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Spectral methods for the estimation of fatigue life

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SPECTRAL METHODS FOR THE ESTIMATION OF FATIGUE LIFE

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SPECTRAL METHODS FOR THE ESTIMATION OF FATIGUE LIFE

1. NOTATION

		UNITS	
		<i>SI</i>	<i>British</i>
a	half length of beam	m	in
a_n	independent random variable of process, $S(t)$		
b	exponent associated with $S - N$ curve (material constant)		
b_n	independent random variable of process, $S(t)$		
C_n	independent random variable		
D	cumulative damage in Miner's rule		
D_{Ray}	cumulative damage for stationary, Gaussian narrow band process (Rayleigh approximation)		
E	Young's modulus for coupon material	Pa	lbf/in ²
F_p	number of peaks per second	Hz	Hz
F_{zc}	number of (zero) mean crossings with positive slope per second	Hz	Hz
$\tilde{F}_1$	input force to beam	N	lbf
F_1	characteristic frequency (m_1/m_0)	Hz	Hz
F_2	characteristic frequency (m_2/m_0) ^{1/2}	Hz	Hz
f	frequency	Hz	Hz
f_n	lowest natural frequency in bending; also, frequency, $f_n = n\Delta f$	Hz	Hz
h	thickness of beam	m	in
$G(f)$	power spectral density function as function of f	(--) ² /Hz (units depend on process)	(--) ² /Hz
$G(f_n)$	power spectral density function of acoustic pressure at frequency f_n	Pa ² /Hz	(lbf/in ²) ² /Hz
$G_x(f)$	input power spectral density function (force)	N ² /Hz	lbf ² /Hz

$G_x(\omega)$	input power spectral density function (force)	$N^2/(\text{rad/s})$	$\text{lbf}^2/(\text{rad/s})$
$G_y(f)$	output power spectral density function (stress)	Pa^2/Hz	$(\text{lbf/in}^2)^2/\text{Hz}$
$G_y(\omega)$	output power spectral density function (stress)	$\text{Pa}^2/(\text{rad/s})$	$(\text{lbf/in}^2)^2/(\text{rad/s})$
$g(f)$	normalised power spectral density function		
K_a	constant in $S_a - N$ curve relationship (material constant)	Pa^b	$(\text{lbf/in}^2)^b$
K_r	constant in $S_r - N$ curve relationship (material constant)	Pa^b	$(\text{lbf/in}^2)^b$
K_{rms}	constant in $S_{rms} - N$ curve relationship (material constant)	Pa^b	$(\text{lbf/in}^2)^b$
K_ζ	damping ratio correction factor		
k_b	bending wavenumber for beam	m^{-1}	in^{-1}
k_b^*	complex bending wavenumber for beam	m^{-1}	in^{-1}
m_b	mass of beam	kg	*
m_k	k^{th} moment of power spectral density function		
N	limit of summation index		
n	number of cycles; summation index		
p	general notation for probability density function of stress range for different models	Pa^{-1}	$(\text{lbf/in}^2)^{-1}$
p_p	probability density function of peaks	Pa^{-1}	$(\text{lbf/in}^2)^{-1}$
p_p^+	probability density function of positive peaks	Pa^{-1}	$(\text{lbf/in}^2)^{-1}$
p_{pr}	probability density function of cycle stress range in peak model	Pa^{-1}	$(\text{lbf/in}^2)^{-1}$
p_{pr}^+	probability density function for stress range of cycles associated with positive peaks	Pa^{-1}	$(\text{lbf/in}^2)^{-1}$
p_{rfr}	probability density function of rainflow cycle stress range	Pa^{-1}	$(\text{lbf/in}^2)^{-1}$
q	dimensionless coefficient ($\sigma_f F_2$)		
R_0	ratio of stress at rivet line in assumed mode shape to applied uniform static pressure		
$S(t)$	random process (for example, stress as a function of time)		
$\dot{S}(t)$	derivative of the process $S(t)$		

* See footnote at end of Notation Section.

$\ddot{S}(t)$	second derivative of the process $S(t)$		
S_{av}	average value of stress	Pa	lbf/in ²
S_{eq}	equivalent range of stress cycles	Pa	lbf/in ²
$\left[S_{eq}^b \right]_{Ray}$	mean value of S_r^b for narrow band process (Rayleigh model)	Pa ^b	(lbf/in ²) ^b
S_p	stress level of positive peak	Pa	lbf/in ²
S_r	range of stress cycle	Pa	lbf/in ²
S_{rms}	root mean-squared value of stress	Pa	lbf/in ²
S_1	stress at input point 1 on beam	Pa	lbf/in ²
T	duration of damage process	s	s
T_{fl}	fatigue life	s	s
t	time	s	s
w	displacement of beam	m	in
v_2	velocity at input point 2 on beam	m/s	in/s
x	coordinate		
y	coordinate		
Y_{21}	transfer mobility, force at point 1, velocity at point 2	s/kg	*
Γ	gamma function		
γ	irregularity factor		
Δf	frequency bandwidth	Hz	Hz
ε	spectral width parameter		
ε_{rms}	root mean-squared value of strain		
ε_1	strain at point 1 on beam		
ε_2	strain at point 2 on beam		
ζ	viscous damping ratio		
λ	damage correction factor		
μ	factor defining position of normal force applied to beam		

* See footnote at end of Notation Section.

φ_n	random phase angle		
σ_f	standard deviation $\sqrt{F_2^2 - F_1^2}$	Hz	Hz
σ_s	standard deviation of process $S(t)$	(units depends on process)	
σ_n	standard deviation of independent random variable		
$\sigma_{\dot{s}}$	standard deviation of process $\dot{S}(t)$		
σ_s^2	variance of process $S(t)$		
ω	frequency ($\omega = 2\pi f$)	rad/s	rad/s
ω_n	frequency of component of random Gaussian process	rad/s	rad/s

* In British units, for the system of units to be coherent, the mass in lb must be divided by the constant 386.4.

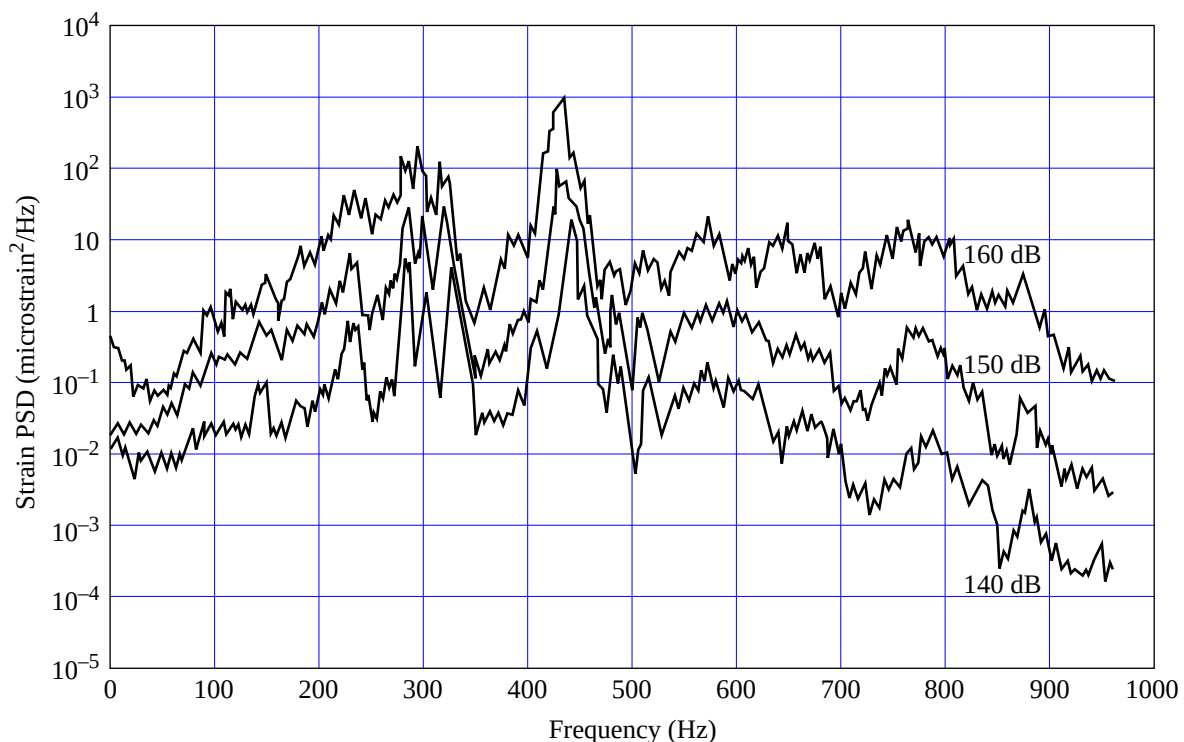
Abbreviations used in the Item:

ASME	American Society of Mechanical Engineers
ESDU	Engineering Sciences Data Unit
FRF	frequency response function
FFT	fast Fourier transform
GLARE	glass/aluminium reinforced laminate
IDFT	inverse discrete Fourier transform
NASA	National Aeronautics and Space Administration
PDF	probability density function
PSD	power spectral density
PWT	progressive wave tube
SDOF	single degree of freedom system
$S - N$	stress v. number of cycles to failure

2. INTRODUCTION

Fatigue life estimations, obtained by analysing processes in the time domain, have been considered in ESDU 06010. In this Item the estimation of the fatigue life of structures is based on the spectrum of the stress response. When a frequency response function (FRF) or transfer function is known, the fatigue damage and fatigue life can be estimated as a function of the excitation spectrum. Spectral methods are beneficial, because they require less storage of experimental data than time domain methods and, because they are computationally efficient. In comparison with the lengthy computations associated with the analysis of rainflow cycles, spectral methods can be used with simpler and quicker calculation procedures. In the case of a Gaussian random process, there is also the possibility of regenerating the time signal from the power spectral density (PSD). The PSD is a real function which lacks phase information and without phase it is not normally possible to carry out the conversion to the time domain by an inverse transform. However, random time signals statistically equivalent to the primary signal can be simulated by assuming a random phase angle uniformly distributed over the range $(0-2\pi)$ (Reference 1).

In the case where acoustic fatigue damage is caused by multi-modal vibration, the stress is most suitably represented by the PSD in the frequency domain. The PSD function is a characteristic way of displaying the information contained in a wide band random process (References 3, 6 and 15). Typical power spectral densities of strain are shown in Sketch 2.1 for a box made of GLARE, representative of the type used as a flap box (Reference 36, page 209), which has been tested in a progressive wave tube (PWT) at sound levels from 140 to 160 dB. The output shown in Sketch 2.1 is from a strain gauge attached to the middle of the lower surface of the box. In this case the units of PSD are $(\text{strain})^2/\text{Hz}$. The area under the curve is the mean-squared value of the strain (which is the same as variance for strain process with a zero mean). A PSD function for stress has the units of Pa^2/Hz . The PSD strain function may be measured during the acoustic loading of a panel or a box (References 36 and 41). The random signal from a strain gauge can be analysed in terms of its frequency components by equipment based on, for example, the Fast Fourier Transform (FFT) algorithm.



Sketch 2.1 Power spectral density functions of strain response of a box made from GLARE (adapted from Reference 36)

The application of spectral methods to the determination of the damage and fatigue life has been widely used, especially in the oil and gas industries, from the 1980s onwards. The prediction of the fatigue life of semi-submersible structures (offshore platforms) subjected to wave loading is vitally important. The PSD of stress (strain) on an offshore platform usually exhibits two predominant frequencies, one between 0.05 and 0.1 Hz, that is due to the wave motion (rigid body response) and depends upon the sea state, and the second, occurring at frequencies of about 0.3 Hz, which is typically the lowest resonant frequency of an offshore platform. Several methods have been devised to estimate the fatigue life from various PSD functions, especially from spectra with two peaks, analysed in the context of the fatigue life of offshore platforms. These methods have been assessed (References 28, 29 and 39) and in all cases it was concluded that the method proposed by Dirlik (Reference 17) gave for wide band processes the most accurate results when compared with rainflow counting.

Spectral methods have been applied to clamped-clamped beams subjected to random excitation (Reference 34). Wallace (Reference 18) has applied the method of Wirsching and Light (Reference 13) to the estimation of the fatigue life of honeycomb panels on the Space Shuttle, when subjected to noise from

the rocket engines at lift-off. Bishop *et al* have recommended the method of Dirlik for acoustic fatigue problems in general (Reference 31).

3. SIMULATION OF RANDOM SIGNALS FROM THE PSD FUNCTION

In general, $G(f)$, the one-sided power spectral density (PSD) function of a time signal does not provide sufficient information to regenerate a time signal, since it lacks phase values.

There are two representations of random Gaussian processes, as pointed out by Rice (Reference 1). The first one can be expressed as

$$S(t) = \sum_{n=1}^N (a_n \cos \omega_n t + b_n \sin \omega_n t), \quad (3.1)$$

where a_n and b_n are independent random variables which satisfy the normal distribution about zero with standard deviation $\sigma_n = \sqrt{G(f_n)\Delta f}$, and where $f_n = n\Delta f$.

The second representation for the random Gaussian signal is:

$$S(t) = \sum_{n=1}^N C_n \cos(\omega_n t - \phi_n), \quad (3.2)$$

where ϕ_n is a random phase angle, distributed uniformly over the range $(0-2\pi)$, and where

$$C_n = [2G(f_n)\Delta f]^{1/2}, \quad (3.3)$$

and $f_n = n\Delta f$.

The representation (3.2) is useful in computational signal simulation and is broadly applied, see, for example, works by Yang (Reference 9) and R.N. Miles (Reference 27). Using Equation (3.3), Equation (3.2) can be presented in the form

$$S(t) = \operatorname{Re} \left\{ \sqrt{2} \sum_{n=1}^N [G(f_n)\Delta f]^{1/2} e^{-i\phi_n} e^{i2\pi f_n t} \right\} \quad (3.4)$$

which implies the possibility of applying the inverse discrete Fourier transform (IDFT) to simulate the random signal. Use of the IDFT can significantly decrease the computation time for the signal simulation.

After generation of the time series the next step in the establishment of fatigue life is the conventional analysis of rainflow cycles of the time signal. It finally leads to the determination of the stress range probability density function for rainflow cycles and to fatigue life estimated from Equation (4.14) in ESDU 06009. The procedure is computationally lengthy and complicated. Hence, there is a requirement to introduce a method which is able to evaluate fatigue damage and fatigue life directly from the PSD function of the structure's response to random loadings.

4. SPECTRAL PARAMETERS

Spectral parameters play an essential role in the evaluation of the fatigue life of structures subjected to random loading. The concept of spectral parameters has already been introduced in ESDU 06009. The spectral parameters are expressed by the moments of the one-sided spectral density function, $G(f)$,

$$m_k = \int_0^{\infty} f^k G(f) df . \quad (4.1)$$

For a process $S(t)$ with zero mean value,

$$\overline{S^2} = \sigma_s^2 = m_0 = \int_0^{\infty} G(f) df , \quad (4.2)$$

where σ_s^2 denotes the variance of the process. Useful spectral parameters are those which have dimensions of frequency (Hz). They are defined as the characteristic frequencies of the process,

$$F_k = \left(\frac{m_k}{m_0} \right)^{1/k} . \quad (4.3)$$

As has been found by Rice (Reference 1) for Gaussian processes, see also Equation (5.12) in ESDU 06009,

$$F_2 = F_{zc} = \left(\frac{m_2}{m_0} \right)^{1/2} , \quad (4.4)$$

where F_{zc} is the number of zero crossings with positive slope per unit time. The characteristic frequency, F_2 , could also be presented for a stationary process (not necessarily Gaussian) as

$$F_2 = \frac{1}{2\pi} \frac{\sigma_{\dot{S}}}{\sigma_s} , \quad (4.5)$$

where $\sigma_{\dot{S}}$ is the standard deviation of the process formed from the differential of the stress, $\dot{S}(t)$.

4.1 Analogy between PSD and PDF

Vanmarcke (Reference 8) observed the analogy between probability density function (PDF) for a process $S(t)$ and the normalised power spectral density function (PSD), $g(f) = G(f)/m_0$. The analogy can be illustrated by means of Table 4.1.

TABLE 4.1 Analogy between PSD and PDF

Process $S(t)$	Frequency f (Hz)
$p(S)$ (PDF for random process $S(t)$)	$g(f) = G(f)/m_0$ (Normalised PSD)
$\int_0^{\infty} p(S) dS = 1$	$\int_0^{\infty} g(f) df = 1$
Mean value $S_{av} = \int_0^{\infty} S p(S) dS$	Characteristic frequency F_1 $F_1 = \int_0^{\infty} f g(f) df$
R.m.s. value $S_{rms} = \left[\int_0^{\infty} S^2 p(S) dS \right]^{1/2}$	Characteristic frequency F_2 $F_2 = \left[\int_0^{\infty} f^2 g(f) df \right]^{1/2}$
Variance $\sigma_s^2 = \int_0^{\infty} S^2 p(S) dS - \left[\int_0^{\infty} S p(S) dS \right]^2$	Variance $\sigma_f^2 = F_2^2 - F_1^2$
Dimensionless coefficient of variation $\frac{\sigma_s}{S_{av}}$	Dimensionless coefficient of variation $\frac{\sigma_f}{F_1} = \left(\frac{F_2^2 - F_1^2}{F_1^2} \right)^{1/2}$
Dimensionless coefficient of variation $\frac{\sigma_s}{S_{rms}}$	Dimensionless coefficient of variation $q = \frac{\sigma_f}{F_2} = \left(\frac{F_2^2 - F_1^2}{F_2^2} \right)^{1/2}$

Table 4.1 can help in the interpretation of spectral parameters. The characteristic frequencies of a random process, F_1 and F_2 , are the mean and r.m.s. frequencies averaged in the frequency domain and σ_f denotes the standard deviation, which is expressed in terms of the characteristic frequencies. The dimensionless coefficients of variation, σ_f/F_1 and σ_f/F_2 , are measures of the distribution of the power spectral density

function. Vanmarcke (Reference 8) has introduced the dimensionless coefficient, q , such that

$$q = \frac{\sigma_f}{F_2} = \left(1 - \frac{m_1}{m_0 m_2}\right)^{1/2}, \quad (4.6)$$

as an indicator of the degree of variability in the frequency content of the process and as one of the three principal parameters of the spectral density function, namely, q , m_0 and F_2 (F_{zc}). The coefficient q can be presented in terms of the characteristic frequencies, $F_1 = m_1/m_0$ and $F_2 = (m_2/m_0)^{1/2}$ as,

$$q = \frac{(F_2^2 - F_1^2)^{1/2}}{F_2} \quad \text{or} \quad q = \frac{(F_{zc}^2 - F_1^2)^{1/2}}{F_{zc}}. \quad (4.7)$$

The choice of the spectral parameters characterising the PSD of the process is, however, not unique. For example, a whole class of bandwidth parameters can be defined in terms of the moments of the PSD as

$$\gamma_k = \frac{m_k}{(m_0 m_{2k})^{1/2}}. \quad (4.8)$$

The particular bandwidth parameter for $k = 2$ is the irregularity factor, see definition in ESDU 06009,

$$\gamma = \gamma_2 = \frac{m_2}{(m_0 m_4)^{1/2}}, \quad (4.9)$$

which can be determined directly from the time signal. The irregularity factor, γ , tends to unity for a narrow band process and approaches zero for a wide band process. The factor, γ , as well as the spectral width parameter, $\varepsilon = (1 - \gamma^2)^{1/2}$, are the most often used parameters to characterise random processes, as they can be interpreted in terms of the time as well as the frequency domains. For random processes with zero means, the moments m_0 , m_2 and m_4 are the variances of the random processes $S(t)$, $\dot{S}(t)$ and $\ddot{S}(t)$ respectively. Since $m_0 = \sigma_s^2$, $m_2 = \sigma_{\dot{s}}^2$, and $m_4 = \sigma_{\ddot{s}}^2$, the irregularity factor can also be expressed as

$$\gamma = \frac{\sigma_{\dot{s}}^2}{\sigma_s \sigma_{\ddot{s}}}. \quad (4.10)$$

The properties of a stationary random process can be defined by a small number of spectral parameters which depend upon the moments of the one-sided PSD function, $G(f)$.

In the Vanmarcke approach (Reference 8), where m_0 , F_{zc} and q are chosen as principal spectral parameters, only three moments of the PSD, m_0 , m_1 and m_2 , are involved in determining the statistical properties of the structural response. More often the group of parameters m_0 , F_{zc} and γ are applied to characterise the random Gaussian process. In this case the spectral moments, m_0 , m_2 and m_4 , are used. It should be noted that the latter group of moments provides not only the mean value of the rate of positive slope zero crossings, F_{zc} , but also the mean value of the peak rate, F_p . The observation that the most

important statistical features of the response signal, as caused by random loading, can be determined by several spectral parameters leads to the concept of an equivalent narrow band process.

In the case of structures which can be represented by a SDOF system, the moment of the PSD of the response to force excitation can be found analytically. Consequently, expressions for the most important spectral moments, m_0 , m_2 and m_4 , are given in Appendix A.

5. SPECTRAL METHODS

To estimate fatigue damage and fatigue life several spectral methods have been developed (References 13-14, 16, 17, 19 and 21-29). Most are semi-empirical methods which are based on a comparison of the results obtained using the spectral models with results from the analysis of the rainflow cycles of the simulated process. The main methods applied in failure analysis are the Rayleigh approximation, the rainflow damage factor method, which is associated with the Rayleigh approximation, the peak approximation, the single moment method and, finally, semi-empirical methods in which the probability density function for the stress ranges of the rainflow cycles is postulated as a function of spectral parameters and weighting factors of the PDF are deduced from a comparison between the assumed PDF and the PDF derived from the histograms for the stress ranges of rainflow cycles of simulated processes.

Validation of the semi-empirical methods is mostly associated with a comparison of numerical results for: either

- (a) expected value of the stress range to the power b (b -th moment), or equivalent stress range,

$$S_{eq}^b = \int_0^{\infty} S_r^b p(S_r) dS_r, \quad (5.1)$$

where S_r denotes the stress range, $p(S_r)$ is the PDF of the stress range and S_{eq} is the equivalent stress range,

or

- (b) mean fatigue damage of the structure subjected to random loading,

$$D(n) = \frac{n}{K_r} S_{eq}^b, \quad (5.2)$$

which is achieved after a total number of stress cycles, n . The material constant is denoted by K_r and S_{eq} is the equivalent stress range.

The number of cycles is associated in spectral models with the rate of occurrence of mean (zero) crossings or the rate of occurrence of peaks in the process. Equation (5.2) is applied to stationary as well as non-stationary processes. Using Equation (5.1), Equation (5.2) can be presented in the form

$$D(n) = \frac{n}{K_r} \int_0^{\infty} S_r^b p(S_r) dS_r. \quad (5.3)$$

The spectral methods applied to the determination of fatigue damage and fatigue life, and considered in this Item, are based on:

- (1) linear damage rule (Palmgren-Miner, Reference 2),
- (2) stress life approach associated with high cycle fatigue and on the application of fatigue curves ($S-N$) to stresses within the elastic range,
- (3) assumption that the considered processes (narrow band as well as wide band) are stationary and Gaussian.

5.1 Rayleigh Approximation

For a stationary, Gaussian and narrow band process the fatigue damage, Equation (5.3), is expressed in terms of the gamma function, Γ , and the standard deviation of the stress process, σ_s , as

$$D_{Ray}(n) = \frac{n}{K_r} (2\sqrt{2}\sigma_s)^b \Gamma(1 + b/2), \quad (5.4)$$

since in this case the stress range PDF, $p(S_r)$, is

$$p(S_r) = p_{rfr}(S_r) = \frac{S_r}{4\sigma_s^2} e^{-\frac{S_r^2}{8\sigma_s^2}}. \quad (5.5)$$

The PDF for the stress amplitude is the Rayleigh function, in the form shown in Equation (5.23) of Section 5.3 in ESDU 06009. Consequently, the rate of damage accumulation for a stationary, narrow band and Gaussian process can be expressed in analytical form as a function of the mean (zero) crossing rate (with positive slope), F_{zc} ,

$$\dot{D}_{Ray} = \frac{F_{zc}}{K_r} (2\sqrt{2}\sigma_s)^b \Gamma(1 + b/2), \quad (5.6)$$

or for a zero mean process

$$\dot{D}_{Ray} = \frac{2^{3b/2}}{K_r} m_0^{(b-1)/2} m_2^{1/2} \Gamma(1 + b/2). \quad (5.7)$$

The estimation of fatigue damage and fatigue life for a stationary, narrow band, Gaussian process are essential elements of the Rayleigh approximation. The name of the method originates from the fact that the PDF of stress amplitude is in this case a Rayleigh function. The Rayleigh approximation is the simplest and most widely used method, applied in different models of stochastic fatigue. (Note that the definition of the gamma function is given in Appendix A of ESDU 06009.)

5.2 Rainflow Damage Factor Method

Equations (5.4) to (5.7) are strictly correct for narrow band stochastic processes, but the Rayleigh

approximation is also applied to wide band processes. For wide band applications the concept of an equivalent narrow band approximation has to be introduced, see Wirsching and Light (Reference 13). It is assumed that the fatigue damage to a structure caused by a wide band random process is the same as the damage resulting from a hypothetical (equivalent) narrow band process, provided that parameters such as $\sqrt{S^2}$, σ_s and the expected rate of zero crossing, F_{zc} , have the same values for both processes. It should be noted that in the Rayleigh approximation the mean rate of occurrence of cycles in a process is assumed to be equal to the rate of zero crossings with positive slope. The results of the equivalent narrow band approximation can differ significantly from those obtained by rainflow counting, especially for processes with bimodal spectra. On the other hand the model can provide reasonable agreement with rainflow counting procedures for small values of b (References 13 and 35). See ESDU 06010 for a description of rainflow cycle counting.

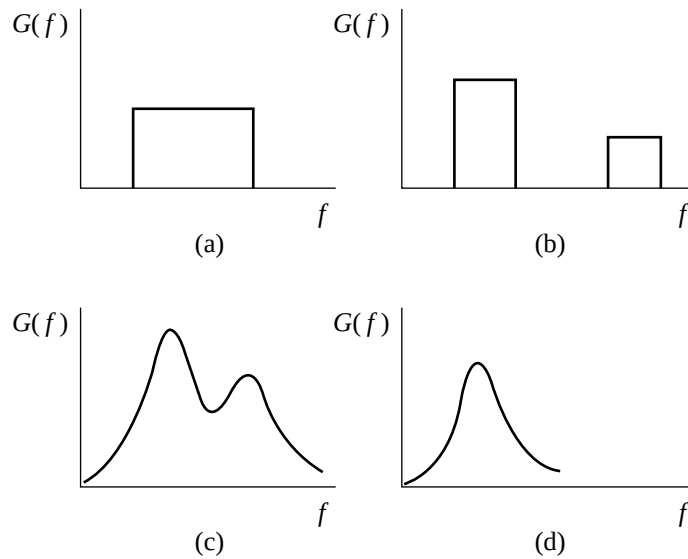
To improve the Rayleigh model of the equivalent narrow band approximation, Wirsching and Light (Reference 13) have introduced a damage correction factor, also called the rainflow damage factor (Reference 19),

$$\lambda = \frac{D}{D_{Ray}}, \quad (5.8)$$

where D_{Ray} is the Rayleigh approximation of fatigue damage caused by the response of the structure to random loading. The damage correction factor is a correction applied to the estimate of fatigue damage, D_{Ray} , (or $[S_{eq}^b]_{Ray}$, see Reference 24) caused by an equivalent narrow band process. This type of semi-empirical approach is applied to wide band random processes. In general, the damage correction factor is assumed to be a function of the spectral parameters. The form of its relationship with the spectral parameters is established usually by combining Equation (5.8), applied to rainflow counting results for simulated processes, with the estimation of spectral parameters derived on the basis of PSD moments.

The essential elements in the Wirsching and Light model is an assumption that a single spectral parameter, either spectral width parameter, ϵ , or irregularity factor, γ , defines the characteristics of the PSD of the process, in the context of the damage correction factor. In this case the damage correction factor, λ , is assumed to be a function of the spectral width parameter, ϵ , and material constant, b , (Reference 13), i.e., $\lambda = \lambda(\epsilon, b)$.

Four different types of spectra are analysed by Wirsching and Light (Reference 13), as shown in Sketch 5.1 (a to d). Sketches 5.1a and 5.1b are described (Reference 13) as unimodal and bimodal spectra, respectively. Sketch 5.1c is typical spectrum of the stress at the welded joints of offshore structures. Similar spectra, especially of the type shown in Sketch 5.1c, have been adopted in other models (Reference 29) based on the rainflow damage factor. Sketch 5.1d is another form of unimodal spectrum.



Sketch 5.1 Spectra used for verification of models

In the Wirsching and Light model (Reference 13) the damage correction factor, λ , is a function of a single spectral parameter, $\varepsilon = \sqrt{1 - \gamma^2}$. Other works (Reference 26 and 29) based on the concept of the damage correction factor suggest a strong correlation between λ and the other bandwidth parameters, such as $\gamma_k = m_k / (m_0 m_{2k})^{1/2}$, where k is empirically related to b . These models require a knowledge of four spectral moments, m_0, m_2, m_4, m_k and m_{2k} . In the model adopted in Reference 19, the irregularity factor and the moments, m_0, m_2, m_k and m_{k+2} , where $k = 2/b$, are needed to estimate the damage correction factor. In this case the damage correction factor is expressed by:

$$\lambda = \left(\frac{m_2 m_k}{m_0 m_{2+k}} \right)^{b/2} \gamma. \quad (5.9)$$

The models of Ortiz and Chen (Reference 19) and Larsen and Lutes (Reference 26) represent a tendency to increase the number of moments in the damage correction factor in order to achieve a greater accuracy in the prediction of the fatigue damage. However, following Wirsching and Light (Reference 13), as well as Vanmarcke (Reference 8), there is also a trend to minimise the number of moments that are required for the correction to the Rayleigh approximation. It has been found empirically (Reference 14) that, for $b = 3$, λ is strongly correlated with the single spectral parameter, $\gamma_{0.75} = m_{0.75} / (m_0 m_{1.5})^{1/2}$. This observation and the additional extensive simulation and rainflow analysis of different spectra (particularly bimodal PSDs) led Lutes and Sarkani (Reference 35) to introduce a damage accumulation model in terms of a single moment. The expected rate of damage accumulation is postulated as:

$$\dot{D} = \frac{2^{3b/2} m_{2/b}^{b/2}}{K_r} \Gamma(1 + b/2). \quad (5.10)$$

5.3 Peak Approximation Method

The Rayleigh approximation method is the simplest and the most frequently used technique in the estimation of the fatigue life of structures for wide band processes. As was already discussed, the method is based on the rate of mean (zero) crossings with positive slope, F_{zc} , and on the concept of an equivalent narrow band process with the same variance, σ_s , (r.m.s. value) and F_{zc} , as in the case of a wide band Gaussian process. Another spectral method, which is an attempt to improve on the Rayleigh approximation, is the peak-approximation method (Reference 35). In this method the rate of occurrences of peaks, F_p , of the Gaussian wide band process is associated with the rate of rainflow cycles. The probability density function of stress amplitude of rainflow cycles is approximated by the PDF for peaks, with exclusion of negative peaks in some models (References 23 and 35).

The probability density function for peaks and the mean peak rate for a stationary, random, and Gaussian wide band process was considered by Rice (Reference 1) and later by Bendat (Reference 7). The probability that a peak will occur during a zero mean process, $S(t)$, between values S and $S + dS$, is $\text{Prob}[S < S_p \leq S + dS] = p_p(S)dS$, where the peak PDF, $p_p(S)$, is expressed by the formula:

$$p_p(S) = \frac{\epsilon}{\sqrt{2\pi}\sigma_s} e^{-\frac{S^2}{2\sigma_s^2\epsilon^2}} + \gamma \frac{S}{\sigma_s^2} \left[1 - \frac{1}{\sqrt{2\pi}} \int_{\frac{S\gamma}{\sigma_s\epsilon}}^{\infty} e^{-y^2/2} dy \right] e^{-\frac{S^2}{2\sigma_s^2}} \quad (5.11)$$

or, after some transformation, in the more convenient form (see also Equation (5.18), ESDU 06009):

$$p_p(S) = \frac{\epsilon}{\sqrt{2\pi}\sigma_s} e^{-\frac{S^2}{2\sigma_s^2\epsilon^2}} \left(\gamma \frac{S}{2\sigma_s^2} \left[1 + \text{erf}\left(\frac{S\gamma}{\sqrt{2}\sigma_s\epsilon}\right) \right] e^{-\frac{S^2}{2\sigma_s^2}} \right), \quad (5.12)$$

where the error function, $\text{erf}(x)$ is defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \quad (5.13)$$

The peak probability density function depends only upon the spectral parameters, such as the irregularity factor, γ , and the moment, m_0 , since the bandwidth parameter $\epsilon = \sqrt{1 - \gamma^2}$ and for a zero mean process, $\sigma_s^2 = m_0$. For a narrow band process, $\gamma = 1$, $\epsilon = 0$ and

$$p_p(S) = \frac{S}{\sigma_s^2} e^{-\frac{S^2}{2\sigma_s^2}}, \quad (5.14)$$

i.e., the peak probability density function is a Rayleigh function. In the case of a wide band process

$\gamma = 0$, $\varepsilon = 1$ and the peak PDF is expressed by the Gaussian function

$$p_p(S) = \frac{1}{\sqrt{2\pi}\sigma_s} e^{-\frac{S^2}{2\sigma_s^2}}. \quad (5.15)$$

In the peak approximation method it is assumed that the stress range of a cycle is twice the value of the peak stress level in the process. In models (Reference 35) which consider only positive peaks during the process the negative peaks, *i.e.*, the fraction $(1-\gamma)/2$ of all peaks, are excluded from the damage analysis. Consequently, mean fatigue time in different models of peak approximation is expressed as:

$$T_{fl} = \frac{2K_r}{(1+\gamma)F_p S_{eq}^b} \quad (5.16)$$

or

$$T_{fl} = \frac{2K_r\gamma}{(1+\gamma)F_{zc} S_{eq}^b}, \quad (5.17)$$

where

$$S_{eq}^b = \int_0^\infty S_r^b p_{pr}^+(S_r) dS_r \quad (5.18)$$

and

$$p_p^+(S_p) = p_p^+(S_r/2) = 2p_{pr}^+(S_r).$$

It should be noted that in general the equivalent stress range is defined as:

$$S_{eq} = \left[\int_0^\infty S_r^b p(S_r) dS_r \right]^{1/b}, \quad (5.19)$$

where $p(S_r)$ denotes the PDF of the ranges of stress cycles which in different models approximates the $p_{pr}(S_r)$, the PDF of the stress ranges of the rainflow cycles.

The peak approximation approach has been used in several models (Reference 16, 21, 23 and 29). In the semi-empirical method of Kam and Dover (References 21 and 23), the peak approximation is combined with the concept of equivalent stress range. Their equivalent stress range is postulated as the weighted function of the equivalent stress ranges for narrow band and wide band processes:

$$S_{eq} = 2\sqrt{2}\sigma_s \left[\frac{\varepsilon}{2\sqrt{\pi}} \Gamma\left(\frac{b+1}{2}\right) + 0.75\gamma \Gamma\left(\frac{b}{2} + 1\right) \right]^{1/b}. \quad (5.20)$$

Abdo and Rackwitz (Reference 23) have obtained analytically the expected value of the equivalent stress