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An introduction to rigid aeroplane response to gusts and atmospheric turbulence

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The Dynamics Committee, which first met in January 1962, has the co-operation of many engineers and scientists in industry, research establishments and universities from whom much assistance and unpublished information is being received.

AN INTRODUCTION TO RIGID AEROPLANE RESPONSE TO GUSTS AND ATMOSPHERIC TURBULENCE

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AN INTRODUCTION TO RIGID AEROPLANE RESPONSE TO GUSTS AND ATMOSPHERIC TURBULENCE

1. INTRODUCTION

This Item presents an introduction to rigid aeroplane response to gusts and atmospheric turbulence, together with the design criteria required to satisfy airworthiness requirements for both types of disturbance. The Item includes the vertical response of an aeroplane to a simple discrete sharp-edged gust, extension of this approach to accommodate the delays due to the build-up of lift and incidence and the further extension to a two degree-of-freedom model to provide lift and pitch response. Aeroplane response to atmospheric turbulence is provided through the relation between the frequency response function of an aeroplane and the methods of the power spectral density of a random variable.

The basic equations for the lift and normal acceleration on an aeroplane due to the simple sharp-edged gust, that is assumed instantaneously to change the velocity and angle of incidence of the aeroplane, are developed. The equations are extended to overcome the restrictive nature of an infinite gust gradient by introducing a gust alleviation factor that also accommodates for the lag effects of the build-up of lift in response to gust entry and sudden changes of incidence. The analysis is further extended to cover two degree-of-freedom motion in pitch and plunge. The gust design criteria for the purposes of satisfying airworthiness requirements are presented.

Continuous atmospheric turbulence is assumed to be a stationary random process and so the results of power-spectral analysis, dependant upon a scale length and an intensity or standard deviation of the turbulence, can be applied to the analysis of gust loads on aeroplanes. The general relationship for linear systems between the power spectrum of a random input variable and an output response are used to relate the spectrum of aeroplane loads to the spectrum of atmospheric gust velocity. For the case of aeroplane output loads having a normal or Gaussian distribution, as the standard deviation is used to describe the probability distribution of aeroplane loads. In order to assess the probability of the highest peak within a time interval or the number of peaks expected in a time interval, frequency of exceedance data methods provide an alternative to cumulative probability methods. Airworthiness requirements for aeroplanes subject to continuous turbulence are described using the power spectral method and the frequency of exceedance method.

The methods presented provide a fairly simplistic approximation to realistic atmospheric disturbances, both for discrete gusts and for turbulence, however they do provide a satisfactory means by which representative aeroplane loading can be estimated and certification satisfied.

Appendix A presents a brief review of random variable theory introducing a description of the Fourier series and integral, introduction of the power-spectral density function and its relationship to the properties of stationary random processes including the mean, standard deviation and auto-correlation function.

The relatively new statistical discrete gust method, which is a way to describe the more extreme gusts in a spectrum, is outlined in Appendix B. This method, which does not form part of the airworthiness requirements, caters for gusts within the high frequency range of the spectrum, involves aeroplane responses in the time domain and uses sequences or families of discrete vertical gusts to formulate the probability of occurrence of a given sequence of gusts.

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2. NOTATION

Any coherent system of units may be used.

$\underline{A}, \underline{B}$	matrices containing coefficients in equations for short period oscillation
A_{rms}	ratio of <i>rms</i> values (see Equation (4.10))
A_1	parameter associated with numerator of normal acceleration transfer function
a	lift curve slope of wing ($= dC_L/\alpha$)
a_n, b_n, c_n	coefficients of Fourier series of $F(t)$, $n = 0, 1, 2, \dots$
b_1, b_2	parameters defining probability distributions of <i>rms</i> gust velocity (see Sketch 4.5)
c	mean chord of wing
F_g	flight profile alleviation factor
F_{gm}, F_{gz}	parameters associated with F_g
$F(t)$	a random function of time
f	acceleration in z-direction
f, g	probability density functions
$G(\omega)$	frequency representation of $F(t)$
g	acceleration due to gravity
H, H_{ref}	gradient distance of discrete gusts and reference gradient distance, respectively
i_Y	inertia parameter (see ESDU Item 67002 ^[1])
K_g	gust alleviation factor
K_u	Küssner function
L, C_L	lift and lift coefficient of aeroplane, respectively, where $L = \frac{1}{2} \rho V_{trim}^2 S C_L$
L	scale of turbulence
$L_g(t), L_\alpha(t)$	defined as $W_a L_\alpha(\infty)$ and $K_u L_\alpha(\infty)$, respectively
$L_\alpha(\infty)$	steady-state dimensional lift curve slope
$M_q, M_w, M_{\dot{w}}$	aero-normalized pitching moment derivatives (see ESDU Item 67002 ^[1])

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N	frequency of exceedance (see Equation (4.9)) and probability of occurrence of discrete gust
N_0	radius of gyration of PSD of a random quantity
n	load factor in units of g
n	probability of occurrence of normal acceleration response peaks
PSD	denotes power-spectral-density
P	cumulative probability distribution
P_1, P_2	parameters defining probability distributions of rms gust velocity
p	Laplace variable relating to non-dimensional time, s
$p_1, p_2, \dots, p_n$	probability factors
q	pitch rate of aeroplane ($= d\theta/dt$)
$R(\tau)$	auto-correlation function
R_1, R_2	factors associated with F_{gm}
S	wing area
SDG	denotes statistical-discrete-gust
s	distance travelled in aeroplane mean chord lengths ($= V_{trim}t/c$)
T	denotes aeroplane transfer function
T	period
t, t_1, t'	time
U_{ref}	reference gust velocity in equivalent airspeed (EAS) (see Section (3.3))
U_σ	design gust velocity (see Section (4.5))
u, v, w	aeroplane velocities in the x -, y - and z -directions, respectively
u_g	variable gust velocity
u_k	intensity variable (see Equation (B.2))
V	aeroplane velocity
V_g	gust velocity
W	weight of aeroplane

W_a	Wagner function
$\underline{X}, \underline{Y}$	matrices containing dependent and independent variables in equations for short period oscillation
x, y, z	body axis directions for overall aeroplane with respect to origin O
$y(t)$	a stationary random function of time
Z_w	aero-normalized Z-force derivatives (see ESDU Item 67002 ^[1])
α	aeroplane incidence
γ	flight path angle
γ	peak normal acceleration amplitude response / u_k
ε	error function
ζ	viscous damping ratio
θ	pitch angle
μ	relative density parameter
μ_g	mass parameter
ν	frequency associated with distance s ($= \omega c / V_{trim}$)
ξ_{DII}	ratio of design peak, or limit, load to <i>rms</i> load
ρ	density of air
σ	standard deviation of stationary random process
τ	unit of aerodynamic time
τ	increment of time
$\Phi(\omega)$	power-spectral-density (PSD) function; frequency representation of $F(t)$
$\Phi^*(\omega)$	complex conjugate of $\Phi(\omega)$
Ω	frequency ($\equiv \omega / V_{trim}$)
ω, ω_0	frequency and fundamental frequency, respectively, associated with time t

Subscripts

B, C, D	relate to design speeds for maximum gust intensity, cruising and dive, respectively
G, H	relate to design speeds from Defence Standards

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<i>Dll</i>	denotes design limit load
<i>g</i>	denotes a gust
<i>i, o</i>	denotes input and output, respectively
<i>Im</i>	denotes unit impulse
<i>SP</i>	denotes short period oscillation
<i>St</i>	denotes unit step
<i>trim</i>	denotes trimmed flight

Operators

<i>D</i>	differential operator (= d/dt) (also one or two dots above a quantity indicate first and second differential, respectively, of that quantity with respect to time)
<i>Det{...}</i>	determinant of {...}
<i>F{...}</i>	Fourier transform of {...}
$\lim_{t \rightarrow \infty} \{...\}$	denotes limiting value of {...} as t approaches ∞
<i>Num{...}</i>	numerator of {...}
<i>p{...}</i>	probability of {...}
<i> ... </i>	denote a modulus

a bar above a quantity denotes the Laplace transform of that quantity

a bar below a quantity denotes a matrix

angled brackets surrounding a quantity, *i.e.* $\langle \dots \rangle$, indicates the mean of that quantity

3. SHARP-EDGED GUSTS

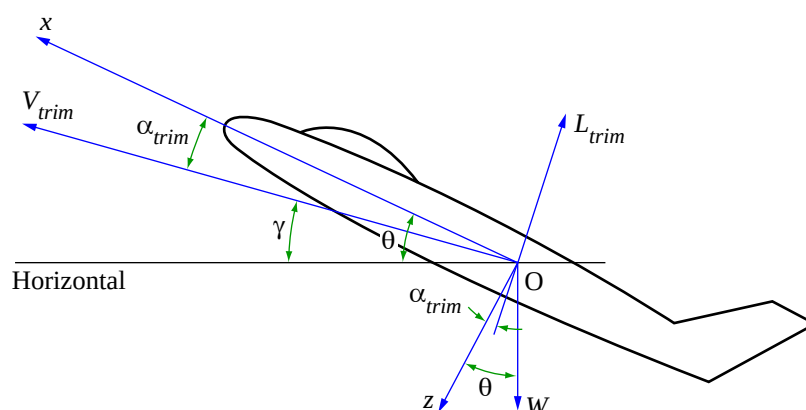
3.1 Basic Equations

An elementary treatment of the response of an aeroplane to a gust is to assume a simple sharp-edged gust, which when encountered changes the wind direction on an aeroplane instantaneously and completely over the whole aeroplane structure, and to assume that there is no angular displacement of the aeroplane from its initial attitude, so that only a change of angle of attack of the aeroplane is induced.

Consider a rigid aeroplane in an initial trimmed climbing flight condition at flight path angle, γ , pitch angle, θ , with an angle of attack, α_{trim} , and flight velocity, V_{trim} , with respect to an undisturbed atmosphere and an orthogonal axis system, Oxyz, as illustrated in Sketch 3.1. The lift on the aeroplane, L_{trim} , may be written as

$$L_{trim} = \frac{1}{2} \rho V_{trim}^2 S a_{2D} \alpha_{trim} \equiv W \cos \gamma, \quad (3.1)$$

in which $a_{2D} (= (dC_L / d\alpha)_{2D})$ is the lift curve slope of the infinite aspect ratio or two-dimensional wing, ρ is the air density and S is the wing area.



Sketch 3.1 Aeroplane in trimmed flight in undisturbed atmosphere

Upon encountering a gust of velocity, V_g , inclined at angle θ_g to the aeroplane flight direction, the increased lift, L , on the aeroplane, illustrated in Sketch 3.2, becomes

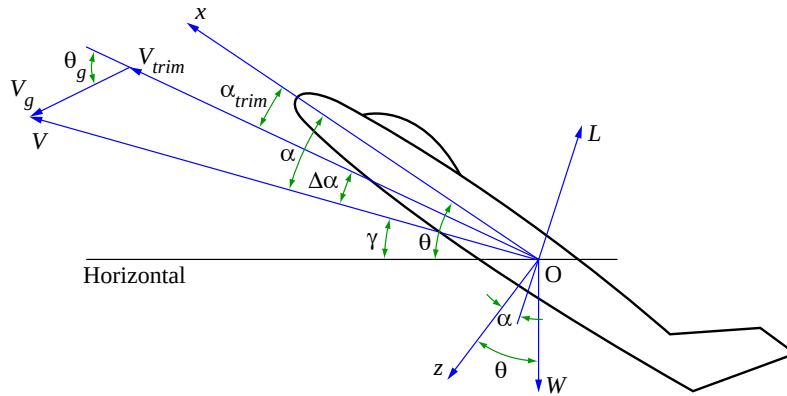
$$L = \frac{1}{2} \rho V^2 S a_{2D} \alpha \equiv \frac{1}{2} \rho V^2 S a_{2D} (\alpha_{trim} + \Delta\alpha), \quad (3.2)$$

in which V is the updated velocity of the aeroplane relative to the disturbed atmosphere and $\Delta\alpha$ is the change of incidence due to the gust.

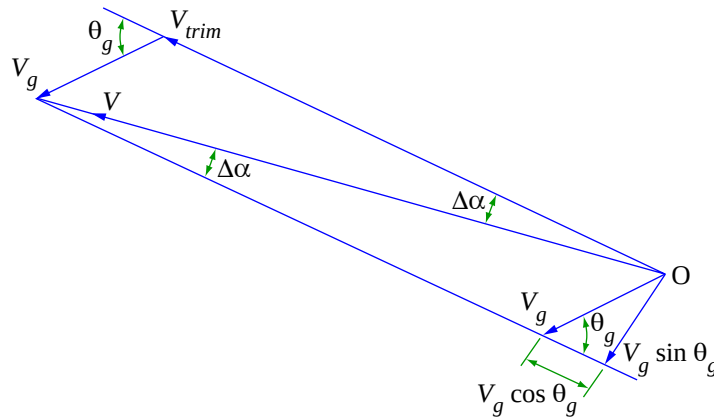
From the parallelogram of velocity vectors from Sketch 3.2, redrawn in Sketch 3.3, the velocity V may be written as

$$\begin{aligned} V^2 &= V_{trim}^2 + V_g^2 - V_{trim} V_g \cos(\pi - \theta_g) \\ &= V_{trim}^2 + V_g^2 + V_{trim} V_g \cos \theta_g. \end{aligned} \quad (3.3)$$

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Sketch 3.2 Aeroplane in disturbed atmosphere from gust of velocity V_g



Sketch 3.3 Parallelogram of velocity vectors from Sketch 3.2

Denoting the incremental load factor due to the gust as Δn then, with the use of Equations (3.1) to (3.3), the load factor, n , on the aeroplane may be rewritten as

$$n = 1 + \Delta n = \frac{L}{W} \equiv \left(1 + \left(\frac{V_g}{V_{trim}} \right)^2 + 2 \frac{V_g}{V_{trim}} \cos \theta_g \right) \left(\cos \gamma + \frac{\frac{1}{2} \rho V_{trim}^2 a_{2D} \Delta \alpha}{W/S} \right). \quad (3.4)$$

The incremental change of incidence, $\Delta \alpha$, may be written in terms of θ_g using the following expression, obtained from Sketch 3.3,

$$\Delta \alpha = \tan^{-1} \left(\frac{V_g \sin \theta_g}{V_{trim} + V_g \cos \theta_g} \right) \equiv \tan^{-1} \left(\frac{(V_g/V_{trim}) \sin \theta_g}{1 + (V_g/V_{trim}) \cos \theta_g} \right) \quad (3.5)$$

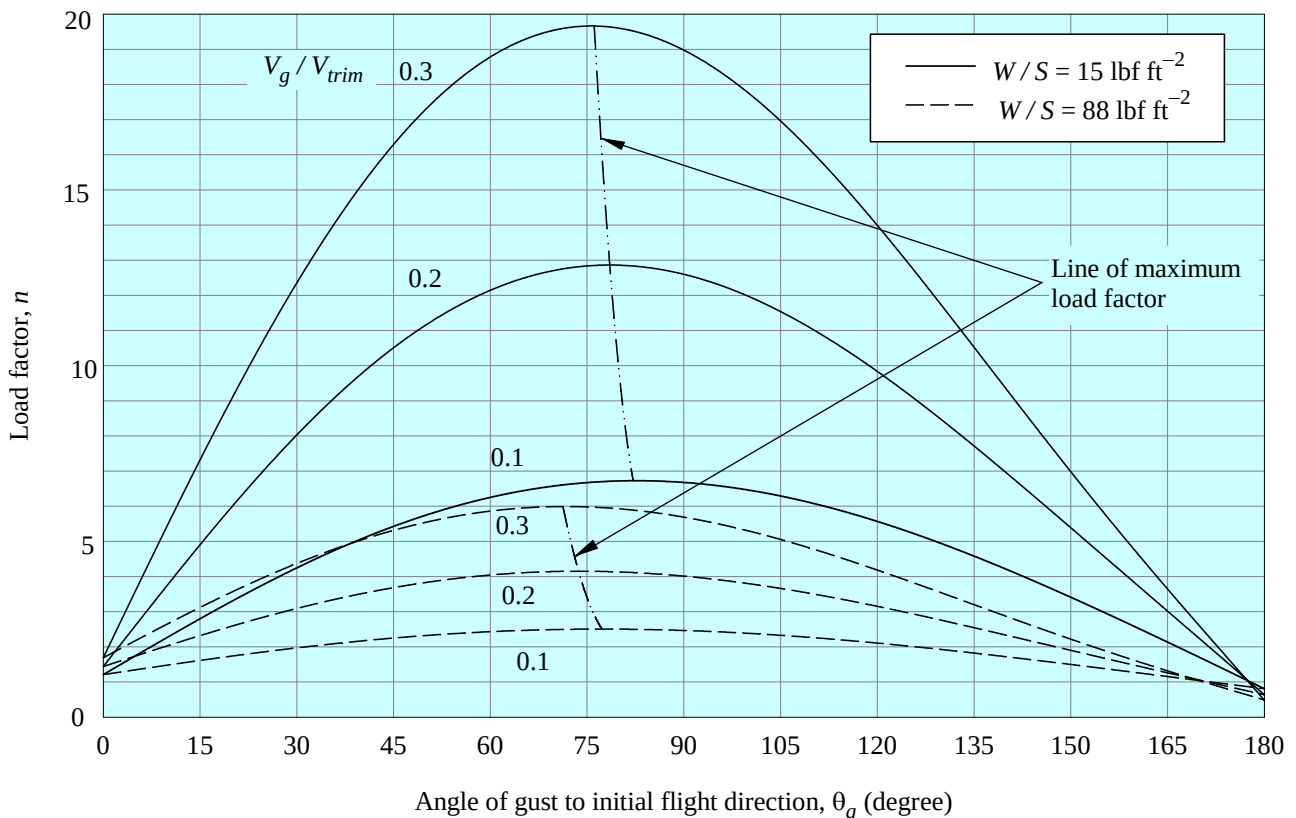
and so Equation (3.4) becomes

$$n = 1 + \Delta n \equiv \left(1 + \left(\frac{V_g}{V_{trim}} \right)^2 + 2 \frac{V_g}{V_{trim}} \cos \theta_g \right) \times \left(\cos \gamma + \frac{\frac{1}{2} \rho V_{trim}^2 a_{2D}}{W/S} \tan^{-1} \left(\frac{(V_g/V_{trim}) \sin \theta_g}{1 + (V_g/V_{trim}) \cos \theta_g} \right) \right). \quad (3.6)$$

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The term in the first set of large brackets on the right-hand-side of Equation (3.6) represents the effect of the increased resultant speed and the term in the second set of large brackets represents the effects of the lift change, *i.e.* the lift in trimmed flight plus the lift increment due to the change of angle of attack.

Sketch 3.4 shows a plot of load factor, n , versus the angle of the gust to the trimmed flight direction, θ_g , for two wing loadings, $W/S = 15$ and 88 lbf/ft^2 , and three gust velocities, $V_g/V_{trim} = 0.1, 0.2$ and 0.3 for an aeroplane flying at level flight at 400 ft/s at sea level.



Sketch 3.4 Variation of load factor, n , versus angle of gust, θ_g

Sketch 3.4 shows that the maximum acceleration occurs at values of θ_g between 70° and 85° and is almost independent of wing loading and gust strength, which implies that the change in angle of attack is of greater significance than the change in airspeed. Furthermore the magnitude of the load factor is fairly constant over a range of θ_g of about 30° about the maximum and so little error is introduced by assuming a gust angle of 90° which simplifies the load factor of Equation (3.6) to

$$n = 1 + \Delta n \approx \cos \gamma + \frac{\frac{1}{2} \rho a_{2D} V_g V_{trim}}{W/S}, \quad (3.7)$$

in which $(V_g/V_{trim})^2$ has been assumed negligibly small and $\tan^{-1}(V_g/V_{trim})$ has been approximated as V_g/V_{trim} . In this expression the gust velocity, V_g , may be interpreted as the vertical component of the gust.

Most often the aeroplane will be in level flight, or in gliding or climbing flight at small angles, *i.e.* $\cos\gamma \approx 1$, and so the incremental load factor for a vertical gust becomes

$$\Delta n \approx \frac{\frac{1}{2} \rho a_{2D} V_g V_{trim}}{W/S} \quad (3.8)$$

and is seen to be proportional with V_{trim} , V_g , a_{2D} , and ρ , and inversely proportional with the wing loading, W/S .

For a horizontal gust when $\theta_g = 0$ and for gliding flight when $\cos\gamma \approx 1$, the incremental load factor simplifies to

$$\Delta n \approx \frac{2V_g}{V_{trim}}. \quad (3.9)$$

3.2 Extended Equations

In order to correct for the restrictive assumptions of the basic analysis, that includes an infinite gust gradient and instantaneous and complete immersion in the gust disturbed atmosphere, Equation (3.8) is regarded as a reference value and a gust alleviation factor, K_g , is introduced and defined as

$$K_g = \frac{\text{peak normal acceleration from a discrete gust}}{\text{reference value of peak acceleration from Equation (3.8)}}. \quad (3.10)$$

The gust alleviation factor is used to accommodate for the type of discrete gust as well as the aerodynamic lag during the build-up of the gust lift and the initial aeroplane response due to the sudden change of aeroplane incidence. With the further assumption that the gust acts uniformly across the wing, a_{2D} is replaced by the three dimensional lift curve slope of the wing, a , and the revised incremental load factor for a vertical gust is then given by

$$\Delta n \approx \frac{\frac{1}{2} \rho a V_g V_{trim}}{W/S} K_g. \quad (3.11)$$

Both the aerodynamic lag and initial aeroplane response is estimated from the work of Küssner^[2] which forms the basis of a solution of an approximate model which removes the assumptions of infinite gust gradient, steady lift and no vertical motion.

The build-up of lift due to entry into a sharp-edged gust will theoretically take an infinite time to reach its final steady-state value, but it is reasonably close to this value after about 10 wing chord lengths. The function that approximates this build-up of lift is known as the Küssner function^[2], K_u . There is also a build-up of lift due to the sudden change of incidence. This is experienced by the aeroplane first as an initial impulse in which the lift jumps to approximately 50% of its final steady-state value and then as an exponential change, in which the lift rises to a value very close to its final value again after about 10 wing chord lengths. The function that approximates this lift variation is known as the Wagner function^[2], W_a . Note that the Wagner function contains a delta function at the initial instant corresponding to the apparent mass effect^[3].

One way of dealing with the build-up of lift is to treat the variations with time of the gust velocity, V_g , and the aeroplane angle of incidence, α , as a large number of incremental step changes and to regard the Küssner

function, K_u , and Wagner function, W_a , as indicial functions^[4], defined by

$$K_u = \frac{L_g(t)}{L_\alpha(\infty)} \quad \text{and} \quad W_a = \frac{L_\alpha(t)}{L_\alpha(\infty)}.$$

Expressing incremental changes in gust velocity and incidence, respectively, as

$$\Delta V_g = \dot{V}_g \Delta t \quad \text{and} \quad \Delta \alpha = \Delta \left(\frac{\dot{z}}{V_{trim}} \right) = \frac{1}{V_{trim}} \ddot{z} \Delta t,$$

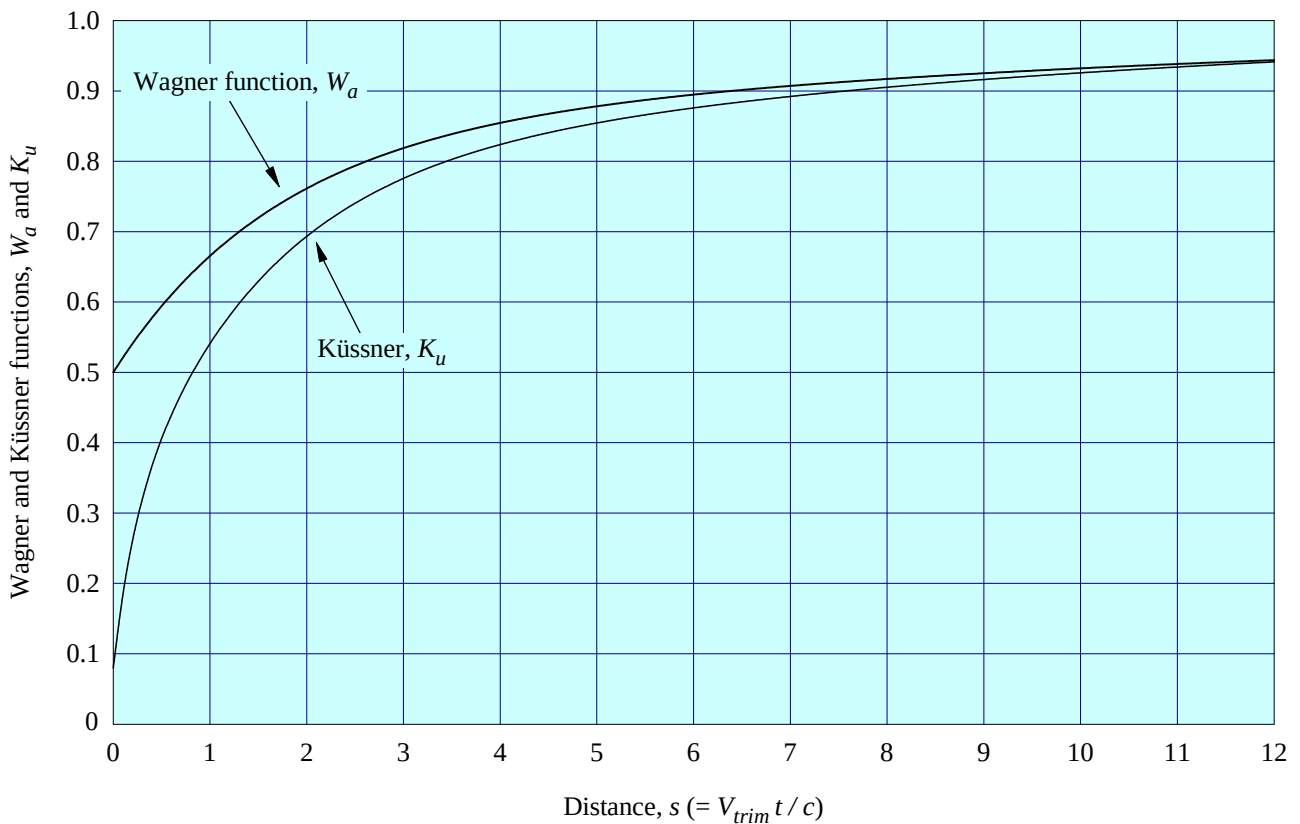
then, with $L_\alpha(\infty) = \frac{1}{2} \rho V_{trim}^2 S a$, the total lifts at time t are the sum of the incremental lifts, given by

$$L_g(t) = \frac{1}{2} \rho V_{trim}^2 S a \sum_0^t K_u \{t-t'\} \dot{V}_g \{t'\} \Delta t' \equiv \frac{1}{2} \rho V_{trim}^2 S a \int_0^t K_u \{t-t'\} \dot{V}_g \{t'\} dt'$$

and

$$L_\alpha(t) = \frac{1}{2} \rho V_{trim} S a \sum_0^t W_a \{t-t'\} \ddot{z} \{t'\} \Delta t' \equiv \frac{1}{2} \rho V_{trim} S a \int_0^t W_a \{t-t'\} \ddot{z} \{t'\} dt'.$$

These unsteady lift functions of Küssner and Wagner that were derived for a two-dimensional wing but are assumed to be applicable to the finite wing, are illustrated in Sketch 3.5[†].



Sketch 3.5 Küssner and Wagner functions

[†] The values for the Küssner and Wagner functions were taken from ESDU Item 84020^[3].

With the inclusion of the unsteady flow effects and assuming that the gust now increases from zero with distance in the direction of flight until the maximum gust velocity, V_g , is obtained, and that the aeroplane can rise but does not pitch under the action of the gust, *i.e.* a single degree-of-freedom, the equation of vertical motion may be written as

$$(\tau D - Z_w) W_a\{t\} w(t) = Z_w K_u\{t\} u_g(t), \quad (3.12)$$

where D is the differential operator ($= d/dt$), Z_w is the aero-normalised Z -force derivative ($\approx a$), $W_a\{t\}$ and $K_u\{t\}$ are the Wagner and Küssner functions, respectively, u_g is the gust velocity and τ is the unit of aerodynamic time given by

$$\tau = \frac{W/g}{\frac{1}{2}\rho V_{trim} S}. \quad (3.13)$$

This equation is usually transformed from time, t , to distance travelled in chord lengths, $s = V_{trim} t/c$, and becomes a function of the mass parameter, μ_g , defined as

$$\mu_g = \frac{W/g}{\frac{1}{2}\rho S c a}. \quad (3.14)$$

A solution of Equation (3.12) has been obtained by Pratt and Walker^[5] using expressions for the transient lift functions given by

$$W_a\{s\} = 1.000 - 0.165e^{-0.090s} - 0.335e^{-0.600s} \quad (3.15)$$

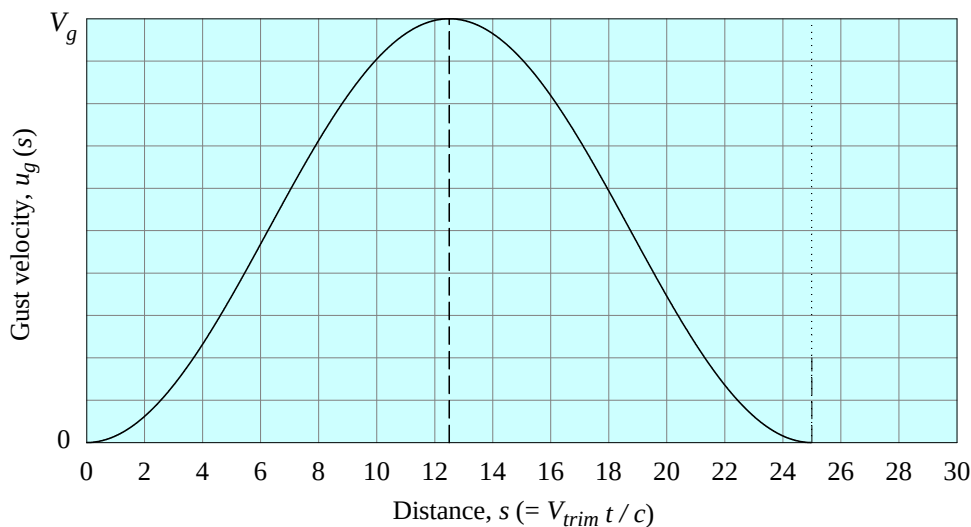
$$\text{and} \quad K_u\{s\} = 1.000 - 0.236e^{-0.116s} - 0.513e^{-0.728s} - 0.171e^{-4.840s},$$

with a gust shape of

$$u_g(s) = \frac{V_g}{2} \left(1 - \cos\left(\frac{\pi s}{H}\right) \right), \quad \text{for } 0 \leq s \leq 2H \quad (3.16)$$

$$u_g(s) = 0, \quad \text{for } 0 > s > 2H,$$

in which H is the horizontal distance in chord lengths from zero to the maximum gust velocity, taken as 12.5 wing chord lengths. This gust shape is illustrated in Sketch 3.6.



Sketch 3.6 Gust shape

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