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Fatigue damage and life under random loading



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FATIGUE DAMAGE AND LIFE UNDER RANDOM LOADING

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FATIGUE DAMAGE AND LIFE UNDER RANDOM LOADING

1. NOTATION

Abbreviations used in the Item:

CFRP	carbon fibre reinforced plastic
ESDU	Engineering Sciences Data Unit
FRF	frequency response function
GLARE	glass/aluminium reinforced laminate
HCF	high cycle fatigue
LCF	low cycle fatigue
NASA	National Aeronautics and Space Administration
PDF	probability density function
PSD	power spectral density
RB	reversed bending
SDOF	single degree of freedom system
SL	spectrum level
SPL	sound pressure level (in dB re. 2×10^{-5} Pa)

		UNITS	
		SI	British
Symbols			
a_x	length of plate in x-direction	m	in
b	exponent associated with $S-N$ curve (material constant)		
b_y	length of plate in y-direction (arc length of curved side)	m	in
C_{rms}	constant in $\epsilon_{rms}-N$ relationship (material constant)		
D	cumulative damage in Miner's rule (Equation (4.3)), expected (mean) value of fatigue damage		
D_i	fractional damage caused by stress cycles with amplitude S_i		
E	Young's modulus for coupon material	Pa	lbf/in ²
F_l	expected (mean) number of upwards crossings with stress level S_l per second	Hz	Hz

F_p	expected (mean) number of peaks per second	Hz	Hz
F_{zc}	expected (mean) number of zero crossings with positive slope per second	Hz	Hz
f	frequency	Hz	Hz
f_a	lower frequency of bandwidth	Hz	Hz
f_b	upper frequency of bandwidth	Hz	Hz
f_c	centre frequency of bandwidth	Hz	Hz
f_n	resonance frequency	Hz	Hz
$G(f)$	power spectral density function as function of f	$(\text{--})^2/\text{Hz}$	
$G(f_n)$	power spectral density function of the acoustic pressure at frequency f_n .	Pa^2/Hz	$(\text{lbf/in}^2)^2/\text{Hz}$
h	plate thickness	m	in
K_a	constant in $S_a - N$ relationship (material constant)	$(\text{Pa})^b$	$(\text{lbf/in}^2)^b$
K_r	constant in $S_r - N$ relationship (material constant)	$(\text{Pa})^b$	$(\text{lbf/in}^2)^b$
K_{rms}	constant in $S_{rms} - N$ relationship (material constant)	$(\text{Pa})^b$	$(\text{lbf/in}^2)^b$
K_v	resonance frequency parameter	m/s	in/s
K_ζ	damping ratio correction factor		
m_n	n^{th} moment of PSD		
N	number of cycles to failure		
N_i	number of cycles to failure at stress S_i		
n	total number of cycles in process		
n_i	number of cycles of stress with given amplitude S_i		
P_i	probability of occurrences of cycles with stress amplitude, S_i		
$p(x)$	probability density function for random variable x		
p_e	PDF for the envelope of process (stress process assumed for purpose of introducing units)	Pa^{-1}	$(\text{lbf/in}^2)^{-1}$
p_G	probability density function of Gaussian process	Pa^{-1}	$(\text{lbf/in}^2)^{-1}$
p_p	probability density function of peaks	Pa^{-1}	$(\text{lbf/in}^2)^{-1}$
p_{rfa}	probability density function of stress amplitudes of rainflow cycles	Pa^{-1}	$(\text{lbf/in}^2)^{-1}$

p_{rfr}	probability density function of stress ranges of rainflow cycles	Pa ⁻¹	(lbf/in ²) ⁻¹
$\tilde{p}_{rfa}$	nondimensional (normalised) PDF of stress amplitudes of rainflow cycles		
$\tilde{p}_{rfr}$	nondimensional (normalised) PDF of stress ranges of rainflow cycles		
R	radius of curvature	m	in
R_0	ratio of stress at rivet line in assumed mode shape to applied uniform static pressure		
$S(t)$	sample record (signal) as a function of time		
$\{S(t)\}$	random process as a function of time		
S	stress	Pa	lbf/in ²
S_a	stress amplitude	Pa	lbf/in ²
S_{av}	the expected value of the process, $\{S(t)\}$	Pa	lbf/in ²
S_{a0}	stress amplitude for damage-equivalent fully reversed alternating stress	Pa	lbf/in ²
S_f	true fracture strength	Pa	lbf/in ²
S_m	mean stress (of rainflow cycle)	Pa	lbf/in ²
S_{max}	maximum stress	Pa	lbf/in ²
S_{min}	minimum stress	Pa	lbf/in ²
S_r	stress range	Pa	lbf/in ²
S_{rms}	root mean-squared value of stress	Pa	lbf/in ²
S_i	i^{th} stress level	Pa	lbf/in ²
S_{ts}	ultimate tensile strength	Pa	lbf/in ²
S_y	yield stress	Pa	lbf/in ²
S_0	stress range for damage-equivalent fully reversed alternating stress	Pa	lbf/in ²
$\bar{S}$	mean stress (averaged over time)	Pa	lbf/in ²
T	duration of damage process	s	s
t	time	s	s
T_{fl}	mean fatigue life of component	s	s

V_s	velocity parameter		
W	power spectral density	Pa^2/Hz	$(\text{lbf}/\text{in}^2)^2/\text{Hz}$
X	random variable		
x_i	values of random variable X		
Γ	gamma function		
γ	irregularity factor		
Δf	bandwidth	Hz	Hz
ε	spectral width parameter		
ε_{rms}	root mean-squared value of alternating microstrain		
ζ	modal viscous damping ratio		
ρ	density of plate material	kg/m^3	*
σ_s	standard deviation of stress process	Pa	lbf/in^2
σ_s^2	variance of stress process	Pa^2	$(\text{lbf}/\text{in}^2)^2$
ω	frequency ($\omega = 2\pi f$)	rad/s	rad/s
ω_n	resonance angular frequency ($\omega_n = 2\pi f_n$)	rad/s	rad/s

Subscripts used:

a	refers to amplitude
p	refers to peaks
r	refers to range
rfa	refers to rainflow cycle amplitude
rfr	refers to rainflow cycle range

Overscores used:

—	denotes averaging in time domain
~	denotes nondimensional form

* In British units, for the system to be coherent the mass in lb must be divided by the constant 386.4. In the case of a density known, for example, to be $0.097 \text{ lb}/\text{in}^3$ the value to be used should be $0.097/386.4 = 2.51 \times 10^{-4} \text{ lbf s}^2/\text{in}^4$, where lbf is the unit of force.

2. INTRODUCTION

This Item, the first in a series of related Items, is concerned with fatigue of structures under random loading with special reference to acoustic fatigue. Acoustic fatigue occurs on structural components where the sound pressures are very high, for example, in the bay of the space shuttle orbiter or on the wings of short take-off aircraft. It can be associated in the case of aircraft with those parts of the aircraft that are close to the engines, such as the fuselage panels, engine nacelle and the box structures of the wing. Large values of sound pressures, caused by the engines, result in bending stresses in the skin panels and other components. ESDU provides a series of Items for estimating the response stresses in panels subjected to random acoustic loading, based on the work of Miles, Powell and Clarkson (References 3, 5, 6 and 11). For example ESDU 72005 is used to estimate the stresses in stiffened panels; in ESDU 72017 stresses are considered in curved sandwich panels and ESDU 84008 is used to estimate the stresses in laminated skin panels (References 14, 16 and 20). These Items are intended to be used in the case of a source of random noise, such as jet noise, although acoustic fatigue can be caused by tones from the fans or compressors of the engines. The amplitudes of the response are relatively small and the frequencies moderately high. Some of the basic concepts and damage models, applied to fatigue problems, are the same for both random and deterministic loading. However, the analysis of acoustic fatigue of a structure in the case of more complex random loading requires more specific methods, such as rainflow cycle counting. In this Item some definitions and model assumptions in relation to acoustic fatigue are discussed. The way in which probability density functions are used to estimate fatigue life is introduced.

In the method presented in ESDU 72005 it is assumed that a single mode – the lowest bending mode – is responsible for the damage caused by the stress response of the structure to a random pressure field. For some structures (mostly skin panels) the response spectra usually have one predominant mode (Clarkson, Reference 11). Structures, such as control surface skin plating (Reference 11) or fuselage frames (Reference 6), may have many comparable peaks in the response spectra. One possible method of assessing the damage to a structure caused by multi-modal stress is to consider the process in the time domain and to establish the probability density function for the stress ranges of the rainflow cycles, a procedure which involves rainflow cycles counting. Rainflow cycle counting is a method of obtaining the number of cycles of stress or strain and is described in the second Item of this series, ESDU 06010. A second possibility is to consider the stress response in the frequency domain. A number of spectral methods are available for estimating fatigue life from multi-modal spectra and these methods are considered in the third Item, ESDU 06011. Thirdly, the damage from random loading can be related to crack growth (fourth Item of the Series).

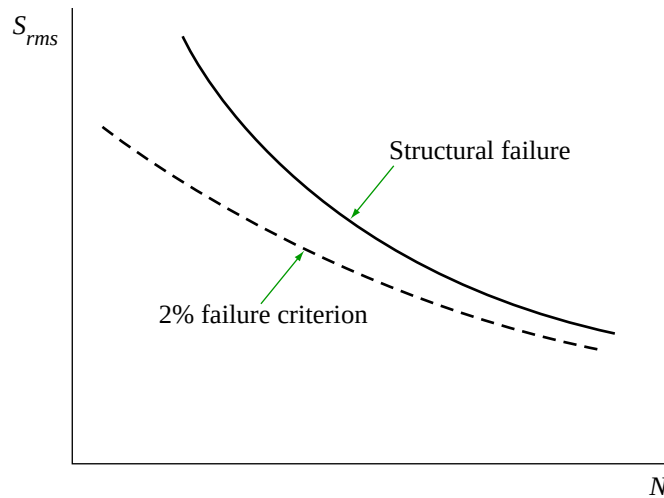
The purpose of this series of Items is to provide methods for estimating the fatigue life of structures which show a multi-modal response to random acoustic loading. The earlier ESDU method, relating to a SDOF response to acoustic loading, has been described in detail in several data Items and outlined in ESDU 74037 (Reference 17). An essential part of the method described in Item 74037 is the estimation of the lowest resonance frequency of the structure and many data Items are available for calculating the resonance frequencies in bending of many forms of plates, cylinders and curved panels, for example, ESDU 78004, ESDU 87002 and ESDU 99016 (References 18, 21 and 28). Information on the lowest resonance frequency has to be used in conjunction with the relationships between stress, S , and number of cycles to failure for the material, N , ($S-N$ curves) to estimate the fatigue life of the structure. The $S-N$ curves are obtained from the simulated random acoustic loading of small components (coupons) which might represent the most highly stressed part of an aircraft structure. The usual procedure is to place a coupon on an electrodynamic vibrator that is driven by a narrow band random signal centred on the lowest resonance frequency of the coupon, as described in ESDU 93027 (Reference 24). Typically, the coupons are riveted T-pieces, free-free beams or other small components that have only bending vibrations.

The basic assumptions relating to acoustic fatigue, which are introduced throughout the Items, are:

1. Acoustic fatigue is high cycle, low stress fatigue caused by a deformation within the structure which is essentially elastic in the macroscopic sense.
2. The stress-life ($S-N$) approach, mainly related to high cycle fatigue, is introduced as a basis upon which to estimate fatigue life. (Reference 27).
3. The definitions of fatigue life are associated with the criteria of failure chosen for the ($S-N$) curves. Two criteria are considered; (a) 2% decrease in the settling value of the fundamental resonance frequency of a coupon subjected to random loading and (b) structural failure (structure unable to take loads). The first criterion is predominantly used in acoustic fatigue (Reference 24) and defines approximately the duration of the crack initiation phase. In terms of fatigue life, the process of crack propagation, which leads to complete failure, is much shorter than that of crack initiation for high cycle fatigue. (Reference 30).
4. The main secondary factor which influences high cycle fatigue is mean stress.
5. Rainflow cycle counting is considered as a base for estimating the probability density function for the rainflow cycle stress range.

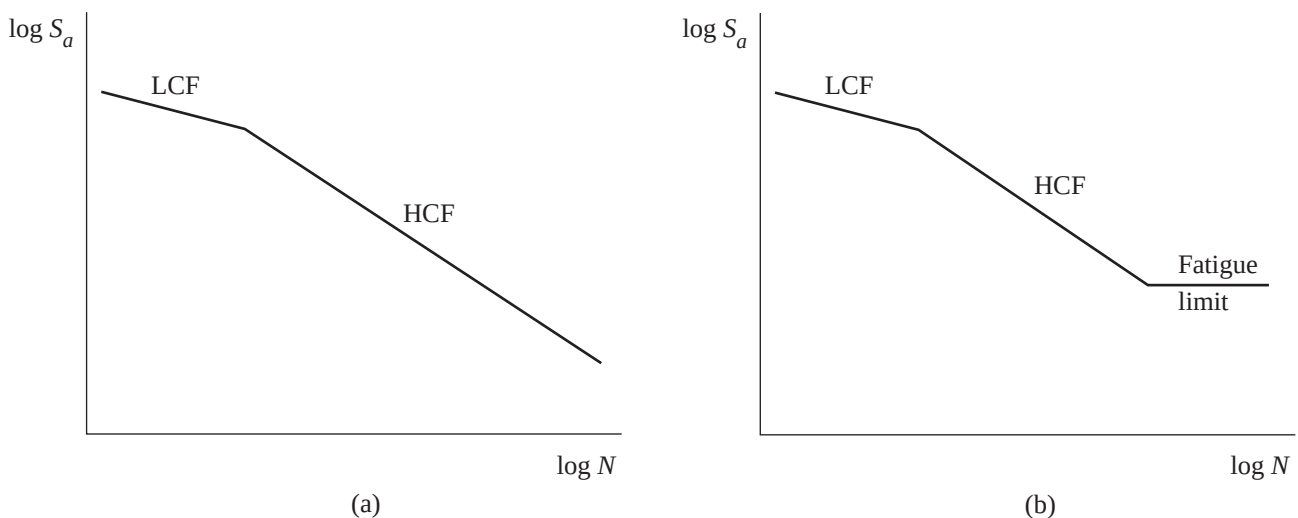
3. $S-N$ FATIGUE CURVES

Information on the fatigue failure of a material, or a particular structural component, which undergoes essentially elastic deformation during loading, is usually provided by an experimentally obtained curve ($S-N$ curve) of a nominal stress in the component versus number of cycles which lead to failure. The stress can be either in terms of the alternating stress amplitude, S_a , the stress range, $S_r (= 2S_a = S_{max} - S_{min})$ or the r.m.s. value of the alternating stress, S_{rms} . In the case of acoustic fatigue, when the 2% criterion of failure is used, the relationship between S_{rms} and the number of cycles to failure, N , is applied. For a fully reversed bending test the S_a-N curve is normally derived. It is important to establish the criterion for failure which has been used for the $S-N$ curves. A fully reversed bending test leads finally to structural failure and in this case the test coupon is not capable of carrying loads. In the case of simulated random loading for acoustic fatigue testing, the meaning of failure is less precise. For tests on some materials, the resonance frequency of the test coupon, excited by narrow band noise, decreases initially and settles to a constant value. After the settling period, evidence of fatigue is indicated by a decrease in resonance frequency. The condition for failure is assumed to be a 2% decrease in the settling value of the resonance frequency. Visual cracks often occur after the 2% decrease of resonance frequency. Subsequently, the cracks propagate, leading to failure of the specimen. Damage mechanisms in composite materials are different from those in metals, where the process of microcrack initiation is followed by crack growth. However, in both cases the criterion of the decrease in the resonance frequency is applied. The schematic $S-N$ curves shown in Sketch 3.1 are based on structural failure and the 2% decrease criteria (in this case r.m.s. stress is used for the ordinate).



Sketch 3.1 S – N curves with different failure criteria

The $S-N$ (stress – fatigue life) curves, which are obtained experimentally by an applied cyclic loading to a smooth specimen, are mostly presented in logarithmic form, as shown in Sketches 3.2a and 3.2b.

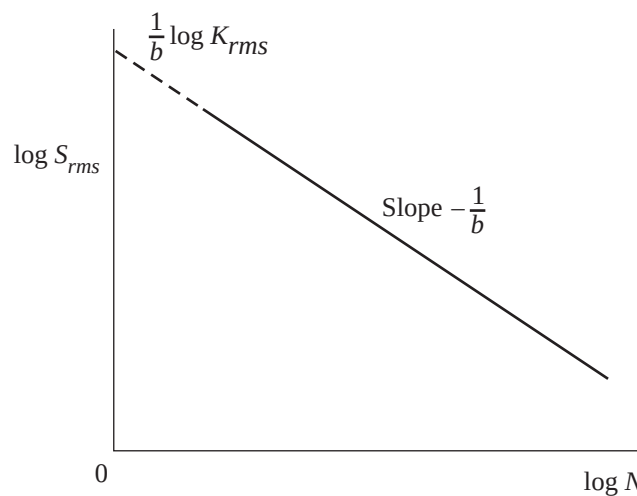


Sketch 3.2 Idealised types of $S-N$ curves (a) without fatigue limit (b) with fatigue limit

Such curves result from a reversed bending test, or from a simulated acoustic fatigue test. For most materials there are changes in slopes of $S-N$ curves when the number of cycles to failure is of order of magnitude 10^3 to 10^4 . In the case of some materials, *e.g.*, low and medium strength steels and ferrous alloys, the $S-N$ curves exhibit also a so-called ‘knee’, usually located in the range from 10^6 to 10^7 cycles to failure, where the slopes of the curves change from negative to zero, as shown in Sketch 3.2b. The stress range (or amplitude) for which the $S-N$ curve becomes horizontal is called the fatigue limit or endurance limit. The fatigue limit is associated with the arrest of crack nucleation in the specimen (References 27, 30 and 33). High strength steels and aluminium alloys belong to a class of materials which do not exhibit a clear endurance limit. For these materials the term endurance limit is often taken to be in engineering practice the stress amplitude for a chosen number of cycles to failure, usually 10^7 cycles, which can be sustained by a specimen. The part of the curve (straight line when constructed as $\log S$ against $\log N$) with negative

slope is described as the finite life region and the part with the horizontal line defines the infinite life region. The shapes of the $S-N$ curves define also regions of low-cycle fatigue (LCF) with higher values of alternating stress amplitude and high-cycle fatigue (HCF) with lower values of alternating stress amplitudes.

Acoustic fatigue is high cycle fatigue caused by a random stress process associated with essentially elastic deformation. $S-N$ curves in logarithmic form may be represented for HCF by a straight line, as shown in Sketch 3.3 which shows typical results for a simulated acoustic fatigue test. For acoustic fatigue the results are presented as a straight line of slope $-1/b$ on a $\log S_{rms} - \log N$ graph, where S_{rms} is the root mean squared value of the narrow band stress process. The slope $-1/b$ in Sketch 3.3 is sometimes called the Basquin exponent (Reference 27, page 223). The straight line intercepts the ordinate at the point $(0, (1/b)\log K_{rms})$. In Sketch 3.3, N represents the number of cycles to failure, although sometimes the number of stress reversals, instead of cycles, is used for the abscissa. There are two reversals for every cycle.



Sketch 3.3 Typical $\log S_{rms} - \log N$ line for acoustic fatigue

In using a statistical method to determine fatigue life, it is important to have a single well-defined $S-N$ curve for the material. The line shown in Sketch 3.3 is represented by the equation

$$\log N + b \log S_{rms} = \log K_{rms}, \quad (3.1)$$

or, in the $S_{rms} - N$ coordinate system the curve is described by

$$N S_{rms}^b = K_{rms}. \quad (3.2)$$

If the $S-N$ curves are in terms of stress amplitude, S_a , then

$$N S_a^b = K_a, \quad (3.3)$$

and, if in terms of stress range, S_r , where $S_r = 2S_a$,

$$N S_r^b = K_r, \quad (3.4)$$

where $K_r = 2^b K_a$. It is important to be aware of the difference between amplitudes and ranges, and to use the appropriate constant. Some values of the constants are shown in Table 3.1.

TABLE 3.1

Type of test	material	S_{rms} or S_r	K_{rms} or K_r (Pa ^b)	b	Reference
Acoustic	Al alloy (DTD 2024)	S_{rms}	$K_{rms} = 3.31 \times 10^{43}$	4.78	15
RB	Al alloy (7075-T6)	S_r	$K_r = 4.42 \times 10^{43}$	4.81	31
Acoustic	SPF/DB* Titanium	S_{rms}	$K_{rms} = 2.00 \times 10^{47}$	5.21	26

* SPF/DB – Super-plastic forming/diffusion bonding.

In Table 3.1 the stress, S_{rms} or S_r , has the units of Pa, but it may be more convenient to express the stress in MPa. With the stress in MPa the value of K_{rms} (or K_r) in Table 3.1 is divided by 10^{6b} . In the Item on titanium and titanium alloys the data on acoustic fatigue is provided with the stress in units of MPa (Reference 36)

N.B. For composite materials the acoustic fatigue information is provided in terms of strain rather than stress. Thus, in terms of r.m.s. microstrain, ϵ_{rms} , there is a similar relationship to Equation (3.2) which is

$$N \epsilon_{rms}^b = C_{rms} \quad (3.5)$$

For coupons made from beams of isotropic material, the stress and strain are directly related through E , the Young's modulus of the material. Table 3.2 provides some information on the constants, C_{rms} and b for two composite materials. (See Reference 32 for their construction.)

TABLE 3.2

Type of test	material	C_{rms}	b	Reference
Acoustic	GLARE	3.40×10^{23}	6.03	32
Acoustic	CFRP	1.60×10^{29}	6.99	32

3.1 Use of S – N Curves in Estimation of Fatigue Life of Panels

The $S-N$ curves which result from the testing of several coupons may be used to estimate the number of cycles to failure for a given stress in a larger structure with the same material as the coupon. In the case of sinusoidal loading with constant amplitude and frequency, the $S-N$ curves provide direct indication of the fatigue life or endurance of a component, if the stress response amplitude is known. For wide band random acoustic loading of a structure, a similar approach is possible if the multi-modal stress response is dominated by a single bending mode with the lowest resonance frequency, f_n . This type of stress response occurs in skin panels of aircraft. Two variants of the method can be applied. Either the damage associated with the higher frequency modes of stress response can be neglected, or it is assumed that the damaging cycles occur with frequency, f_n , as before, but the r.m.s. value of the stress response is based on the wide band process. It is also possible to treat a structure, for example a stiffened panel, as a lightly damped SDOF system and to obtain the r.m.s. value of the stress response to random acoustic loading (References 3 and 22) The r.m.s. value of the stress response of a stiffened panel may be estimated with the aid of ESDU 72005, see References 10 and 16. It is assumed (Reference 14) that the panel is subjected to random acoustic loading

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on one side. Acoustic loading, for example, the noise of a jet engine exhaust, causes a certain sound pressure level on the panel. The sound pressure is assumed to be uniform and in phase over the whole panel and the panel responds at frequency f_n as a SDOF system to broad band loading. In such a case the r.m.s. stress, S_{rms} , for a stiffened panel can be approximated at the rivet line or at the panel edge by (Reference 14)

$$S_{rms} = \left(\frac{\pi}{4\zeta} f_n G(f_n) \right)^{1/2} K_\zeta R_0, \quad (3.6)$$

where f_n is the resonance frequency of the lowest bending mode,

$G(f_n)$ is the power spectral density function of the acoustic pressure at frequency f_n for random acoustic loading of the panel,

ζ is the modal viscous damping ratio,

K_ζ is the damping ratio correction factor, as described in ESDU 72005, and

R_0 is the ratio of the stress at the rivet line in the assumed mode shape to the applied uniform static pressure.

Next, the appropriate $S-N$ curve for the material and construction method of the stiffened plate is selected and, with the value of S_{rms} estimated, the number of cycles to failure in acoustic fatigue can be determined. Once the number of cycles to failure has been established, the acoustic fatigue life is estimated, since the lowest resonance frequency in bending, f_n , is known. Thus, the fatigue life, T_{fl} , of the structural component with lowest resonance frequency in bending, f_n , is given by

$$T_{fl} = N/f_n. \quad (3.7)$$

Substituting for N , the number of cycles to failure, from Equation (3.2) gives

$$T_{fl} = \frac{K_{rms}}{f_n S_{rms}^b}. \quad (3.8)$$

Finally, combining Equations (3.6) and (3.8) leads to

$$T_{fl} = \frac{K_{rms}}{f_n^{b/2+1} \left(\frac{\pi}{4\zeta} G(f_n) \right)^{b/2} K_\zeta^b R_0^b}. \quad (3.9)$$

3.1.1 Example – life estimation for stiffened panel

This example is similar to Example 7.1 of ESDU 72005 in which a curved panel is subjected to jet noise and the Item is used for the estimation of stress in the panel.

An estimate is to be made of the fatigue life of a stiffened curved panel, subjected to jet noise on one side. The sound pressure level (SPL) of the noise in one-third octave bands on the panel has been estimated and is given in the table. The noise is assumed to be random without any discrete tones.

One-third octave centre frequency (Hz)	400	500	630	800	1000
One-third octave SPL (dB)	149	148	147	142	133