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Dynamic characteristics of cylindrical helical springs. Part 1: deformation, stresses and stability

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The Mechanisms Committee, which first met in 1975, is continuing the work that previously was the responsibility of the former Gearing and Mechanisms Committee, which first met in November 1964. Like the earlier Committee it has the cooperation of many engineers and scientists in industry, research establishments and universities from whom much assistance and information is being received.

DYNAMIC CHARACTERISTICS OF CYLINDRICAL HELICAL SPRINGS. PART I: DEFORMATION, STRESSES AND STABILITY

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DYNAMIC CHARACTERISTICS OF CYLINDRICAL HELICAL SPRINGS. PART I: DEFORMATION, STRESSES AND STABILITY

1. NOTATION

Units used in this Item are predominantly SI but any coherent system of units may be used. Dimensions are measured to the centre line of circular section spring wire.

A_w	cross-section area of spring wire ($\equiv \pi d_w^2/4$)	m^2
c	spring index ($= D_s / d_w$)	–
D_s	mean diameter of spring coil	m
d_w	diameter of spring wire	m
E_w	Young's modulus of spring wire material	$N\ m^{-2}$
$E_w I_w$	bending stiffness of spring	$N\ m^2$
e	eccentricity associated with F_z (see Equation (5.3))	m
F_x, F_z	external force on spring in x- and z-directions, respectively	N
G_w	shear modulus of spring wire material	$N\ m^{-2}$
$G_w J_w$	torsional stiffness of spring	$N\ m^2$
H	effective height of unloaded spring	m
H_0	working height of spring	m
h	height of a single coil of an unloaded spring	m
I_w	second moment of area of round cross-section of spring wire	m^4
J_w	polar second moment of area of round cross-section of spring wire	m^4
K_b, K_t	section moduli for bending and torsion, respectively ($= \pi d_w^3/32$ and $\pi d_w^3/16$, respectively)	m^3
K'_b, K'_t, K'	factors associated with stresses in tension and compression springs (see Equations (4.4))	–
K''_b, K''_t, K''_n	factors associated with stresses in torsion springs (also $K'' = K''_t$) (see Equations (4.9))	–
k	spring stiffness	$N\ m^{-1}$

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k_b, k_t, k_{tb}	coefficients associated with maximum stress (see Equations (7.8))	–
L_0	free length of spring	m
l_w	effective, or working, length of spring wire	m
M	external moment on spring about z-axis	N m
N	normal force along spring wire	N
n_s	number of working coils	–
n_u	number of turns of coil from top of spring to Section C (see Sketch 7.3)	–
Q	shear force across spring wire	N
r	general radius	m
r_H	radius of tension spring in vicinity of hook or eye (see Sketch 5.3)	m
r_1, r_2	radii associated with tension springs in vicinity of hook or eye (see Sketch 5.3)	m
s	$= d_w / (2r_1)$	–
T	torsion in spring coil	N m
$\bar{t}, \bar{n}, \bar{b}$	unit vector triad in which $\bar{t}$ is a tangent to helix, $\bar{n}$ is perpendicular to $\bar{t}$ and parallel to y-axis and $\bar{b}$ is binormal, perpendicular to both $\bar{t}$ and $\bar{n}$ (see Sketch 4.1)	–
u	distance in z-direction of M and Q (see Sketch 7.3)	m
x, y, z	axis coordinate system of spring (see Sketch 3.1)	–
α	spring helix angle	radian
δ	spring deflection	m
η	parameter for end condition of spring (see Equations (6.1) to (6.3) and Table 6.1)	–
θ	relative rotation of spring ends ($= \Delta\phi$)	radian
κ	curvature of spring coil	m^{-1}
μ	Poisson's ratio	–
ξ	factor defined by Equation (5.8)	–
ρ_w	density of wire material	kg m^{-3}
σ	tensile stress	N m^{-2}
σ_b, σ_t	tensile stress in directions $\bar{b}$ and $\bar{t}$, respectively	N m^{-2}

τ	shear stress	N m^{-2}
τ_B	shear stress at Point B (see Equation (5.12))	N m^{-2}
τ'_{max}	modified maximum shear stress (see Equation (5.3))	N m^{-2}
ϕ	polar angle of spring	radian
ψ	angle between Moment M and tangential component M_T	radian

Subscripts

b	bending
bb, bt, tt, tb	indicates directions associated bb , bt , tt and tb , respectively
$crit$	critical value
equ	equivalent
F	denotes association with external force on spring in z-direction
f	friction
min, max	minimum and maximum, respectively
N	normal
n	denotes association with number of coils
P	permissible
R, res	resultant
red	reduced
s	spring
T	tangential
t	torque
w	spring wire
0	unloaded condition
1	relates to unit magnitude

Operators

Δ	denotes increment
$\wedge$	denotes a cross-product of vectors

a bar above a bold quantity denotes a vector

2. INTRODUCTION, ASSUMPTIONS AND LOADING

This Data Item is the first in a series that deal with the dynamic characteristics of cylindrical helical springs, comprising both tension/compression and torsion springs. The assumptions and limitations that apply to the calculation procedure are defined together with the prescribed loading conditions assumed to apply to the spring. The Item provides derivation of the deformation, stresses and transverse loading on the spring and the form design of the spring ends which will affect the loading characteristics. The elastic stability of compression and torsion springs are discussed and formulae given for ensuring stability.

In engineering practice cylindrical helical springs are used in three different ways, as

compression springs,
tension springs and
and torsion springs.

This Item contains design data for all three kinds of helical spring as listed above. Provided that the following assumptions and limitations are observed in the design process, the given methods of calculation yield very close approximations to the actual values.

2.1 Assumptions and Limitations

The following assumptions and limitations apply to the calculation procedures in this Item:

- (a) the cross-section of the wire is constant,
 - (b) the helix angle, α , remains small (normally $\alpha < 8^\circ$ to 10°),
 - (c) the spring material is not stressed beyond its elastic limit,
 - (d) the stress condition of all cross-sections of the working coils is the same (a consequence of axial symmetry),
 - (e) the deflection, δ , of the spring under load remains small (normally $\delta < D_s / 4$, in which D_s is the mean diameter of the coil)
- and (f) the resultant force, F_z , loading the spring acts along the spring axis, *i.e.* the geometrical centre-line, and also the loading moment, M , acts in a plane perpendicular to the axis.

2.2 Loading

In general the loads on the spring consist of an axial force, F_z , with a line of action along the z -axis, and a moment, M , acting in a plane perpendicular to the z -axis (see Sketches 2.1 and 3.1).

The force F_z is positive if it causes tension in the spring and negative if it causes compression. Similarly, the moment M is positive if it increases the curvature of the spring and negative if it decreases the curvature.

The force vector, $\vec{F}_z$, can be resolved into two mutually perpendicular components, a normal force, N , along the spring wire and a shear force, Q , across the spring wire as shown in the exploded view of Sketch 2.1 (b), and are given by

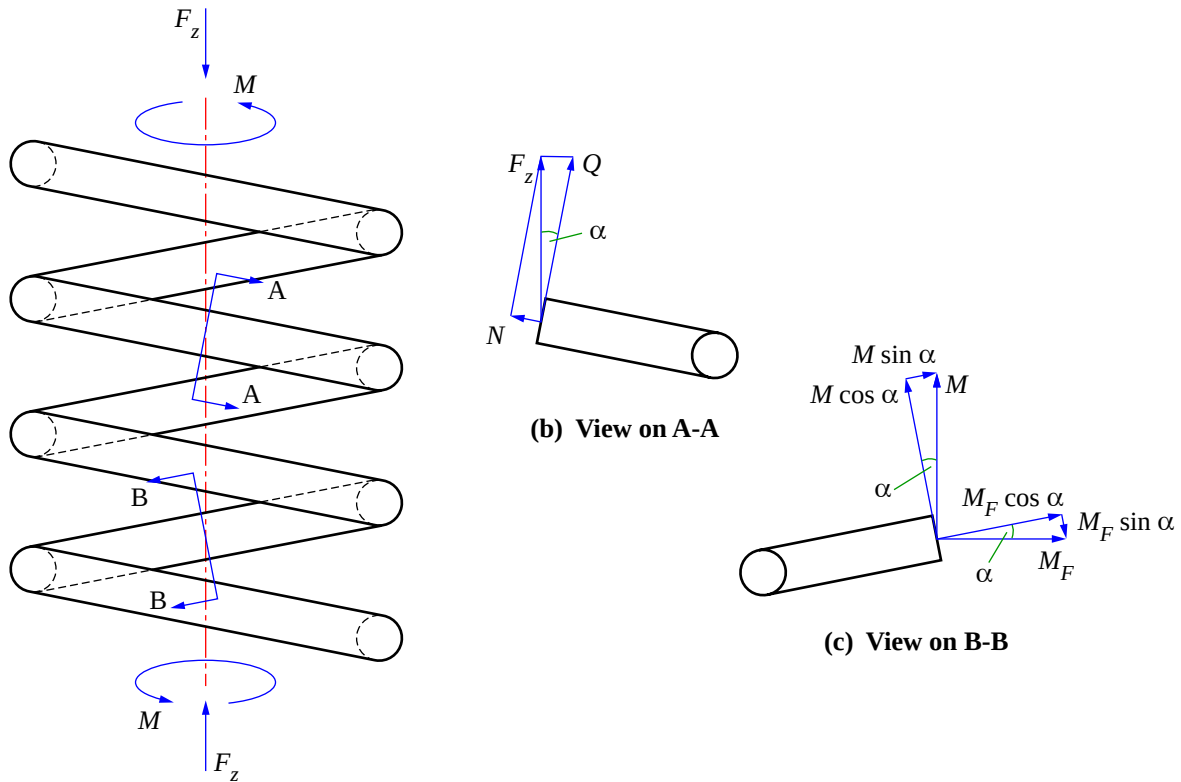
$$N = F_z \sin \alpha \quad \text{and} \quad Q = F_z \cos \alpha.$$

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The resultant moment vector, $\bar{\mathbf{M}}_R$, acting on the cross-section of the wire, is the sum of the external moment, $\bar{\mathbf{M}}$, and the moment due to the force, $\bar{\mathbf{F}}_z$, i.e. $\bar{\mathbf{M}}_F$, which is written as

$$\bar{\mathbf{M}}_R = \bar{\mathbf{M}} + \bar{\mathbf{M}}_F, \quad \text{where} \quad \bar{\mathbf{M}}_F = \bar{\mathbf{r}} \wedge \bar{\mathbf{F}}_z,$$

in which $\wedge$ indicates a cross-product. In scalar form, $M_F = F_z D_s / 2$.



(a) External force and moment on spring

Sketch 2.1 Resolution of forces and moments acting on spring

The resultant moment can be resolved into two mutually perpendicular components, a bending moment, M_b , and a torque, M_t , resolved from Sketch 2.1 (c), and are

$$M_b = M \cos \alpha - \frac{F_z D_s}{2} \sin \alpha \quad (2.1)$$

$$\text{and} \quad M_t = M \sin \alpha + \frac{F_z D_s}{2} \cos \alpha. \quad (2.2)$$

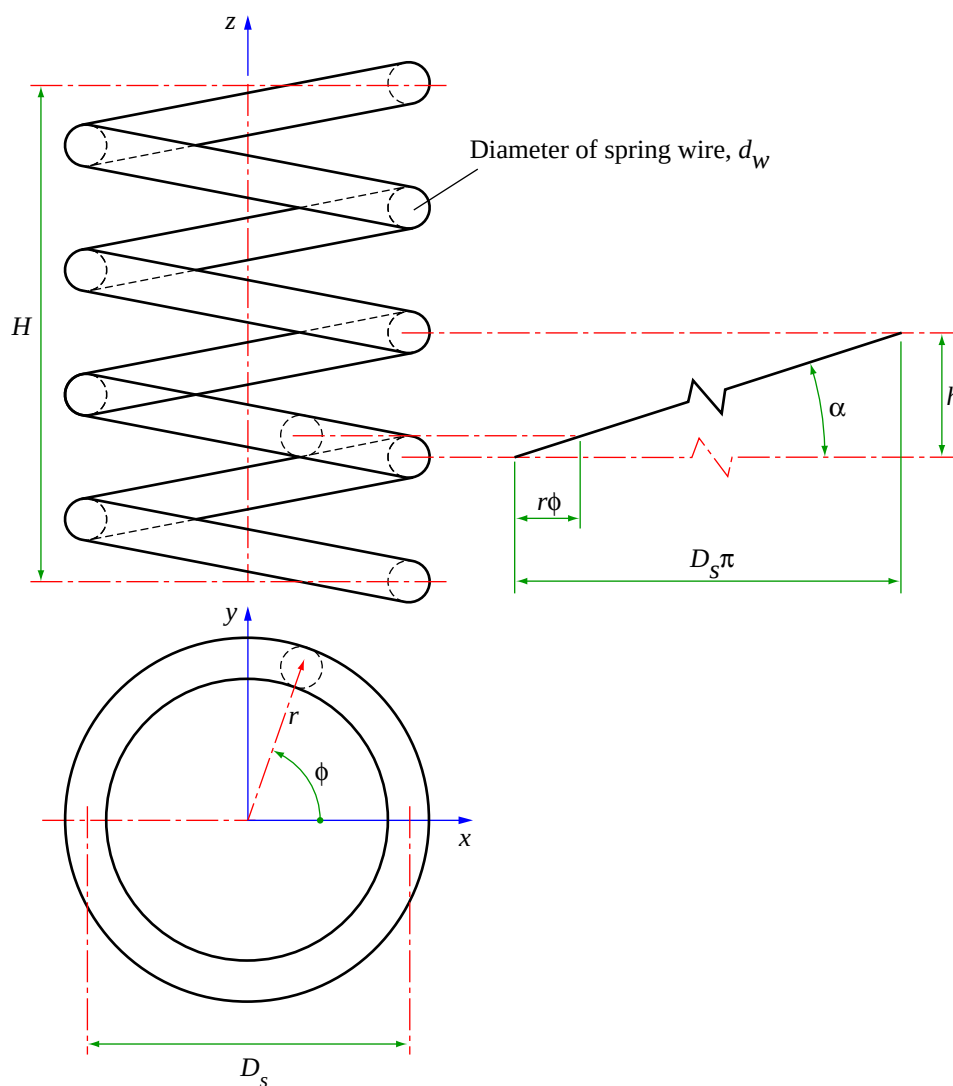
In stress calculations the effects of the normal and shear forces are neglected, however the shear force, Q , has a significant effect on the elastic stability of compression springs.

3. ELASTIC DEFORMATION OF HELICAL SPRINGS

The geometry of helical springs is defined by the following three independent parameters:

- D_s the mean diameter of the coil,
- α the helix angle (which can be right-handed or left-handed)
- and l_w the effective, or working, length of the spring wire.

These are the fundamental geometrical parameters of the spring in terms of which all the other parameters used in the calculations can be expressed. These parameters are shown in Sketch 3.1 in the usual cartesian coordinate system xyz .



Sketch 3.1 Spring axis system and geometrical parameters

The coordinates, x , y and z , of any point on the helix of the spring is given by

$$x = \frac{D_s}{2} \cos \phi, \quad y = \frac{D_s}{2} \sin \phi, \quad z = \frac{D_s}{2} \phi \tan \alpha, \quad (3.1)$$

where the polar angle, ϕ , is measured anti-clockwise from the positive direction of the x -axis.

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The maximum value of the polar angle is

$$\phi_{max} = \phi_n = 2 \pi n_s, \quad (3.2)$$

in which n_s is the number of working coils of the spring. Since

$$\pi D_s n_s = \frac{1}{2} D_s \phi_n = l_w \cos \alpha, \quad (3.3)$$

the maximum value of the polar angle can also be expressed from Equation (3.3) as

$$\phi_{max} = \phi_n = \frac{2 l_w \cos \alpha}{D_s} \quad (3.4)$$

and using Equations (3.2) and (3.4) the number of effective turns of the coil is

$$n_s = \frac{l_w \cos \alpha}{\pi D_s}. \quad (3.5)$$

The height, h , of a single coil of the helix is

$$h = \pi D_s \tan \alpha \quad (3.6)$$

and the effective height of the spring, H , is

$$H = h n_s \quad (\text{or, } H = l_w \sin \alpha). \quad (3.7)$$

As a measure of the curvature of the spring wire, the spring index, c , is introduced as

$$c = \frac{D_s}{d_w},$$

in which d_w is the diameter of the wire. Normally, the value of c is chosen in the range, $4 \leq c \leq 12$.

3.1 Small Deformations

Under the influence of the force, F_z , and moment, M , acting at the ends of the spring, the dimensions of the spring will alter but the centre-line of the wire remains helical in shape. The change of the three parameters can be written in the form

$$\left. \begin{aligned} D_s &= D_0 + \Delta D_s \\ \alpha &= \alpha_0 + \Delta \alpha \\ l_w &= l_0 + \Delta l_w \end{aligned} \right\} \quad (3.8)$$

The most important of these three parameters is the helix angle, α , which has far the largest change, $\Delta \alpha$. The change in the mean diameter of the coil, ΔD_s , is much smaller but still significant, sometimes even in design. In contrast, the change in length of the wire, Δl_w , is extremely small, consequently it will be neglected in the following calculations and the substitution of $l_w = l_0$ will be made.

The quantities $\Delta\alpha$ and ΔD_s are functions of not only the spring dimensions but also the elastic properties of the spring material. In order to find the relationship between these quantities the formulae for the curvature, κ , and the torsion, T , of a helix are required and found from geometrical considerations as

$$\kappa = \frac{D_s / 2}{(D_s / 2)^2 + (D_s / 2)^2 \tan^2 \alpha} = \frac{2}{D_s(1 + \tan^2 \alpha)} = \frac{2 \cos^2 \alpha}{D_s} \quad (3.9)$$

and

$$T = \frac{D_s / 2 \tan \alpha}{(D_s / 2)^2 + (D_s / 2)^2 \tan^2 \alpha} = \frac{2}{D_s} \frac{\cos^2 \alpha \sin \alpha}{\cos \alpha} = \frac{\sin 2\alpha}{D_s}, \quad (3.10)$$

from which the following incremental quantities are

$$\Delta\kappa = \frac{4 \cos \alpha \Delta(\cos \alpha)}{D_s} - \frac{2 \cos^2 \alpha \Delta D_s}{D_s^2} \quad (3.11)$$

and

$$\Delta T = \frac{2 \cos \alpha \Delta(\sin \alpha)}{D_s} + \frac{2 \sin \alpha \Delta(\cos \alpha)}{D_s} - \frac{2 \sin \alpha \cos \alpha \Delta D_s}{D_s^2}, \quad (3.12)$$

where $\Delta(\sin \alpha) = \sin(\alpha + \Delta\alpha) - \sin \alpha$ and $\Delta(\cos \alpha) = \cos(\alpha + \Delta\alpha) - \cos \alpha$.

A third equation, obtained from the increment of the following identity,

$$\sin^2 \alpha + \cos^2 \alpha = 1,$$

is given by $\Delta(\sin \alpha) = -\frac{\cos \alpha}{\sin \alpha} \Delta(\cos \alpha).$ (3.13)

Since Equations (3.11) to (3.13) are linear in the increments, $\Delta(\sin \alpha)$, $\Delta(\cos \alpha)$ and ΔD_s , their solution can easily be found in terms of $\Delta\kappa$ and ΔT , i.e. as

$$\left. \begin{aligned} \Delta D_s &= -\frac{D_s^2 \cos 2\alpha}{2 \cos^2 \alpha} \Delta\kappa - \frac{D_s^2 \sin \alpha}{\cos \alpha} \Delta T, \\ \Delta(\sin \alpha) &= -\frac{D_s \sin \alpha}{2} \Delta\kappa + \frac{D_s \cos \alpha}{2} \Delta T \\ \text{and} \quad \Delta(\cos \alpha) &= \frac{D_s \sin^2 \alpha}{2 \cos \alpha} \Delta\kappa - \frac{D_s \sin \alpha}{2} \Delta T. \end{aligned} \right\} \quad (3.14)$$

The changes in curvature and torsion are functions of the bending moment and torque, respectively, and are given by

$$\Delta\kappa = \frac{1}{\rho_w} - \frac{1}{\rho_0} = \frac{M_b}{E_w I_w} \quad (3.15)$$

and

$$\Delta T = T - T_0 = \frac{M_t}{G_w J_w},$$

in which $E_w I_w$ and $G_w J_w$ are the bending and torsional stiffnesses, respectively.

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The second moment of area, I_w , and the polar second moment of area, J_w , for round wire, are given by

$$I_w = \frac{\pi d_w^4}{64}, \quad J_w = \frac{\pi d_w^4}{32} \quad (3.16)$$

and for any homogenous and isotropic body the relationship between Young's modulus, E_w , and shear modulus, G_w , is given by

$$E_w = 2(1 + \mu) G_w,$$

in which μ is Poisson's ratio and d_w is the diameter of the spring wire.

The elastic deformations considered so far are small in comparison with the spring dimensions and so in the following calculations the parameters of the unloaded spring will be used, i.e. $D_s \approx D_0$ and $\alpha = \alpha_0$.

With this approximation the values of ΔD_s , $\Delta(\sin \alpha)$ and $\Delta(\cos \alpha)$ can now be readily determined. With the insertion of Equations (3.15) into Equations (3.14) the following relationships are obtained

$$\left. \begin{aligned} \Delta D_s &= -\frac{D_0^2 \cos(2\alpha_0)}{2 \cos^2 \alpha_0} \frac{M_b}{E_w I_w} - \frac{D_0^2 \sin \alpha_0}{\cos \alpha_0} \frac{M_t}{G_w J_w}, \\ \Delta(\sin \alpha) &= -\frac{D_0 \sin \alpha_0}{2} \frac{M_b}{E_w I_w} + \frac{D_0 \cos \alpha_0}{2} \frac{M_t}{G_w J_w}, \\ \Delta(\cos \alpha) &= \frac{D_0 \sin^2 \alpha_0}{2 \cos \alpha_0} \frac{M_b}{E_w I_w} - \frac{D_0 \sin \alpha_0}{2} \frac{M_t}{G_w J_w}. \end{aligned} \right\} \quad (3.17)$$

After inserting the values of M_b and M_t , from Equations (2.1) and (2.2), Equations (3.17) take the form

$$\left. \begin{aligned} \Delta D_s &= -\frac{D_0^2 \cos(2\alpha_0)}{2 \cos^2 \alpha_0} \left(\frac{M \cos \alpha_0 - \frac{1}{2} F_z D_0 \sin \alpha_0}{E_w I_w} \right) \\ &\quad - \frac{D_0^2 \sin(2\alpha_0)}{\cos \alpha_0} \left(\frac{M \sin \alpha_0 + \frac{1}{2} F_z D_0 \cos \alpha_0}{G_w J_w} \right), \\ \Delta(\sin \alpha) &= -\frac{D_0 \sin \alpha_0}{2} \left(\frac{M \cos \alpha_0 - \frac{1}{2} F_z D_0 \sin \alpha_0}{E_w I_w} \right) \\ &\quad + \frac{D_0 \cos \alpha_0}{2} \left(\frac{M \sin \alpha_0 + \frac{1}{2} F_z D_0 \cos \alpha_0}{G_w J_w} \right), \\ \Delta(\cos \alpha) &= \frac{D_0 \sin^2 \alpha_0}{2 \cos \alpha_0} \left(\frac{M \cos \alpha_0 - \frac{1}{2} F_z D_0 \sin \alpha_0}{E_w I_w} \right) \\ &\quad - \frac{D_0 \cos \alpha_0}{2} \left(\frac{M \sin \alpha_0 + \frac{1}{2} F_z D_0 \cos \alpha_0}{G_w J_w} \right). \end{aligned} \right\} \quad (3.18)$$

A re-arrangement of Equation (3.18) and inserting Equation (3.16) finally yields

$$\left. \begin{aligned} \Delta D_s &= -\frac{32}{\pi} \frac{M D_0^2}{E_w d_w^4} \left(\frac{1 + 2 \mu \sin^2 \alpha}{\cos \alpha_0} \right) - \frac{16}{\pi} \frac{F_z D_0^3 \sin \alpha_0}{E_w d_w^4} \left(\frac{1}{\cos^2 \alpha_0} + 2 \mu \right), \\ \Delta(\sin \alpha) &= \frac{32}{\pi} \frac{M D_0}{E_w d_w^4} (\mu \sin \alpha_0 \cos \alpha_0) + \frac{16}{\pi} \frac{F_z D_0^2}{E_w d_w^4} (1 + \mu \cos^2 \alpha_0), \\ \Delta(\cos \alpha) &= -\frac{32}{\pi} \frac{M D_0}{E_w d_w^4} (\mu \sin^2 \alpha_0) - \frac{16}{\pi} \frac{F_z D_0^2}{E_w d_w^4} \frac{\sin \alpha_0}{\cos \alpha_0} (1 + \mu \cos^2 \alpha_0). \end{aligned} \right\} \quad (3.19)$$

Since the height of the working coils, from Equation (3.7), is $H_0 = l_0 \sin \alpha_0$, the spring deflection is given by

$$\delta = \Delta H = l_0 \Delta(\sin \alpha),$$

which with the use of Equation (3.3), i.e. $l_0 = \pi D_0 n_0 / \cos \alpha_0$, gives

$$\delta = \frac{\pi D_0 n_0}{\cos \alpha_0} \Delta(\sin \alpha). \quad (3.20)$$

With the value of $\Delta(\sin \alpha)$ obtained from Equations (3.19) and inserted into Equation (3.20), δ becomes

$$\delta = \frac{\pi D_0 n_0}{\cos \alpha_0} \left(\frac{32}{\pi} \frac{M D_0}{E_w d_w^4} (\mu \sin \alpha_0 \cos \alpha_0) + \frac{16}{\pi} \frac{F_z D_0^2}{E_w d_w^4} (1 + \mu \cos^2 \alpha_0) \right),$$

or finally

$$\delta = \frac{32 M D_0^2 n_0}{E_w d_w^4} (\mu \sin \alpha_0) + \frac{16 F_z D_0^3 n_0}{E_w d_w^4} \left(\frac{1 + \mu \cos^2 \alpha_0}{\cos \alpha_0} \right). \quad (3.21)$$

The relative rotation of the spring ends is given by the increment of Equation (3.4) as

$$\theta = \Delta \phi = 2 l_0 \left(\frac{\Delta(\cos \alpha)}{D_0} - \frac{\cos \alpha_0}{D_0^2} \Delta D_s \right). \quad (3.22)$$

Inserting the values of ΔD_s and $\Delta(\cos \alpha)$ from Equations (3.19) gives

$$\begin{aligned} \theta = \Delta \phi &= \frac{2 \pi D_0 n_0}{\cos \alpha_0} \left(-\frac{1}{D_0} \left(\frac{32}{\pi} \frac{M D_0}{E_w d_w^4} \mu \sin^2 \alpha_0 \right) \right. \\ &\quad - \frac{1}{D_0} \frac{16}{\pi} \frac{F_z D_0^2}{E_w d_w^4} \frac{\sin \alpha_0}{\cos \alpha_0} (1 + \mu \cos^2 \alpha_0) \\ &\quad \left. + \frac{\cos \alpha_0}{D_0^2} \left(\frac{32}{\pi} \frac{M D_0^2}{E_w d_w^4} \frac{(1 + 2 \mu \sin^2 \alpha_0)}{\cos \alpha_0} + \frac{16}{\pi} \frac{F_z D_0^3 \sin \alpha_0}{E_w d_w^4} \left(\frac{1}{\cos^2 \alpha_0} + 2 \mu \right) \right) \right). \end{aligned}$$

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