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Dynamic characteristics of cylindrical helical springs. Part 2: vibration

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DYNAMIC CHARACTERISTICS OF CYLINDRICAL HELICAL SPRINGS. PART 2: VIBRATION

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DYNAMIC CHARACTERISTICS OF CYLINDRICAL HELICAL SPRINGS. PART 2: VIBRATION

1. NOTATION

Units used in this Item are predominantly SI but any coherent system of units may be used. Dimensions are measured to the centre line of circular section spring wire.

A_e, A_w	cross-section area of equivalent rod and spring wire ($\equiv \pi d_w^2/4$), respectively	m^2
A	amplitude of free vibration of spring-mass system (see Equation (4.7))	m
A, B	coefficients associated with general solution of differential equation of mode shapes (e.g. Equation (3.15))	m
A_0	deflection of spring due to static force, F_0 , known as “static amplitude” ($= F_0/k$)	m
A_1, B_1, A_2, B_2	coefficients of general solutions of differential equations when damping is assumed a force proportional to displacement (Equations (4.40) and (4.41))	m
a	$= (E_e/\rho_e)^{0.5}$	m s^{-1}
a_i, b_i	coefficients associated with general solution of differential equation of displacement (e.g. Equation (3.12))	m
$a_0, a_1, b_1, a_2, b_2, \dots$	coefficients of Fourier series	N
b_0	constant associated with Equation (3.32)	m
C_1, C_2	coefficients of general solution of equations of motion (see Equation (4.4))	m
c	spring index ($= D_w/d_w$)	—
$c_0, c_1, c_2, \dots$	coefficients associated with vibrational displacement of spring	m
c_0'	quasi-static mean stress	N m^{-2}
$c_1', c_2', \dots$	quasi-static stress amplitudes	N m^{-2}
$\bar{c}$	damping factor (see Equation (4.37))	N m^{-1}
D_s	mean diameter of spring coil	m
d_w	diameter of spring wire	m
E_e, E_w	Young's modulus of equivalent rod and spring wire materials, respectively	N m^{-2}

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E	total energy of system ($= T_{KE} + U$)	N m
F_z	external force on spring in z-direction	N
F_0	coefficient of harmonic function (see Equation (4.17))	N
F_d	constant friction force (see Equation (4.27))	N
f_n	first natural frequency	s ⁻¹
f_1, f_2	coefficient functions of ϑ (see Equation (4.85b) and Sketch 4.7)	—
G_e, G_w	shear modulus of equivalent rod and spring wire materials, respectively	N m ⁻²
G	force due to weight of mass, $m (= \delta_{st} k)$	N
g	acceleration due to gravity	m s ⁻²
H	effective working height of unloaded spring (see Sketch 1.1)	m
h	height of an unloaded single coil	m
J	solution of homogeneous equation of motion (see Equation (4.10))	s
k	stiffness of spring and equivalent rod	N m ⁻¹
l_e, l_w	length of equivalent rod and effective, or working, length of spring wire, respectively	m
M, N	coefficients used in particular solution of second-order differential equation	m
m	external mass attached to spring	kg
m_s	mass of spring and equivalent rod	kg
m_{e1}	mass per unit length of equivalent rod	kg m ⁻¹
N	normal force along spring wire	N
N_0	amplitude of normal force ($= E_w A_0 du/dz$) (Equation (4.89))	m
n_s	number of effective, or working, coils	—
n	$= \kappa/(2m)$	s ⁻¹
q	number of oscillations with natural frequency ω_n (see Equation (4.85a))	—
R	damping force (see Equation (4.53))	N
r	general radius	m

T	period of oscillation	s
T_{KE}	kinetic energy	N m
T_0	period of oscillation without damping	s
T_1	period of first natural frequency	s
t	time	s
U	potential energy	N m
$u_i, u_j, i, j = 1, 2, \dots$	modes of vibration	m
W_1	deformation energy per unit volume	N m ⁻²
x, y, z	axis coordinate system of spring (see Sketch 1.1)	—
α	spring helix angle (see Sketch 1.1)	radian
β_w	$= \omega/(\omega_n)_1$ (see Equation (3.47))	—
β	dynamic magnification factor	—
γ	amplification factor (see Equation (4.21))	—
γ_n	angular frequency of damped vibration (see Equation (4.35))	s ⁻¹
δ	spring deflection due to force F_d	m
ϵ	strain of rod ($= \partial \xi / \partial z$)	—
ϵ_0	phase angle	radian
$\epsilon_1, \epsilon_2, \epsilon_3, \dots$	phase angles associated with vibrational displacement of spring	—
ϑ	see Equation (4.86)	—
κ	damping factor (see Equation (4.33))	N m ⁻¹
$\bar{\kappa}$	measure of damping (see Equation (4.68))	—
λ	logarithmic decrement ($= nT$)	—
ν	dissipated energy in one cubic metre of material in one cycle	N m ⁻²
ξ	displacement of the cross-section of an arbitrary element of the equivalent rod in the z -direction	m
ξ_0	displacement at $t = 0$	m
ξ_1	displacement of particular solution of inhomogeneous part of equation of motion (see Equation (4.10))	m

ρ_e, ρ_w	density of equivalent rod and spring wire materials, respectively	kg m^{-3}
τ	time variable	s
τ	shear stress	N m^{-2}
ϕ	polar angle of spring	radian
χ	constant, proportional to rate of change of angular frequency, $\omega(t)$ (see Section 4.10)	s^{-2}
ψ	damping intensity ($= \Delta E/E$)	—
ω	angular frequency	—
ω_n	natural frequency of free vibration of a spring-mass system ($= (k/m)^{0.5}$)	s^{-1}
ω_1, ω_2	frequency of spring-mass system with damping proportional to displacement (see Equations (4.40) and (4.41), respectively)	s^{-1}
$(\omega_n)_i, i = 1, 2, \dots$	angular natural frequencies	s^{-1}

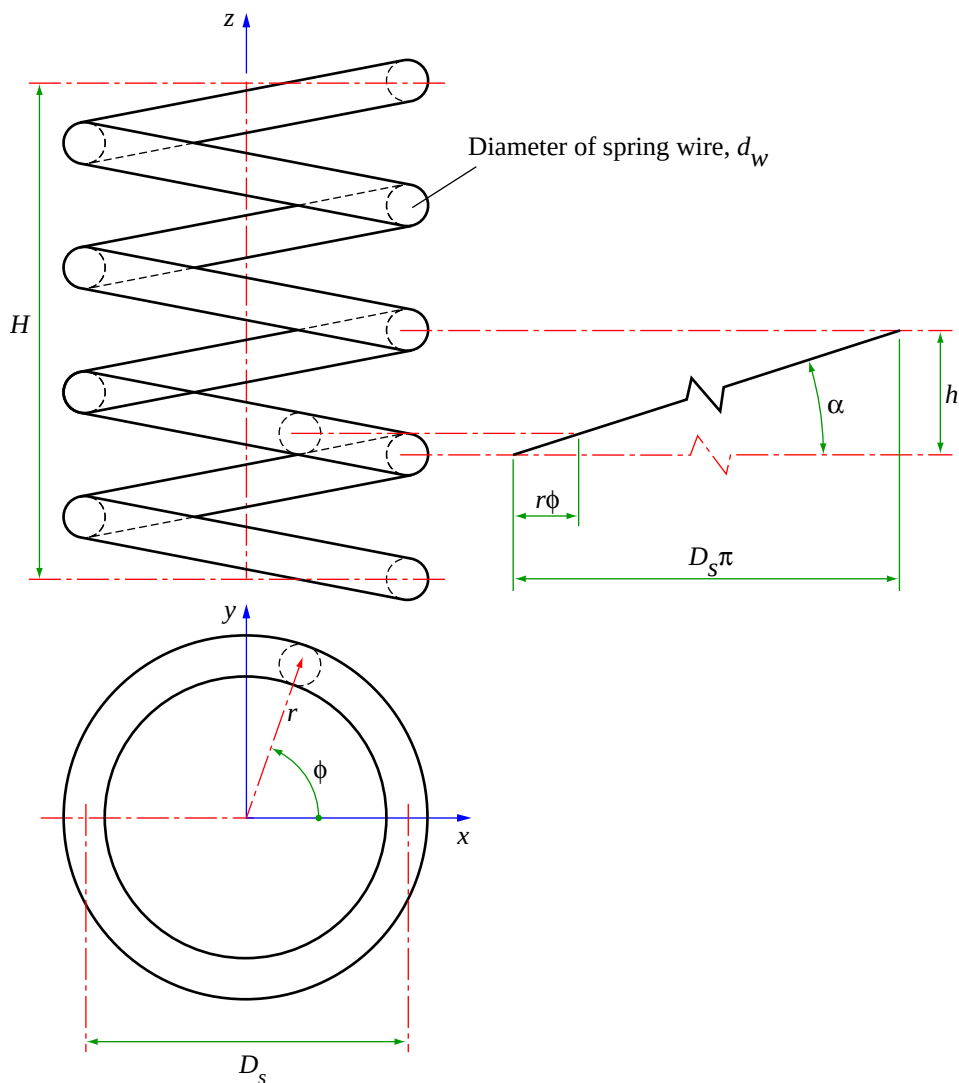
Subscripts

<i>dyn</i>	dynamic
<i>e</i>	equivalent rod
<i>l</i>	equivalent values when $z = l$
<i>mean</i>	mean value
<i>min, max</i>	minimum and maximum, respectively
<i>res</i>	resonance
<i>s</i>	spring
<i>st</i>	static or quasi-static
<i>w</i>	spring wire
0	unloaded condition or $t = 0$

Operators

Δ	increment
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one or two dots above a quantity indicate the first or second differential of that quantity with respect to time.



Sketch 1.1 Notation associated with unloaded spring coil

2. INTRODUCTION

The dynamic behaviour of cylindrical helical springs, comprising both tension/compression and torsion springs, is extremely difficult to calculate since its geometrical shape is a curve in three-dimensional space. To make the calculations manageable, simple but representative mathematical models are required. The simplest of such models is the straight elastic rod, the so called 'equivalent rod' which clearly must have the same elastic properties as the helical spring it represents. It is rather surprising, but fortunate, that the use of this very simple mathematical model should yield such reasonable results, certainly accurate enough for most practical purposes.

An earlier Data Item No. 06024^[2] defined the assumptions and limitations that apply to the calculation procedure for estimating the dynamic characteristics of springs, together with the prescribed loading conditions assumed to apply to the spring. The Item also provided derivation of the deformation, stresses and transverse loading on the spring and the form design of the spring ends which will affect the loading characteristics. The elastic stability of compression and torsion springs are discussed and formulae given for ensuring stability.

The present Item extends the scope of the earlier Item, presenting the vibration characteristics of cylindrical helical springs.

Section 3 discusses the axial vibration of compression/tension helical springs on the basis of the 'equivalent rod' approximation, dealing with both free and forced axial vibration. For free vibration, cases when both ends of the rod are free, one end of the rod is clamped and the other end is free and both ends of the rod are clamped, are considered. For forced vibration, the case when one end of the spring is forced to follow a cyclic motion and the stresses induced by the cyclic motion is discussed.

Section 4, considers the free and forced vibrations of a spring-mass system in reasonable detail, dealing with the cases when the system mass is large compared to the mass of the spring and of comparable size. The influence of various kinds of damping, Coulomb and viscous friction, material hysteresis, *etc.* are also discussed. In conjunction with forced vibration, the resonance phenomenon is dealt with in a number of sections. Although it is an important design principle to avoid resonance whenever possible, in high speed applications it is sometimes inevitable that the elastic system during its normal operation must pass through the resonance domain. In such cases the only practical possibility is to try to avoid sustained resonance. Recognising the engineering importance of this problem a separate section is devoted to the discussion of the transition through resonance.

A further Data Item in this series on springs, No. 09003^[3], considers the dynamic characteristics of cylindrical helical springs due to impact loading, which is an integral part of the normal operation of the majority of machines that execute rapid alternating motion. The Item also provides worked examples that estimate spring dynamic performance.

3. AXIAL VIBRATION OF COMPRESSION SPRINGS

Since the helix is a curve in three dimensions, the accurate calculation of the vibration characteristics of a helical spring is extremely complicated. However, provided the helix angle is small a close approximation can be obtained by replacing the spring with an equivalent straight rod. Assuming that under dynamic load the stress condition of the spring wire remains the same as under static load and the load can be represented by a single resultant force that is proportional to the deflection and acts along the spring axis, *i.e.* the *z*-axis, the spring can be modelled by a straight rod. **This “equivalent rod” must have the same mass and stiffness as the spring.**

The mass of the spring, m_s , is given by

$$m_s = \rho_w A_w \pi D_s n_s, \quad (3.1)$$

where ρ_w is the density of the spring material,
 A_w is the cross-section of the spring wire,
 D_s is the mean diameter of the spring coil
 and n_s is the number of effective, or working, coils.

Spring stiffness, k , in the axial direction (taken from Equation (3.25) of ESDU 06024^[2]) is

$$k = \frac{F_z}{\delta} = \frac{G_w d_w^4}{8 D_s^3 n_s}. \quad (3.2)$$

The mass of the equivalent rod, which is equal to the mass of the spring, is

$$m_s = m_{e1} l_e, \quad (3.3)$$

where m_{e1} is the mass per unit length of the equivalent rod
 and l_e is the length of the equivalent rod.

The stiffness of the equivalent rod, which is equivalent to the stiffness of the spring, is given by

$$k = \frac{E_e A_e}{l_e}, \quad (3.4)$$

where E_e is the Young's modulus of the rod material
 and A_e is the cross-section of the equivalent rod.

Equating these equivalent rod quantities with the corresponding values of the spring, the following equalities result

$$l_e m_{e1} = \rho_w A_w \pi D_s n_s \quad (3.5)$$

$$\text{and} \quad \frac{E_e A_e}{l_e} = \frac{G_w d_w^4}{8 D_s^3 n_s}. \quad (3.6)$$

The length of the equivalent rod, l_e , can be chosen arbitrarily and has, in this case, been taken as the effective height of the spring, i.e. $l_e = H$, which when substituted into Equations (3.5) and (3.6) gives

$$m_{e1} = \frac{\rho_w A_w \pi D_s n_s}{H} \quad (3.7)$$

and
$$E_e A_e = \frac{G_w d_w^4 H}{8 D_s^3 n_s} . \quad (3.8)$$

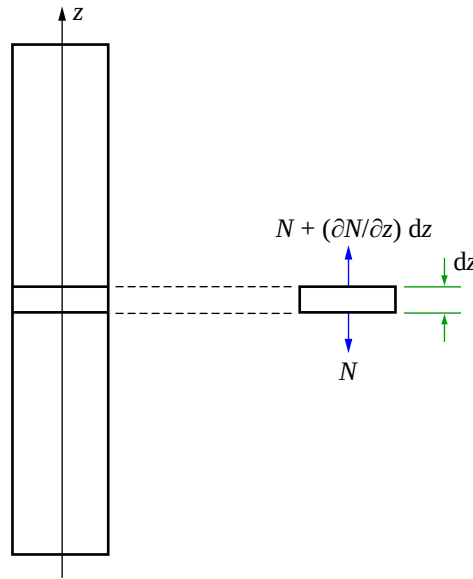
Also since $m_{e1} = \rho_e A_e$ and $k = E_e A_e / H$

then
$$\frac{E_e}{\rho_e} = \frac{Hk}{m_{e1}} = \frac{H^2 k}{l_w A_w \pi D_s n_s} .$$

3.1 Free Axial Vibrations

In the following calculations it is assumed that the cross-sections of the rod remain plane during vibrations. It is also assumed that any displacements perpendicular to the axis can be neglected. This leads to a somewhat higher value of natural frequency than a more accurate calculation would give.

Consider Sketch 3.1 that shows the forces acting on an element of length dz of the equivalent rod.



Sketch 3.1 Forces on an element of the equivalent rod

Applying D'Alembert's principle, the equilibrium of the element dz is given by the following partial differential equation

$$\frac{\partial N}{\partial z} dz - \rho_e A_e dz \frac{\partial^2 \xi}{\partial t^2} = 0 ,$$

or
$$\frac{\partial N}{\partial z} - \rho_e A_e \frac{\partial^2 \xi}{\partial t^2} = 0 , \quad (3.9)$$

where ξ is the displacement of the cross-section in the z -direction.

If the displacement of the bottom of the element, dz , is ξ , and that of the top is $(\partial\xi/\partial z)dz$, then the strain (or elongation) of the rod is

$$\varepsilon = \frac{\partial\xi}{\partial z} . \quad (3.10)$$

Use of this last expression for the strain, the normal force, N , is given by

$$N = E_e A_e \varepsilon = E_e A_e \frac{\partial\xi}{\partial z} . \quad (3.11)$$

Inserting the normal force, N , from Equation (3.11) into Equation (3.9) gives

$$E_e \frac{\partial}{\partial z} \left(A_e \frac{\partial\xi}{\partial z} \right) - \rho_e A_e \frac{\partial^2 \xi}{\partial t^2} = 0 .$$

If the rod is of uniform cross-section, the last equation then becomes

$$\frac{\partial^2 \xi}{\partial z^2} - \frac{1}{a^2} \frac{\partial^2 \xi}{\partial t^2} = 0 , \quad (3.12)$$

where $a^2 = E_e/\rho_e$.

The solution of this partial differential equation, using Fourier's method has the form

$$\xi = u(z) \sin(\omega_n t + \varepsilon_0) , \quad (3.13)$$

where $u(z)$ defines the mode of vibration, ω_n is the natural frequency of vibration of the system and ε_0 is the phase angle.

Double partial differentiation of Equation (3.13) with respect to both z and t yields

$$\begin{aligned} \frac{\partial\xi}{\partial z} &= \frac{\partial u(z)}{\partial z} \sin(\omega_n t + \varepsilon_0) , & \frac{\partial^2 \xi}{\partial z^2} &= \frac{\partial^2 u(z)}{\partial z^2} \sin(\omega_n t + \varepsilon_0) , \\ \frac{\partial\xi}{\partial t} &= u(z) \omega_n \cos(\omega_n t + \varepsilon_0) , & \frac{\partial^2 \xi}{\partial t^2} &= -u(z) \omega_n^2 \sin(\omega_n t + \varepsilon_0) . \end{aligned}$$

Inserting these equations into Equation (3.12) gives

$$\frac{\partial^2 u(z)}{\partial z^2} \sin(\omega_n t + \varepsilon_0) + \frac{\omega_n^2}{a^2} u(z) \sin(\omega_n t + \varepsilon_0) = 0 .$$

Since $u(z)$ is not a function of time, the last equation becomes an ordinary differential equation

$$\frac{d^2 u(z)}{dz^2} + \frac{\omega_n^2}{a^2} u(z) = 0, \quad (3.14)$$

the general solution of which has the form

$$u(z) = A \sin\left(\frac{\omega_n}{a} z\right) + B \cos\left(\frac{\omega_n}{a} z\right). \quad (3.15)$$

The values of the constants A and B , depend upon the boundary conditions, *i.e.* how the ends of the rod are fixed.

3.1.1 Both ends of rod free

In this case when both ends of the rod are free, there is no force at the ends of the rod and consequently the elongation (often referred to as strain) at the rod ends must be zero, which gives the end conditions

$$\left(\frac{\partial \xi}{\partial z}\right)_{z=0} = 0 \quad \text{and} \quad \left(\frac{\partial \xi}{\partial z}\right)_{z=l_e} = 0. \quad (3.16)$$

Since in Equation (3.13) only $u(z)$ is a function of z , Equations (3.16) imply

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = \left(\frac{du}{dz}\right)_{z=0} = 0 \quad \text{and} \quad \left(\frac{\partial u}{\partial z}\right)_{z=l_e} = \left(\frac{du}{dz}\right)_{z=l_e} = 0. \quad (3.17)$$

Differentiating Equation (3.15) gives

$$\frac{du(z)}{dz} = A \frac{\omega_n}{a} \cos\left(\frac{\omega_n}{a} z\right) - B \frac{\omega_n}{a} \sin\left(\frac{\omega_n}{a} z\right), \quad (3.18)$$

from which

$$\left(\frac{du}{dz}\right)_{z=0} = A \frac{\omega_n}{a} \cos 0 - B \frac{\omega_n}{a} \sin 0 = 0, \quad \text{hence } A \frac{\omega_n}{a} = 0, \quad \text{or } A = 0$$

$$\left(\frac{du}{dz}\right)_{z=l_e} = A \frac{\omega_n}{a} \cos\left(\frac{\omega_n}{a} l_e\right) - B \frac{\omega_n}{a} \sin\left(\frac{\omega_n}{a} l_e\right) = 0,$$

$$\text{hence } -B \frac{\omega_n}{a} \sin\left(\frac{\omega_n}{a} l_e\right) = 0.$$

The coefficient B can only be different from zero if $\sin(\omega_n l_e / a) = 0$, from which it follows that

$$(\omega_n)_i = i \frac{\pi a}{l_e}, \quad \text{for } i = 1, 2, 3, \dots. \quad (3.19)$$

Equation (3.19) gives the angular natural frequencies, $(\omega_n)_1, (\omega_n)_2, (\omega_n)_3, \dots$, of the equivalent rod which, when inserted into Equation (3.15), gives

$$u_i = u_i(z) = \cos\left(\frac{i\pi}{l_e} z\right). \quad (3.20)$$

With use of the results obtained, the general solution, Equation (3.13), can be written in the form

$$\begin{aligned} \xi &= \sum_{i=1}^{\infty} u_i \sin\left((\omega_n)_i t + \varepsilon_0\right) \\ &= \sum_{i=1}^{\infty} \left(a_i \cos\left(\frac{i\pi a}{l_e} t\right) + b_i \sin\left(\frac{i\pi a}{l_e} t\right) \right) \cos\left(\frac{i\pi}{l_e} z\right), \end{aligned} \quad (3.21)$$

where the coefficients a_i and b_i can be determined from the initial conditions of the motion.

3.1.2 One end of rod clamped other end free

In this case one end of the rod is clamped and the other end is free. Since there is no displacement at the clamped end of the rod and no elongation of the rod at the free end, the boundary conditions are

$$u(z)_{z=0} = 0 \quad \text{and} \quad \left(\frac{du}{dz}\right)_{z=l_e} = 0. \quad (3.22)$$

From Equations (3.15) and (3.18) the following is obtained

$$\begin{aligned} u(z)_{z=0} &= A \sin 0 + B \cos 0 = 0, \quad \text{hence } B = 0, \\ \left(\frac{du}{dz}\right)_{z=l_e} &= A \frac{\omega_n}{a} \cos\left(\frac{\omega_n}{a} l_e\right) - B \frac{\omega_n}{a} \sin\left(\frac{\omega_n}{a} l_e\right) = 0, \\ &\text{from which } A \frac{\omega_n}{a} \cos\left(\frac{\omega_n}{a} l_e\right) = 0. \end{aligned}$$

The constant A can only be non-zero if

$$\cos\left(\frac{\omega_n}{a} l_e\right) = 0, \quad \text{or} \quad (\omega_n)_i = \frac{(2i-1)}{2} \frac{\pi a}{l_e} \equiv (2i-1) \frac{\pi}{2} \left(\frac{E_e}{\rho_e l_e^2}\right)^{0.5}. \quad (3.23)$$

With use of the above results Equation (3.13), that gives the displacement of any arbitrary cross-section of the equivalent rod in the z -direction, becomes

$$\xi = \sum_{i=1}^{\infty} \left(a_i \cos\left(\frac{(2i-1)}{2} \frac{\pi a}{l_e} t\right) + b_i \sin\left(\frac{(2i-1)}{2} \frac{\pi a}{l_e} t\right) \right) \sin\left(\frac{(2i-1)}{2} \frac{\pi}{l_e} z\right), \quad (3.24)$$

where the coefficients a_i and b_i can be determined from the initial conditions of the motion.

3.1.3 Both ends of rod clamped

When both ends of the rod are clamped, there is no displacement at either end of the rod and so the boundary conditions are

$$u(z)_{z=0} = 0 \quad \text{and} \quad u(z)_{z=l_e} = 0. \quad (3.25)$$

From Equation (3.15) the following is obtained

$$u(z)_{z=0} = A \sin 0 + B \cos 0 = 0, \quad \text{hence} \quad B = 0,$$

$$u(z)_{z=l_e} = A \sin\left(\frac{\omega_n}{a} l_e\right) - B \cos\left(\frac{\omega_n}{a} l_e\right) = 0, \quad \text{hence} \quad A \sin\left(\frac{\omega_n}{a} l_e\right) = 0.$$

The constant A can only be non-zero if $\sin(\omega_n l_e/a) = 0$, from which

$$(\omega_n)_i = i \frac{\pi a}{l_e}, \quad \text{for} \quad i = 1, 2, 3, \dots \quad (3.26)$$

With use of Equation (3.13) the displacement of an arbitrary cross-section of the equivalent rod in the direction of the z -axis is given by

$$\xi = \sum_{i=1}^{\infty} \left(a_i \cos\left(\frac{i\pi a}{l_e} t\right) + b_i \sin\left(\frac{i\pi a}{l_e} t\right) \right) \sin\left(\frac{i\pi}{l_e} z\right). \quad (3.27)$$

The angular natural frequencies, $(\omega_n)_i$, of the equivalent rod are equal to that of the spring.

The boundary conditions most frequently met in practice are when both ends of the spring are clamped. For this case the angular natural frequencies have already been obtained (see Equation (3.26)) and are

$$\begin{aligned} (\omega_n)_i &= i \frac{\pi a}{l_e} = i\pi \left(\frac{E_e}{\rho_e l_e^2} \right)^{0.5} = i\pi \left(\frac{E_e A_e}{\rho_e A_e l_e^2} \right)^{0.5} \\ &= i\pi \left(\frac{E_e A_e}{m_{e1} l_e^2} \right)^{0.5}, \quad \text{for} \quad i = 1, 2, 3, \dots, \end{aligned} \quad (3.28)$$

where $a^2 = E_e/\rho_e$ and $m_{e1} = \rho_e A_e$.

After inserting the values of m_{e1} and $E_e A_e$ from Equations (3.7) and (3.8), respectively, and noting that $H = l_e$, the following expression is obtained

$$\begin{aligned} (\omega_n)_i &= i\pi \left(\frac{G_w d_w^4 H}{8D_s^3 n_s} \frac{H}{\rho_w A_w \pi D_s n_s} \frac{1}{l_e^2} \right)^{0.5} = i\pi \left(\frac{G_w d_w^4}{8D_s^4 \rho_w \frac{d_w^2 \pi}{4} n_s^2 \pi} \right)^{0.5} \\ &= \frac{i}{n_s} \left(\frac{G_w d_w^2}{2\rho_w D_s^4} \right)^{0.5}, \quad \text{for} \quad i = 1, 2, 3, \dots, \end{aligned} \quad (3.29)$$

from which the first, and most important, angular natural frequency is obtained by setting $i = 1$ to give the following

$$(\omega_n)_1 = \frac{1}{n_s} \left(\frac{G_w d_w^2}{2 \rho_w D_s^4} \right)^{0.5} = \frac{1}{n_s c D_s} \left(\frac{G_w}{2 \rho_w} \right)^{0.5} = \frac{0.707}{n_s c D_s} \left(\frac{G_w}{\rho_w} \right)^{0.5}, \quad (3.30)$$

where c is the spring index.

3.2 Forced Axial Vibrations

In this section forced axial vibrations of compression springs are considered for the case when one end of the spring is forced to follow a cyclic motion. This kind of excitation has the greatest practical interest, an example of which is the excitation caused by a rapidly rotating cam. Since any cyclic motion can be represented by an appropriate Fourier series, the time-displacement function of the end of the spring forced by a cam (or any other means) can be written in the form

$$\xi(z)_{z=H} = c_0 + c_1 \sin(\omega t + \epsilon_1) + c_2 \sin(2\omega t + \epsilon_2) + \dots, \quad (3.31)$$

where ω is the angular velocity, or angular frequency, of the cam and $c_0, c_1, c_2, \dots$ and $\epsilon_0, \epsilon_1, \epsilon_2, \dots$ are known constants.

The partial differential equation describing the longitudinal vibration of the equivalent rod was derived in Equation (3.12) and is repeated here as

$$\frac{\partial^2 \xi}{\partial z^2} - \frac{1}{a^2} \frac{\partial^2 \xi}{\partial t^2} = 0,$$

in which $a^2 = E_e / \rho_e = E_e A_e l_e / m_s$.

The general solution of this last equation can be written in the form

$$\xi = b_0 + \sum_{j=1}^{\infty} u_j(z) \sin(j\omega t + \epsilon_j). \quad (3.32)$$

Partial differentiation of Equation (3.32) with respect to z and t yields

$$\begin{aligned} \frac{\partial \xi}{\partial z} &= \sum_{j=1}^{\infty} \frac{\partial u_j(z)}{\partial z} \sin(j\omega t + \epsilon_j) \quad \text{and} \quad \frac{\partial \xi}{\partial t} = \sum_{j=1}^{\infty} u_j(z) j\omega \cos(j\omega t + \epsilon_j) \\ \frac{\partial^2 \xi}{\partial z^2} &= \sum_{j=1}^{\infty} \frac{\partial^2 u_j(z)}{\partial z^2} \sin(j\omega t + \epsilon_j) \quad \text{and} \quad \frac{\partial^2 \xi}{\partial t^2} = - \sum_{j=1}^{\infty} u_j(z) j^2 \omega^2 \sin(j\omega t + \epsilon_j). \end{aligned}$$

After inserting these into Equation (3.12), the following ordinary differential equation is obtained

$$\sum_{j=1}^{\infty} \frac{\partial^2 u_j(z)}{\partial z^2} \sin(j\omega t + \varepsilon_j) + \sum_{j=1}^{\infty} \frac{j^2 \omega^2}{a^2} u_j(z) \sin(j\omega t + \varepsilon_j) = 0 ,$$

or

$$\frac{d^2 u_j(z)}{dz^2} + \frac{j^2 \omega^2}{a^2} u_j(z) = 0 . \quad (3.33)$$

The general solution of this last equation (see Equation (3.15)) can be written in the form

$$u_j(z) = b_j \sin\left(\frac{j\omega}{a} z\right) + e_j \cos\left(\frac{j\omega}{a} z\right) , \quad (3.34)$$

where the constants b_j and e_j are obtained from the boundary conditions. There is no displacement at the clamped end, i.e. at $z = 0$, and so

$$u_j(0) = b_j \sin(0) + e_j \cos(0) = 0 \quad \text{and so} \quad e_j = 0 .$$

With use of this result, Equation (3.34) takes the simpler form

$$u_j(z) = b_j \sin\left(\frac{j\omega}{a} z\right) . \quad (3.34a)$$

Inserting Equation (3.34a) into Equation (3.32) gives the displacement of an arbitrary point of the spring in the z -direction as

$$\xi(z) = b_0 + \sum_{j=1}^{\infty} b_j \sin\left(\frac{j\omega}{a} z\right) \sin(j\omega t + \varepsilon_j) . \quad (3.35)$$

The displacement of the end of the spring, i.e. $z = H$ (see Sketch 1.1), is

$$\xi(H) = b_0 + \sum_{j=1}^{\infty} b_j \sin\left(\frac{j\omega}{a} H\right) \sin(j\omega t + \varepsilon_j) . \quad (3.36)$$

A comparison of Equations (3.36) and (3.31) gives the values of the constants b_0 and b_j as

$$b_0 = c_0 \quad \text{and} \quad b_j = \frac{c_j}{\sin(j\omega H/a)} . \quad (3.37)$$

Inserting these values into Equation (3.35), the final expression for the displacement of an arbitrary point of the spring at a distance z from the clamped end is obtained as

$$\begin{aligned} \xi(z) = & c_0 + \frac{c_1}{\sin(\omega H/a)} \sin(\omega z/a) \sin(\omega t + \varepsilon_1) \\ & + \frac{c_2}{\sin(2\omega H/a)} \sin(2\omega z/a) \sin(2\omega t + \varepsilon_1) + \dots \end{aligned} \quad (3.38)$$

Whenever any of the coefficients $\sin(j\omega H/a) = 0$, the corresponding member of Equation (3.38) becomes infinite. This happens when

$$j\omega = i \frac{\pi a}{H} = i\pi \left(\frac{E_e}{\rho_e H^2} \right)^{0.5}, \quad (3.39)$$

but according to Equation (3.28), this last expression is equal to the angular natural frequency of the spring, *i.e.*

$$i\pi \left(\frac{E_e}{\rho_e H^2} \right)^{0.5} = (\omega_n)_i, \quad \text{or} \quad \omega = (\omega_n)_1. \quad (i = 1) \quad (3.40)$$

This is the so-called resonance condition, when the amplitude of the vibration of the spring is ultimately limited only by the always-present damping.

In practice, the first natural frequency is the most important and is given by

$$f_n = \frac{(\omega_n)_1}{2\pi} \quad \text{and its period is} \quad T_1 = \frac{2\pi}{(\omega_n)_1}. \quad (3.41)$$

3.2.1 Shear stresses induced by vibrations

Under rapid cyclic loading, the dynamic effects that occur produce shear stresses additional to those resulting from static or low frequency loading. These effects depend upon the characteristics of both the spring and the loading. Since it has been assumed (see Assumption (c) in Section 2.1 of ESDU 06024^[2]) that the spring material is not stressed beyond its elastic limit, Hooke's law can be applied to relate stresses with the corresponding strains. In order to do that the displacements will be considered first. Equation (3.38), that gives the displacement of an arbitrary point in the spring, must be rewritten in the form

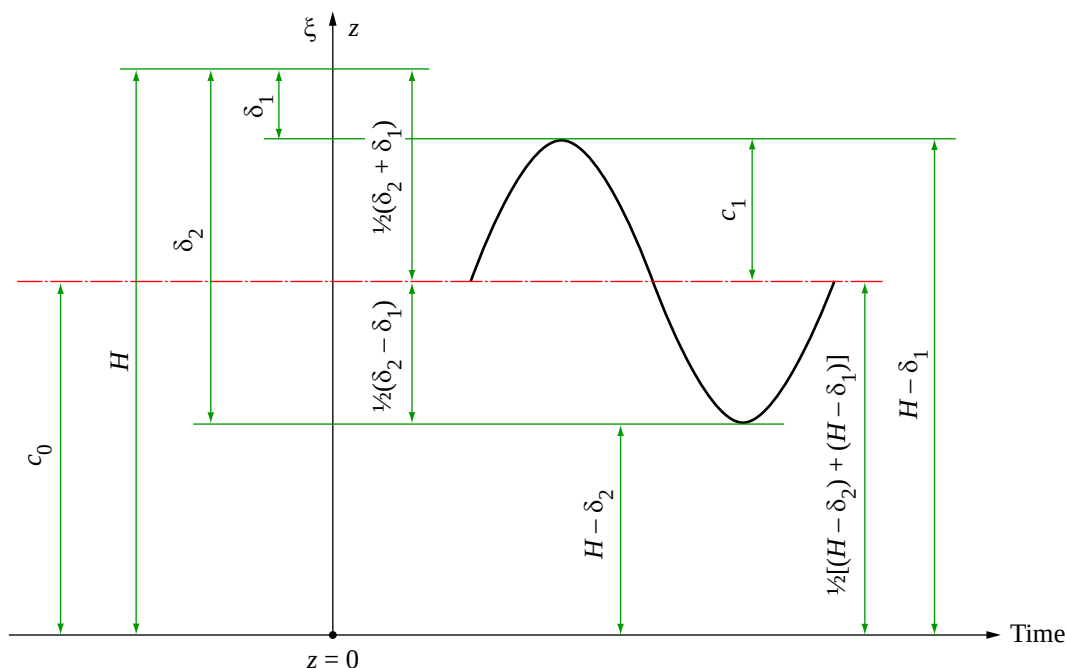
$$\xi(z) = c_0 + c_1 \left(\frac{\sin(\omega z/a)}{\sin(\omega H/a)} \sin(\omega t + \varepsilon_1) + \frac{c_2}{c_1} \frac{\sin(2\omega z/a)}{\sin(2\omega H/a)} \sin(2\omega t + \varepsilon_2) + \dots \right), \quad (3.42)$$

which is more suitable for the purpose. The constants $c_0, c_1, c_2, \dots$, are the same as those in Equation (3.31).

Very often in practice the influence of the higher-order harmonics of the excitation are small enough to be neglected and only the first two terms of Equation (3.31) are used, *i.e.*

$$\xi(H) = c_0 + c_1 \sin(\omega t + \varepsilon_1),$$

where the values of the constants can easily be determined, c_0 is the z -coordinate of the mean displacement and c_1 is the amplitude of the forced oscillation (see Sketch 3.2).



Sketch 3.2 Displacement of an arbitrary point in the spring against time

Under quasi-static conditions the displacement ξ_{st} of an arbitrary point of the spring at a distance z from the fixed end is proportional to the displacement $\xi(H)$ of the moving end, i.e.

$$\xi_{st} = \frac{z}{H} \xi(H), \quad \text{and the reciprocal relationship} \quad \xi(H) = \frac{H}{z} \xi_{st}.$$

Hence the values of the constants c_0 and c_1 are

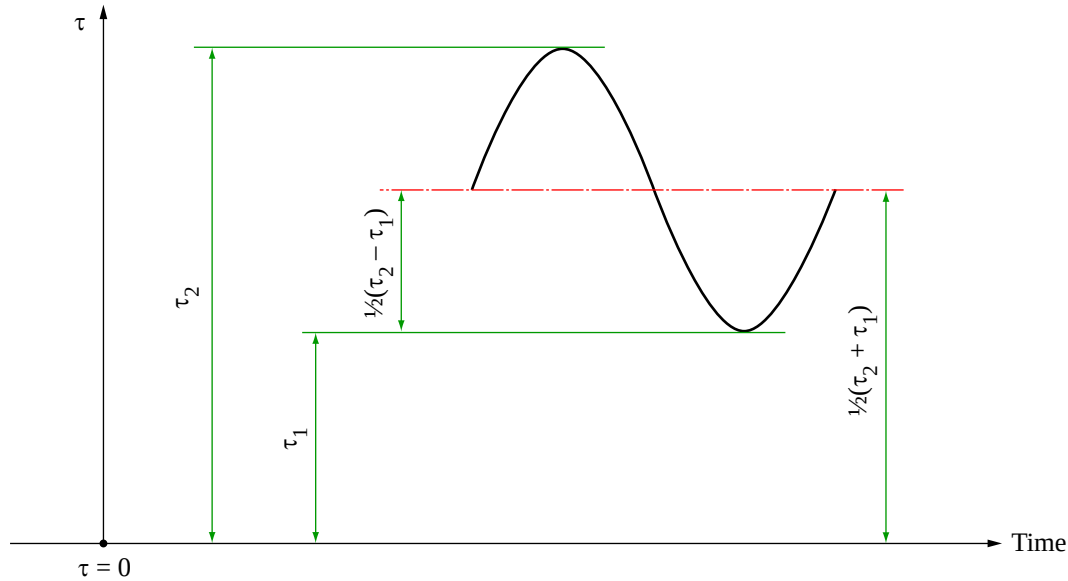
$$c_0 = \frac{z}{H} \left(H - \frac{(\delta_2 + \delta_1)}{2} \right) \quad \text{and} \quad c_1 = \frac{(\delta_2 - \delta_1)}{2}. \quad (3.43)$$

From the results of Equation (3.43), Equation (3.42) takes the simpler form

$$\xi(H) = \frac{z}{H} \left(H - \frac{(\delta_2 + \delta_1)}{2} \right) + \frac{(\delta_2 - \delta_1)}{2} \frac{\sin(\omega z/a)}{\sin(\omega H/a)} \sin(\omega t + \varepsilon_1). \quad (3.42a)$$

A similar relationship can be derived for the dynamic shear stresses in the spring. If the stresses, corresponding to the minimum and maximum static deflections are denoted by τ_1 and τ_2 , respectively, then the mean stress, τ_{mean} , in the working coils of the spring (see Sketch 3.3) and the quasi-static stress amplitude, τ_{st} , are given by

$$\tau_{mean} = \frac{(\tau_2 + \tau_1)}{2} \quad \text{and} \quad \tau_{st} = \frac{(\tau_2 - \tau_1)}{2}.$$



Sketch 3.3 Variation of shear stresses in spring with time

For calculating the dynamic stresses, Equation (3.42) can be used after replacing the displacements with the corresponding stresses as in

$$\tau_{dyn} = c_0' + c_1' \left(\frac{\sin(\omega z/a)}{\sin(\omega H/a)} \sin(\omega t + \varepsilon_1) + \frac{c_2'}{c_1'} \frac{\sin(2\omega z/a)}{\sin(2\omega H/a)} \sin(2\omega t + \varepsilon_2) + \dots \right), \quad (3.44)$$

where the constant c_0' is the quasi-static mean stress, which is the same in all working coils and is

$$c_0' = \tau_{mean} = \frac{(\tau_1 + \tau_2)}{2} \quad (3.45a)$$

and the constants $c_1', c_2', \dots$, are the quasi-static stress amplitudes, which depend upon the point in question along the spring axis and therefore must be scaled. This means that c_1' becomes

$$c_1' = \frac{H}{z} \frac{(\tau_2 - \tau_1)}{2} \quad \text{instead of} \quad \frac{(\tau_2 - \tau_1)}{2}. \quad (3.45b)$$

Also, as a consequence of the linear relationship between stress and strain, the ratio of the corresponding constants c_1', c_2', c_3' and c_1, c_2, c_3 remain the same, i.e.

$$\frac{c_2'}{c_1'} = \frac{c_2}{c_1}, \quad \frac{c_3'}{c_1'} = \frac{c_3}{c_1}, \quad \text{etc.} \quad (3.45c)$$

With use of the foregoing results and neglecting higher harmonics, Equation (3.44) reduces to

$$\tau_{dyn} = \frac{(\tau_2 + \tau_1)}{2} + \frac{H}{z} \frac{(\tau_2 - \tau_1)}{2} \frac{\sin(\omega z/a)}{\sin(\omega H/a)} \sin(\omega t + \varepsilon_1). \quad (3.44a)$$