z_∞	impedance of infinite structure, force excitation	kg/s	*
α	reflection factor ($\alpha = e^{-2\beta l}$)		
α_1	variable in elliptical integral, Equation (4.24)		
β	attenuation coefficient		
γ	shear stiffness ($G h$)	kg s ⁻²	*
δ	radius of disc (connector)	m	in
η	dimensionless loss factor		
θ	angle of twist of bar in torsion	rad	rad
λ_p	bending wavelength in plate	m	in
λ_s	shear wavelength	m	in
μ	effective mass per unit length	kg/m	*
μ_b	mass per unit length of infinite plate or cylinder	kg/m	*
ν	Poisson's ratio		
ν_{xy}	Poisson's ratio for load in x-direction, contraction in y-direction		
ν_{yx}	Poisson's ratio for load in y-direction, contraction in x-direction		
ρ	density	kg/m ³	*
ϕ	circumferential angle in moment excitation of plate	rad	rad
ω	angular frequency ($= 2\pi f$)	rad/s	rad/s
ω_2	frequency when $k_s = \pi/(2\delta)$	rad/s	rad/s

* In British units, for the system to be coherent the mass in lb must be divided by the constant 386.4.

2. INTRODUCTION

Formulae for frequency-dependent mobilities can become long and complicated for many structures, particularly if they are built up from different components, such as plates and beams. Simplified expressions may be obtained by considering infinite structures. In Part II of this Item, point mobilities of infinite structures are discussed.

The concept of the mobility of an infinite structure is particularly useful in Statistical Energy Analysis (SEA). Detailed descriptions of the purpose, main assumptions and applications of SEA may be found (References 11, 24-26). Item 99009 summarizes the most important aspects of the SEA approach (Reference 27). SEA deals with averaged quantities and is concerned with structural systems (subsystems) in terms of energy and power. Averaging can be spatial, over a frequency band, or over an ensemble. SEA is applied to systems excited by either random or deterministic forces or moments in the high frequency range, *i.e.*, to systems with structural wavelengths much smaller than the characteristic dimensions of the system. In this range of frequencies, the averaged structural response is neither sensitive to boundary conditions nor to small changes in structural properties. The power input to a system analyzed by statistical energy is an essential quantity in the power balance equations and in general depends on the excitation and the structural properties of the system. The input power supplied by a broad-band point force to a finite structure is identical to the power supplied by the force to a corresponding infinite structure, a point which underlines the importance of the mobilities of infinite structures.

3. FREQUENCY-AVERAGED MOBILITY

The power supplied to a system by a point force can be defined in terms of the point mobility of the system. The time-averaged power supplied to a system by a harmonic point force is given by the expressions (Reference 22):

$$P = \langle F^2 \rangle \operatorname{Re}(Y) = \frac{1}{2} |F|^2 \operatorname{Re}(Y) \quad (3.1)$$

or

$$P = \langle v^2 \rangle \operatorname{Re}(Z) = \frac{1}{2} |v|^2 \operatorname{Re}(Z), \quad (3.2)$$

where $\langle F^2 \rangle$ and $\langle v^2 \rangle$ are the mean-squared force and mean-squared velocity of the structure and $|F|^2$ and $|v|^2$ denote the squared modulus (squared magnitude) of the applied point force and the squared modulus of the structural velocity, respectively.

When the force is a random function of time, and if the power spectral density of the force is approximately constant within the frequency band $\Delta\omega$, then the input power to the system within the frequency band may be formulated in terms of the mobility averaged over the frequency band $\Delta\omega$ and the mean-squared force within the band (Reference 26).

$$P = \langle F^2 \rangle \langle \operatorname{Re}(Y) \rangle_{\Delta\omega}. \quad (3.3)$$

It was shown (Reference 22) that for a homogeneous system, *e.g.*, a plate, excited by a point force with frequency band $\Delta\omega$, the time-averaged input power, within the frequency band and averaged over space, is proportional to the modal density $n_d(\omega)$ and the mean-squared force $\langle F^2 \rangle$ in the frequency band $\Delta\omega$, *i.e.*,

$$P = \langle F^2 \rangle (\pi/2m) n_d(\omega), \quad (3.4)$$

where m denotes the mass of the system. The frequency band should incorporate at least five resonant

frequencies (Reference 15). The real part of the mobility $Y_{\Delta\omega}$, appearing in Equation (3.3),

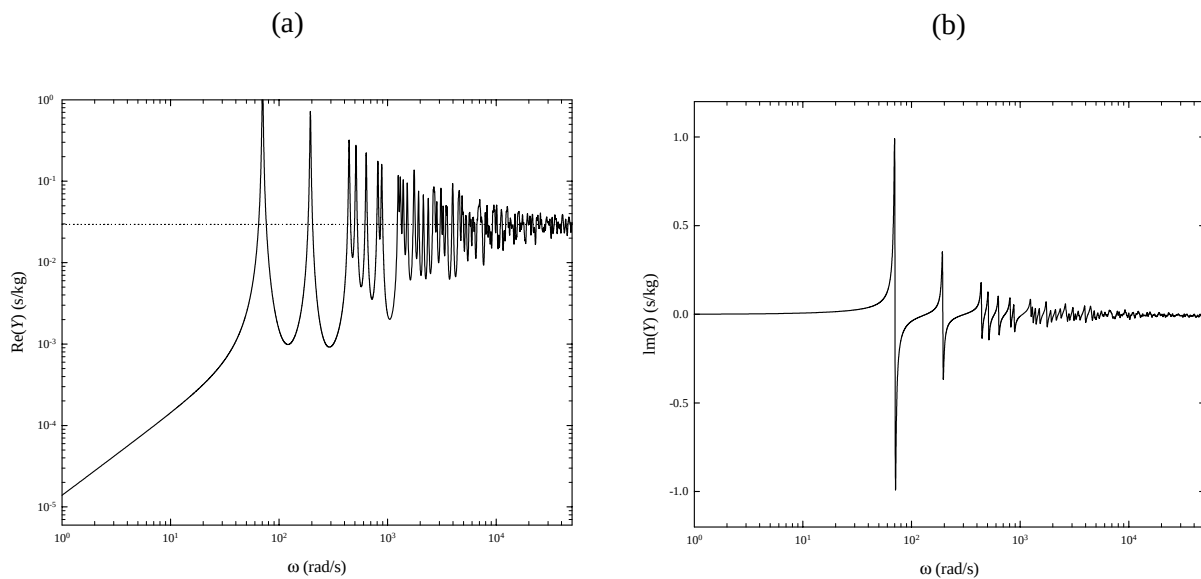
$$\text{Re}(Y_{\Delta\omega}) = (\pi/2m)n_d(\omega) = \text{Re}(Y_\infty), \quad (3.5)$$

does not depend on boundary conditions and is equal to the real part of the point mobility of the corresponding infinite system. The mobility $Y_{\Delta\omega}$ can be interpreted as a point mobility of the system, averaged over space and the frequency band. It should also be noted that in the case of a homogeneous structure, *e.g.*, a finite isotropic plate, the frequency average, as well as the spatial average of the point mobility of the structure, are approximated well by the corresponding infinite structure.

Finally, the power input supplied by a point force with frequency band $\Delta\omega$ to a finite structure can be presented as

$$P = \langle F^2 \rangle \text{Re}(Y_\infty). \quad (3.6)$$

As an example of the relationship between the point mobility of a finite structure and the point mobility of an infinite structure, the real and imaginary parts of the mobility of a simply-supported, isotropic rectangular plate are shown in Sketch 3.1 for a structural loss factor, η , of 0.02. As the frequency becomes large, the real part converges to a constant value and the imaginary part tends to zero.



Sketch 3.1 Point mobility at the midpoint of an aluminium alloy simply-supported plate, (a) real part and (b) imaginary part ($E = 72.4$ GPa, $\nu = 0.3$, $\eta = 0.02$, $a = 1.0$ m, $b = 0.5$ m and $h = 1$ mm.)

3.1 Relationship between Modal Density and Mobility of an Infinite Structure (Frequency-averaged Mobility).

The modal density is the number of resonant modes that occur on average within a unit frequency band. The modal density may be expressed as $n_d(f)$, in terms of the number of modes per Hz, or as $n_d(\omega)$ in modes per (rad/s). The relationship between $n_d(f)$ and the real part of the infinite mobility, see

Equation (3.5), is:

$$\operatorname{Re}(Y_{\infty}) = \frac{n_d(f)}{4m}. \quad (3.7)$$

For an isotropic plate in flexural vibration, the modal density is given by (Reference 32):

$$n_d(f) = \frac{S(\rho h)}{2(D)}^{1/2} = \frac{\sqrt{3}S}{hc'_l}, \quad (3.8)$$

where S is the surface area of the plate, h is the thickness and c'_l is the longitudinal wave velocity ($\sqrt{E/\rho(1-\nu^2)}$); E is Young's modulus, ν is Poisson's ratio, D is the bending stiffness of the plate and ρ is the density of the material of the plate. Thus the real part of the point mobility of the infinite plate from Equation (3.7) is:

$$\operatorname{Re}(Y_{\infty}) = \frac{\sqrt{3}S}{4hmc'_l}, \quad (3.9)$$

or,

$$\operatorname{Re}(Y_{\infty}) = \frac{1}{8\sqrt{D\rho h}} \quad (3.10)$$

in terms of D , where

$$D = \frac{Eh^3}{12(1-\nu^2)}. \quad (3.11)$$

Flexural vibrations of a plate may also be excited by a moment acting on the plate. The moment point mobility is the ratio of angular velocity to bending moment and has units of $\text{s kg}^{-1} \text{m}^{-2}$ or $(\text{rad/s})/\text{N m}$. For a thin plate excited by a moment represented by a vector in the plane of the plate (forces making up the moment, equal and opposite, perpendicular to the plate), the frequency-averaged point moment mobility, $Y_{m\infty}$, has a real part given by (Reference 23):

$$\operatorname{Re}(Y_{m\infty}) = \frac{k_p^2 n_d(f)}{8m}, \quad (3.12)$$

where k_p is the flexural wavenumber of the plate and is given by:

$$k_p = \omega^{1/2} \left(\frac{\rho h}{D} \right)^{1/4}. \quad (3.13)$$

By substitution from Equations (3.8), (3.11) and (3.13), Equation (3.12) may be rewritten as:

$$\operatorname{Re}(Y_{m\infty}) = \frac{\omega}{16D}. \quad (3.14)$$

4. POINT MOBILITIES OF SIMPLE STRUCTURES

4.1 Point Mobilities and Properties of Bars with Longitudinal and Torsional Wave Motion

The point mobilities for infinite structures, such as bars with one-dimensional wave motion, are shown in Tables 8.1a and 8.1b. The bars are infinite or semi-infinite and the wave motion is either longitudinal or torsional. Other properties of the wave motion are included in Table 8.1a and 8.1b, such as the velocity field, wavenumbers k_l and k_s , wave velocity c_l or c_s and the modal density for corresponding bars of finite length.

The parameters in the tables are related. The modal density and frequency-averaged point mobility are related through Equation (3.7). The wavenumber is either ω/c_l for longitudinal waves (Table 8.1a) or ω/c_s for torsional waves (Table 8.1b). Damping may be introduced through the loss factor, η , for the material of the bar, thus the complex Young's modulus E^* , is given by:

$$E^* = E(1 + i\eta) \quad (4.1)$$

and, consequently, the complex wave velocity, c_l^* , and complex wavenumber, k_l^* , are:

$$c_l^* = \sqrt{E^*/\rho} \quad (4.2)$$

and

$$k_l^* = \omega\sqrt{\rho/E^*}. \quad (4.3)$$

Similar expressions apply for the torsional wave motion where the complex shear modulus, G^* , is

$$G^* = G(1 + i\eta) \quad (4.4)$$

and for an isotropic material

$$G = \frac{E}{2(1 + \nu)}. \quad (4.5)$$

4.2 Point Mobilities and Properties of Beams with Flexural Wave Motion

The frequency-averaged point mobilities and related properties for beams with force or moment excitation are shown in Table 8.2 for the case of no damping. If structural damping is introduced, as in Equation (4.1), the complex wavenumber, k_b^* , is given by:

$$k_b^* = \omega^{1/2}(\rho S/E^*I)^{1/4}, \quad (4.6)$$

where S is the cross-sectional area of the beam and I is the second moment of area about the neutral axis. For small values of the loss factor, η , an approximate expression for k_b^* is:

$$k_b^* = k_b(1 - i\eta/4), \quad (4.7)$$

where

$$k_b = \omega^{1/2}(\rho S/EI)^{1/4}. \quad (4.8)$$

The point mobility of an infinite beam with force excitation is given by:

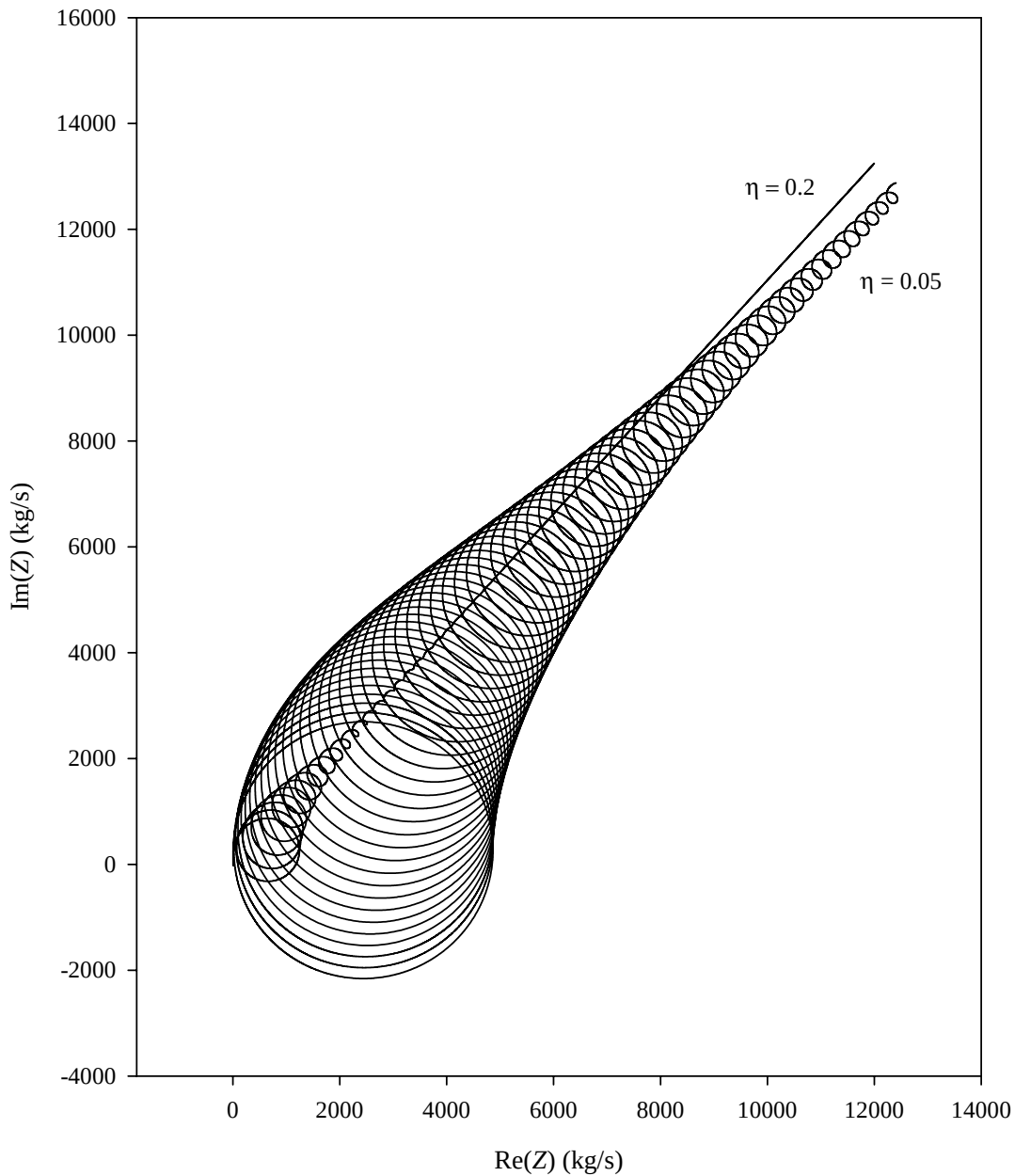
$$Y_{\infty} = \frac{(1-i)k_b^*}{4\rho S\omega}. \quad (4.9)$$

From Equation (4.6) and with $I = Sr^2$, where r is the radius of gyration of the beam cross-section about the neutral axis, the expression for Y_{∞} becomes:

$$Y_{\infty} = \frac{(1-i)\sqrt{c_l^*}}{4S\sqrt{\rho E^*}r\omega}. \quad (4.10)$$

It has been shown in Part I, Section 7, that at high frequencies the modulus of the point mobility of the infinite beam becomes independent of the length of the beam as well as the damping. Thus, the dependence of the modulus of the point mobility on frequency is the same for finite and infinite beams at high frequencies. The other high frequency feature of the beam may be seen in terms of the point impedance of a finite beam. Sketch 4.1 shows Argand diagrams for two different damping ratios. At high frequencies the point impedance of the beam becomes independent of its length. Note that in the complex plane the values of the damping loss factor has a small effect on the point impedance. With high damping the imaginary part of the impedance is greater than the real part. With low damping the impedance converges to a line with a slope of 1, which is to be expected, as the impedance of an infinite beam, Z_{∞} , is given by:

$$Z_{\infty} = \frac{1}{Y_{\infty}} = \frac{2(1+i)S\sqrt{\rho E^*}r\omega}{\sqrt{c_l^*}}. \quad (4.11)$$

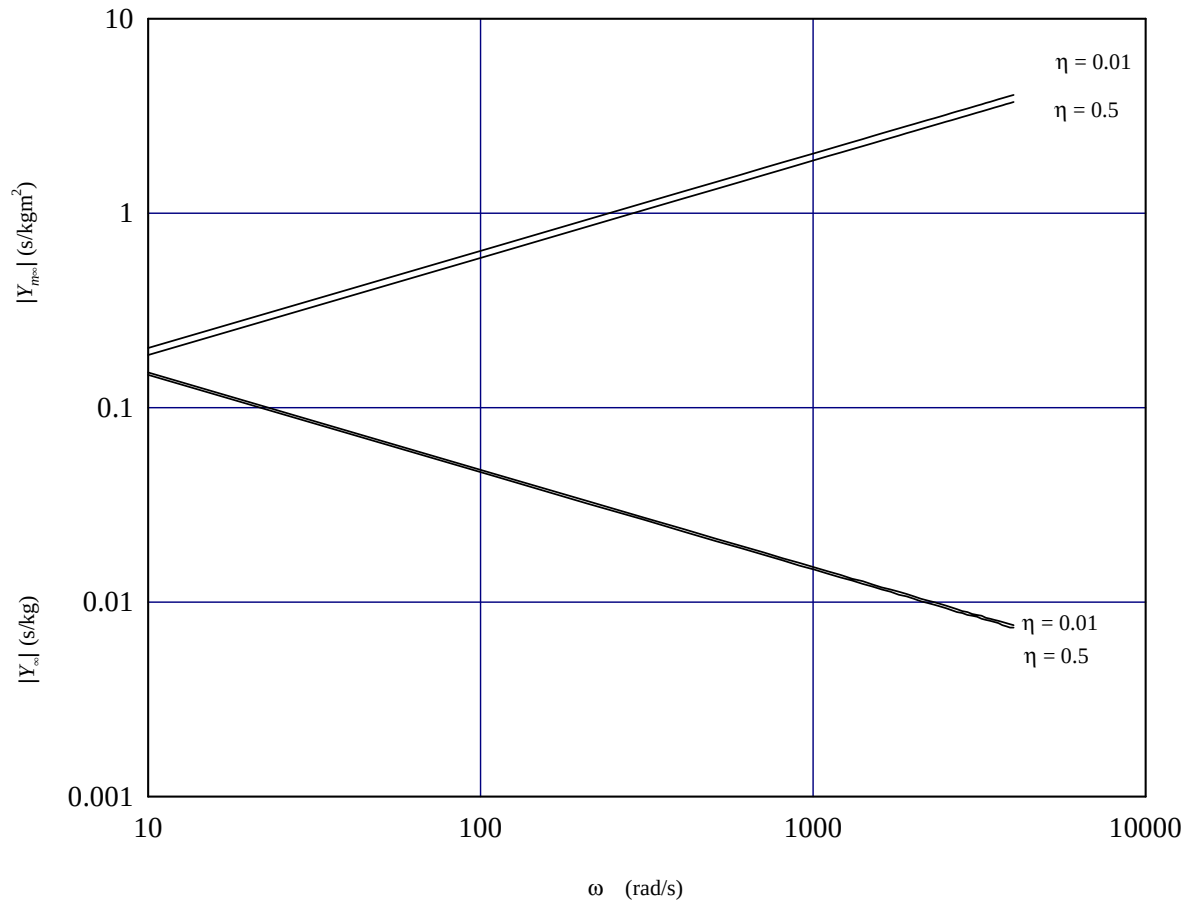


Sketch 4.1 Point impedance in Argand diagram of a thin beam with force excitation for damping loss factor, η , of 0.05 and 0.2 ($E = 191$ GPa, $\rho = 7.8$ kg/m³, thickness is 5 mm, width is 20 mm, length is 1.4 m).

The expression for the point mobility of an infinite beam with moment excitation is given by:

$$Y_{m\infty} = \frac{(1+i)\omega}{4E^*Ik_b^*}. \quad (4.12)$$

Both the force-excited and the moment-excited frequency-averaged point mobilities are frequency dependent. For force excitation $Y_{\infty} \propto \omega^{-1/2}$ and for moment excitation $Y_{m\infty} \propto \omega^{1/2}$. Both Y_{∞} and $Y_{m\infty}$ decrease as the damping increases, although the effect of damping is very small, particularly in the case of force excitation, see Sketch 4.2.



Sketch 4.2 Point mobility of infinite, thin beam for force and moment excitation (Aluminium alloy, thickness is 5 mm and width is 20 mm)

4.3 Point Mobilities and Properties of Isotropic Plates with Flexural Wave Motion

The point mobilities of infinite thin plates are listed in Table 8.3 for normal force excitation and for moment excitation, where the moment vector is in the plane of the plate (References 8, 14, 18, 22, 29). The force or moment has to be applied through a disk or circular connector of radius δ , otherwise the shear stress at the point of application tends to infinity. The disk is assumed to be rigid, massless and small compared with the structural wavelength.

An expression for the point mobility of an infinite plate has been established by Cremer and Heckl (Reference 22) in terms of the bending wavenumber, k_p , (Equation (3.13)), the bending stiffness, D , (Equation (3.11)) and the radius of the connector, δ . Their expression is:

$$Y_{\infty} = \frac{\omega}{8Dk_p^2} \left\{ H_0^{(2)}(k_p\delta) - i\frac{2}{\pi}K_0(k_p\delta) \right\}, \quad (4.13)$$

where $H_0^{(2)}$ is a Hankel function of the second kind and order zero and K_0 is the modified Bessel function. Equation (4.13) has been confirmed by Ljunggren (Reference 18) who has considered the infinite plate as a three-dimensional continuum. His approach has shown that the normal force on the connector or indenter in his terminology produces longitudinal waves as well as the expected bending waves. According to Ljunggren, Equation (4.13) is restricted to low frequencies when the bending wavelength, λ_p , is much

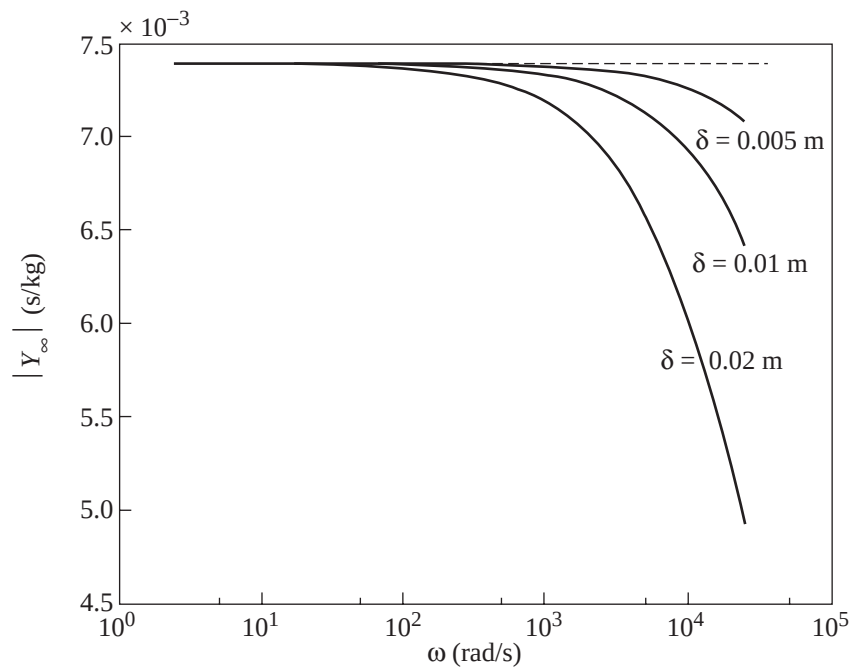
greater than the thickness of the plate, *i.e.*, $h \ll \lambda_p$ or $k_p h \ll 2\pi$. Additionally, it is required that $k_p \delta \ll 1$. If the small term approximations of the Hankel and modified Bessel functions are applied,

$$Y_\infty = \frac{\omega}{8Dk_p^2} \quad (4.14)$$

or

$$Y_\infty = \frac{1}{8\sqrt{D\rho h}}. \quad (4.15)$$

The simplified expression of Equation (4.15) gives a point mobility independent of frequency, whereas the point mobility from Equation (4.13) depends on frequency for large connectors, as illustrated in Sketch 4.3. For most purposes Equation (4.15) provides a good approximation for the point mobility of thin isotropic plates.



Sketch 4.3 Point mobility of an infinite, isotropic plate;-----from Equation (4.15); — from Equation (4.13) for different connector radii δ ($h = 1$ mm, $E = 72.4$ GPa, $\nu = 0.3$)

For excitation of an infinite plate by a moment, where the moment vector is in the plane of the plate (forces making up the couple are perpendicular to the plate), the point mobility has been obtained either by the continuum approach of Ljunggren (Reference 18) or by using the Mindlin theory (Reference 2) for the bending of thick plates (Dyer, Reference 5). Ljunggren's expression for the point moment mobility is:

$$Y_{m\infty} = \frac{\omega}{8Dk_p\delta} \left\{ H_1^{(2)}(k_p\delta) - i\frac{2}{\pi}K_1(k_p\delta) \right\} \quad (4.16)$$

which applies for $k_ph \ll 2\pi$.

Dyer has established a method for estimating the point moment mobility of thick plates. A simplified expression is available for thin plates and small radii of the discs, *i.e.*, $k_ph \ll 1$ and $(k_p\delta)^2 \ll 1$, namely,

$$Y_{m\infty} = \frac{\omega}{8D(1+L)} \left\{ 1 - i\frac{4}{\pi}\ln(k_p\delta) - \frac{8L}{\pi(1-\nu)}\left(\frac{h}{\pi\delta}\right)^2 \right\}, \quad (4.17)$$

where

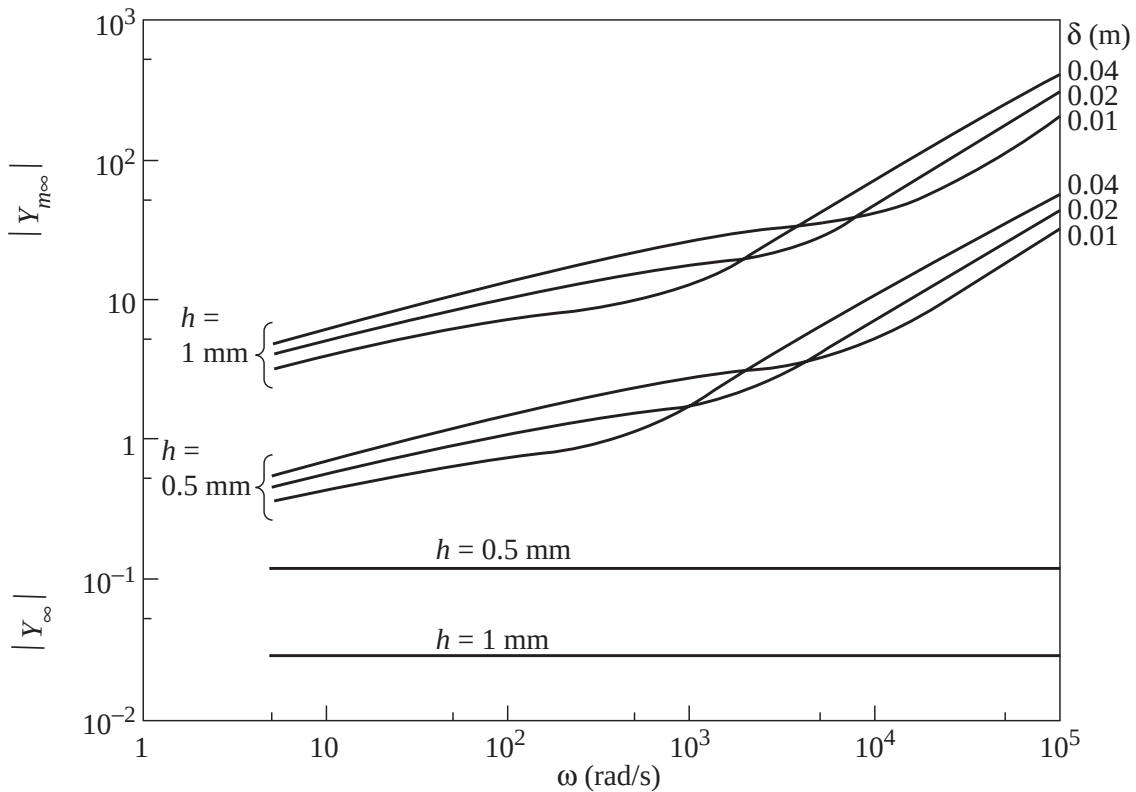
$$L = \frac{1 + K_1\left(\frac{\pi\delta}{h}\right)}{\frac{\pi\delta}{h}K_0\left(\frac{\pi\delta}{h}\right)} \quad (4.18)$$

and where K_n is the modified Bessel function of the second kind and of order n . If a further limitation is imposed so that $\pi\delta \gg h$, then $L = 1$ and

$$Y_{m\infty} = \frac{\omega}{16D} \left\{ 1 - i\frac{4}{\pi}\ln(k_p\delta) \right\}. \quad (4.19)$$

In Equation (4.19) the effect of transverse shear is negligible. In Sketch 4.4 it is shown how the force and moment excited point mobilities vary with frequency ω , based on Equation (4.15) for the force excitation and Equation (4.19) for the moment excitation. The moment mobility depends on the connector radius δ . The modulus of the moment mobility is approximately proportional to ω at high frequencies.

For the point moment mobility the velocity field is directional. The velocity is zero in the direction of the moment vector (y -axis) and maximum in the direction of the x -axis.



Sketch 4.4 Point mobilities of infinite, isotropic plate in bending for force and moment excitation (Aluminium alloy)

4.4 Point Mobilities and Properties of Orthotropic Plates with Flexural Wave Motion

In general, orthotropic materials are characterised by three mutually perpendicular axes, called the principal material axes. For a thin, rectangular, orthotropic plate with edges aligned with the principal material axes, *i.e.*, specially orthotropic plates, the propagation of the bending waves can be described in terms of the stiffnesses, D_x and D_y , in the directions of the two principal material axes, lying in the plane of the plate, and the bending - torsional stiffness, D_{xy} , (References 28 and 32). In the case of a single layer orthotropic plate stiffnesses, D_x and D_y , may be expressed in terms of E_x and E_y , the Young's moduli in the directions of the principal material axes (x, y axes), respectively, and ν_{xy} , and ν_{yx} , the Poisson's ratios for loading in the x - and y -directions. Hence,

$$D_x = \frac{E_x h^3}{12(1 - \nu_{xy}\nu_{yx})}, \quad (4.20)$$

$$D_y = \frac{E_y h^3}{12(1 - \nu_{xy}\nu_{yx})}. \quad (4.21)$$

and

$$D_{xy} = \frac{\nu_{xy}E_y h^3}{12(1 - \nu_{xy}\nu_{yx})} + 2G_{xy}\frac{h^3}{12}, \quad (4.22)$$

where G_{xy} is the shear modulus in the xy plane. For $0 < D_{xy} \leq (D_x D_y)^{1/2}$ the point mobility for a plate of infinite extent, Y_∞ , may be expressed as:

$$Y_\infty = \frac{K(\alpha_1)}{4\pi\sqrt{\rho h(D_x D_y)^{1/2}}}, \quad (4.23)$$

where $K(\alpha_1)$ is an elliptical integral of the first kind (References 6 and 29) and

$$\alpha_1 = \frac{1}{\sqrt{2}} \left| 1 - \frac{D_{xy}}{\sqrt{D_x D_y}} \right|^{1/2}. \quad (4.24)$$

For $(D_x D_y)^{1/2} < D_{xy} < 3(D_x D_y)^{1/2}$ the elliptical integral $K(\alpha_1)$ in Equation (4.23) should be replaced by the term

$$\frac{1}{\sqrt{1 + \alpha_1^2}} K\left(\frac{\alpha_1}{\sqrt{1 + \alpha_1^2}}\right). \quad (4.25)$$

For the special case of $D_{xy} = (D_x D_y)^{1/2}$ the expression for point mobility of infinite plate is reduced to:

$$Y_\infty = \frac{1}{8\sqrt{\rho h(D_x D_y)^{1/2}}}. \quad (4.26)$$

Equation (4.26) can be a sufficient approximation for a wide range of plates (Reference 6).

Corrugated and ribbed plates may be regarded as equivalent orthotropic plates, provided that the separation of the ribs or the wavelength of the corrugations is less than half the structural wavelength ($s < \lambda_p/2$). See ESDU Item No. 02009, Appendix A, for more information on equivalent orthotropic plates.

4.5 Point Mobilities of Rings in Flexure

Rings are able to vibrate in flexure and in torsion (Reference 10). In flexure the motion may either be perpendicular to the plane of the ring or radial. This section is concerned with the latter type of bending, when displacements are normal and in the plane of the ring. The point mobility in Table 8.5 is similar to that of an infinite beam (Table 8.2) and applies for a thin ring above the ring frequency, which is

$$f_r = \frac{c_l}{2\pi R}, \quad (4.27)$$

where c_l is the longitudinal wave velocity and R is the mean radius of the ring.

4.6 Point Mobilities and Properties of Thin Cylindrical Shells

During the vibration of a thin cylinder there is a circumferential wave motion in which the cylinder is in bending and a longitudinal wave motion with axial deformation of the cylindrical shell. Above the ring frequency (Equation (4.27)) bending due to the circumferential motion is predominant and the point mobility of the cylinder becomes the same as that for an infinite, thin plate in bending, see Table 8.6.