



Dynamic characteristics of cylindrical helical springs. Part 3: impact loading on compression springs

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DYNAMIC CHARACTERISTICS OF CYLINDRICAL HELICAL SPRINGS. PART 3: IMPACT LOADING ON COMPRESSION SPRINGS

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DYNAMIC CHARACTERISTICS OF CYLINDRICAL HELICAL SPRINGS. PART 3: IMPACT LOADING ON COMPRESSION SPRINGS

1. NOTATION AND UNITS

Units used in this Item are predominantly SI but any coherent system of units may be used.

A_e, A_w	cross-section area of equivalent rod and spring wire, respectively	m^2
A_j, B_j	constants (see Equation (5.9a))	—
a	velocity of elastic propagation $\left(= (E_e / \rho_e)^{1/2} \right)$	m s^{-1}
C_1, C_2	coefficients of general solution of equations of motion (see Equation (3.1))	N
c	spring index $(= D_s / d_w)$	—
D_s	mean diameter of spring coil	m
D, D_1, D_2	determinants	—
d_w	diameter of spring wire	m
E_e	Young's modulus of material of equivalent rod	N m^{-2}
F_z	external force on spring in z-direction (see Sketch 1.1)	N
F_0	pre-load on spring in z-direction	N
f_1, f_2	functions of $(at - z)$ and $(at + z)$, respectively (see Equation (6.2))	—
G_w	shear modulus of spring wire material	N m^{-2}
G	force due to weight of mass, m $(= \delta_{st} k)$ (see Equation (3.10a))	N
H	effective working height of spring	m
h	height of a single working coil of spring	m
K'	$= (c + 0.5) / (c - 0.5)$	—
k, k_1, k_2	spring stiffnesses	N m^{-1}
l_e	length of equivalent rod	m
m, m_1, m_2	external masses	kg
m_e, m_s	mass of equivalent rod and spring, respectively	kg
n	number of effective, or working, coils	—
p	intermediate variable (see Section 6)	m
T	period of oscillation	s
T_0	kinetic energy of mass, m , with velocity, v_0	N m
T_2	period of free vibration of mass, m_2	s

t	time	s
t_1, t_2	$= T/4$ and $T/2$, respectively	s
U	potential energy	N m
u_1, u_2	eigenvectors of system of Equation (5.2)	m
u_{ij}	eigenvector i relating to eigenfrequency j	m
v, v_c, v_{cr}	velocity, common velocity of masses m_1 and m_2 after non-elastic impact and critical velocity, respectively	m s ⁻¹
v_0, v_1, v_2	initial velocity of masses, m , m_1 and m_2 , respectively	m s ⁻¹
v_{0C}	critical velocity (see Equation (6.14))	m s ⁻¹
$v_{1 2}, v_{2 2}$	velocity of masses m_1 and m_2 just before second impact	m s ⁻¹
$v_{1 j}, v_{2 j}$	velocity of masses m_1 and m_2 just before j^{th} impact	m s ⁻¹
$v'_{1 2}, v'_{2 2}$	velocity of masses m_1 and m_2 just after second impact	m s ⁻¹
v'_1, v'_2	velocity of masses m_1 and m_2 after impact	m s ⁻¹
x, y, z	axis coordinate system of spring (see Sketch 1.1)	—
z_a	intermediate variable $= at$ (see Section 6)	m
α	spring helix angle (see Sketch 1.1)	radian
δ	spring deflection	m
δ_0	initial spring deflection	m
ϵ	strain of rod ($= \partial \xi / \partial z$)	—
ϵ_y	strain associated with yield point of material	—
κ	impact coefficient	—
ξ, ξ_1, ξ_2	displacements	m
ρ_e, ρ_w	density of equivalent rod and wire material, respectively	kg m ⁻³
σ	tensile stress	N m ⁻²
σ_y	yield stress	N m ⁻²
τ	shear stress	N m ⁻²
ϕ	polar angle of spring (see Sketch 1.1)	radian
ω	frequency	s ⁻¹
ω_n	natural frequency of free vibration of a spring-mass system ($= (k / m)^{1/2}$)	s ⁻¹
ω_1, ω_2	eigenfrequencies of system of Equation (5.1) given by Equation (5.6)	s ⁻¹

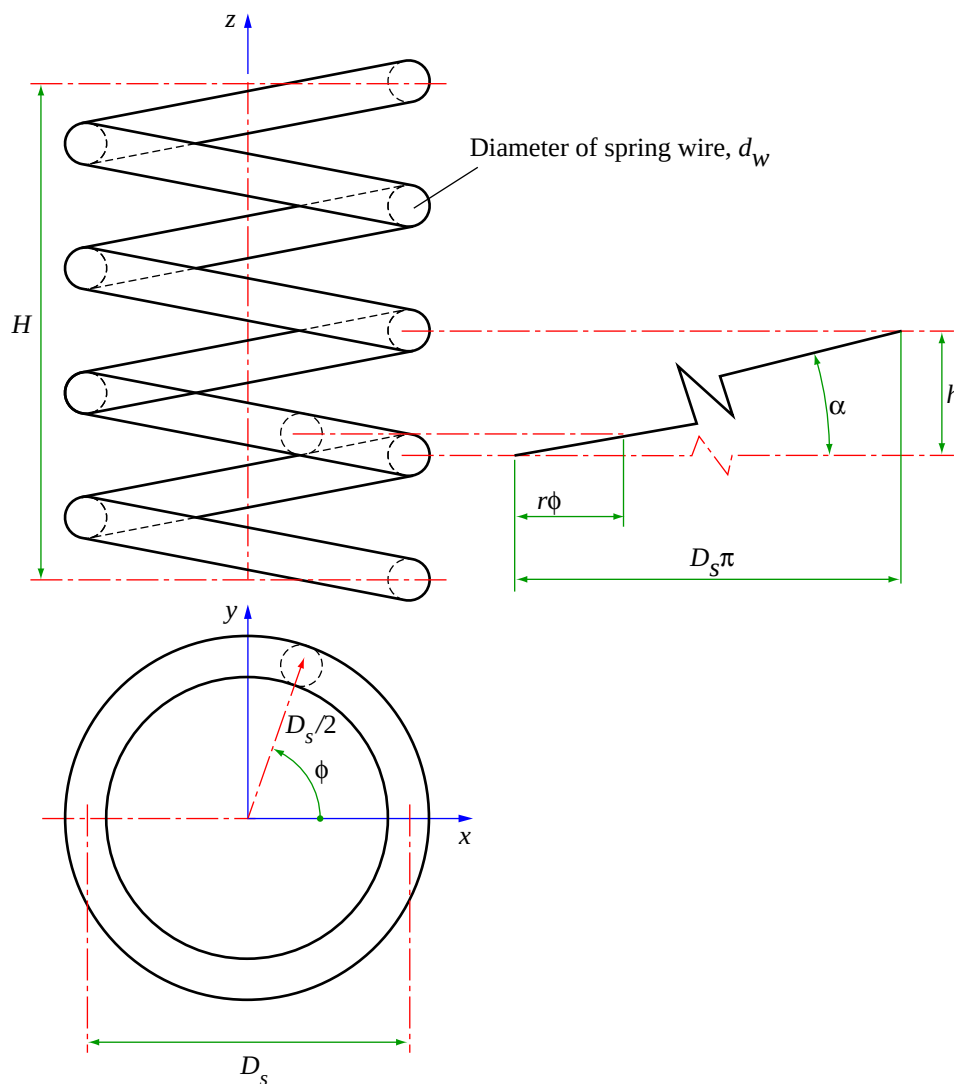
Subscripts

e	equivalent rod
i, j, k	general integers
max	maximum
s	spring
st	static or quasi-static
w	spring wire
0	unloaded condition or value at $t = 0$

Operators

Det[...] denotes determinant of [...]

one or two dots above a quantity indicate the first or second differential of that quantity with respect to time



Sketch 1.1 Notation associated with unloaded spring coil

2. INTRODUCTION

The dynamic behaviour of cylindrical helical springs, comprising both tension/compression and torsion springs, is extremely difficult to calculate since its geometrical shape is a curve in three-dimensional space. To make the calculations manageable, simple but representative mathematical models are required. The simplest of such models is the straight elastic rod, the so called 'equivalent rod' which clearly must have the same elastic properties as the helical spring it represents. It is rather surprising, but fortunate, that the use of this very simple mathematical model should yield such reasonable results, certainly accurate enough for most practical purposes.

The first Data Item in this series on springs, No. 06024^[1], defines the assumptions and limitations that apply to the calculation procedure for estimating the dynamic characteristics of springs, together with the prescribed loading conditions assumed to apply to the spring. The Item also provides derivation of the deformation, stresses and transverse loading on the spring and the form design of the spring ends which will affect the loading characteristics. The elastic stability of compression and torsion springs is discussed and formulae given for ensuring stability.

The second Data Item, No. 08015^[2], extends the scope of the earlier Item, presenting the vibration characteristics of cylindrical helical springs. The Item discusses the axial vibration of compression/tension helical springs on the basis of the 'equivalent rod' approximation, dealing with both free and forced axial vibration. For free vibration, cases when both ends of the rod are free, one end of the rod is clamped and the other end is free and both ends of the rod are clamped, are considered. For forced vibration, the case when one end of the spring is forced to follow a cyclic motion and the stresses induced by the cyclic motion is also discussed.

ESDU 08015 further considers free and forced vibrations of a spring mass system, dealing with the two cases when the system mass is large compared to the mass of the spring and when it is of comparable size. The influence of various kinds of damping, Coulomb and viscous friction, material hysteresis, *etc.* is also discussed. In conjunction with forced vibration, the resonance phenomenon is dealt with in a number of sections. Although it is an important design principle to avoid resonance whenever possible, in high speed applications it is sometimes inevitable that the elastic system during its normal operation must pass through the resonance domain. In such cases the only practical possibility is to try to avoid sustained resonance. Recognising the engineering importance of this problem a separate section is devoted to the discussion of the transition through resonance.

The present Item extends the scope of the earlier Items to impact loading.

In the majority of machines, particularly those which execute alternating motion, impacts occur during their normal operation. Often displacement impacts are small, such as clearances in bearings or joints, in other cases the impacts are an integral part of the normal functioning of the machines or mechanisms, for example in vehicle suspensions systems, valves of internal combustion engines, forging hammers, firearms, *etc.* In these latter cases springs are normally employed to absorb or store the energy of the impact. When designing machine components for impact loading, the stresses and deformations in the components must be considered. It is also necessary to find out what effects these stresses and deformations have on the materials involved.

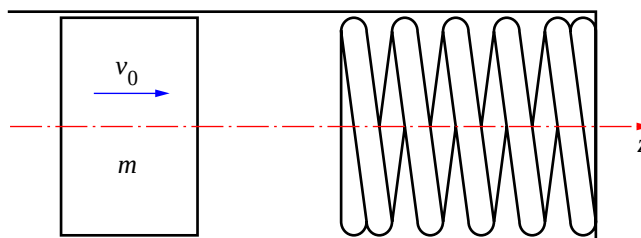
The classical theory of impact regards the bodies involved as rigid and the impact as being instantaneous and so it is suitable only for determining the kinetic consequences of the impact. When the impact process itself is to be investigated, *i.e.* its duration and the deformations, forces and stresses involved, much more advanced theories must be used. However, when the dynamic behaviour of only a spring mass system after impact is the subject of the investigation, and not that of the spring itself, the classical theory of impact proves to be very useful.

As far as the impact process is concerned, there are two extreme idealised cases, the perfectly non elastic impact and the perfectly elastic impact. The reality lies somewhere in between these two extremes. Accordingly, in practical calculations a so-called “impact coefficient”, designated κ , is introduced which has the values in the range $0 \leq \kappa \leq 1$. The value of κ must be determined by experiment. In the case of a perfectly non elastic impact its value is zero, *i.e.* $\kappa = 0$, and in the case of a perfectly elastic impact, $\kappa = 1$.

In the first part of this Item, the impact on spring mass systems is investigated using the classical theory of impact. In the latter part of the Item, the problem of an impact on an elastic rod is considered. This latter problem has a special practical significance, since an elastic rod can be used as an “equivalent rod” representing a helical spring.

3. IMPACT LOAD APPLIED TO FREE END OF SPRING

The dynamic system investigated in this section is shown in Sketch 3.1a. The system consists of a concentrated mass m and a cylindrical helical spring, both constrained to move in the horizontal plane. The mass of the spring is assumed to be negligible in comparison with the concentrated mass. In the Sketch the spring ends are shown closed and ground (see ESDU 06024 ^[1]).



Sketch 3.1a Spring and mass in horizontal plane

The concentrated mass m moves at a velocity v_0 towards the spring as shown in Sketch 3.1a. After making contact with the spring the mass continues to move, compressing the spring. Meanwhile the increasing spring force gradually slows the mass. At the instant when the mass comes to a halt its kinetic energy has been converted completely into potential energy stored in the spring and the force between the mass and the spring has reached a maximum. Subsequently, under the influence of the spring force, the mass starts to move backwards. During this reverse motion, the force of the spring decreases until the spring expands to its original length and contact between the mass and the spring ceases. The mass now moves away from the spring in the opposite direction. If energy losses are neglected, the velocity of the mass becomes $-v_0$, the same value in magnitude as its original velocity, but opposite in direction.

Since the mass and the spring do not lose contact during the impact, the process can be regarded as a free vibration of a spring mass system of stiffness k , with a natural frequency of

$$\omega_n = (k / m)^{1/2}.$$

The displacement of the mass at any moment of time is given by

$$\xi = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t). \quad (3.1)$$

If the impact process starts at time $t = 0$, the initial conditions are

$$\xi_{t=0} = 0 \quad \text{and} \quad \left(\frac{d\xi}{dt} \right)_{t=0} = v_0.$$

The values of the constants, C_1 and C_2 , can be determined from Equation (3.1) and its first time -derivatives as

$$\xi_{t=0} = \xi_0 = C_1 \cos 0 + C_2 \sin 0 = 0, \quad \text{from which } C_1 = 0$$

$$\text{and} \quad \left(\frac{d\xi}{dt} \right)_{t=0} = v_0 = -C_1 \omega_n \sin 0 + C_2 \omega_n \cos 0, \quad \text{hence } C_2 = v_0 / \omega_n.$$

Inserting these into Equation (3.1), the displacement becomes

$$\xi = \frac{v_0}{\omega_n} \sin(\omega_n t). \quad (3.2)$$

The displacement reaches its maximum value when $t = t_1 = \pi / (2\omega_n) = T/4$. Inserting this value of t_1 into Equation (3.2), the maximum value of the displacement is given by

$$\xi_{max} = \frac{v_0}{\omega_n} \sin\left(\omega_n \frac{\pi}{2\omega_n}\right) = \frac{v_0}{\omega_n}. \quad (3.3)$$

At the instant when $t = t_2 = \pi / \omega_n = T/2$, the displacement, ξ , becomes zero and the mass, m , now travelling in the opposite direction leaves the spring behind, moving at a velocity of

$$\left(\frac{d\xi}{dt}\right)_{t=t_2} = \frac{v_0}{\omega_n} \omega_n \cos(\omega_n t) = v_0 \cos \pi = -v_0.$$

The spring force, F_z , has its maximum value when the displacement is also at its maximum, *i.e.*

$$(F_z)_{max} = \xi_{max} k = \frac{v_0}{\omega_n} k. \quad (3.4)$$

The maximum values of both the displacement and the force can also be expressed in terms of the kinetic energy, $T_0 (= \frac{1}{2} m v_0^2)$, of the mass, m , as

$$\xi_{max} = \frac{v_0}{\omega_n} = \frac{v_0}{(k/m)^{1/2}} = \left(\frac{m v_0^2}{k}\right)^{1/2} = \left(\frac{2 T_0}{k}\right)^{1/2} \quad (3.5a)$$

and
$$(F_z)_{max} = \frac{v_0}{\omega_n} k = \left(\frac{2 T_0}{k}\right)^{1/2} k = (2 T_0 k)^{1/2}. \quad (3.5b)$$

From Equations (3.3) and (3.4) it is clear that both ξ_{max} and $(F_z)_{max}$ are explicitly independent of the mass m . Also from Equations (3.5a) and (3.5b) they are seen to be functions of the kinetic energy, T_0 , and the spring stiffness, k , only. This conclusion is true only for single degree-of-freedom systems, since in multi degree-of-freedom systems the individual masses reach their maximum displacements at different times.

If the spring mass system moves in the vertical plane, as shown in Sketch 3.1b, the displacements and forces become larger compared with the horizontal motion. The reason is that, from the instant of the impact not only is the kinetic energy of the concentrated mass being converted into potential energy of the spring, but also into work done by the gravitational force of the mass over the whole displacement, ξ , caused by the impact.

The displacement Equation (3.1) is still valid in this case also, provided the displacements are measured from the static equilibrium position of the spring mass system as indicated in Sketch 3.1b. Let the time at the instant of the impact be $t = 0$. At that moment the mass m has a velocity of

$$\left(\frac{d\xi}{dt}\right)_{t=0} = v_0$$

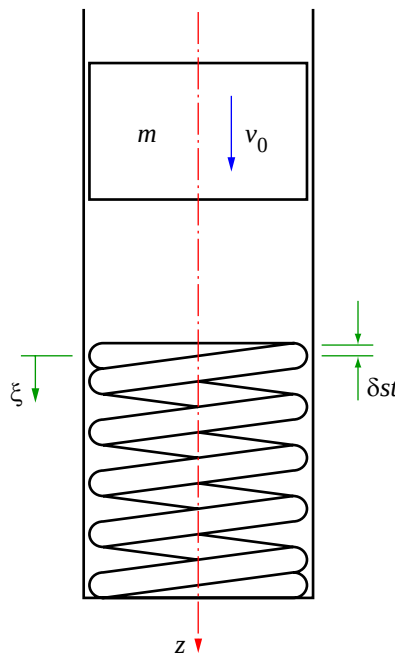
and a displacement, measured from the static equilibrium position (see Sketch 3.1b) of

$$\xi_{t=0} = -\delta_{st},$$

where δ_{st} is the static compression of the spring under the weight of the mass, m . Now, with the use of Equation (3.1) the values of the constants, C_1 and C_2 , can be found. At time, $t = 0$

$$\xi_{t=0} = \xi_0 = C_1 \cos 0 + C_2 \sin 0 = -\delta_{st}, \quad \text{hence } C_1 = -\delta_{st}$$

and $\left(\frac{d\xi}{dt}\right)_{t=0} = -C_1 \omega_n \sin 0 + C_2 \omega_n \cos 0 = v_0$, from which $C_2 = v_0 / \omega_n$.



Sketch 3.1b Spring and mass in vertical plane

Substituting these values back in Equation (3.1), the displacement becomes

$$\xi = -\delta_{st} \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t). \quad (3.6a)$$

With use of the trigonometrical identity

$$A \cos \alpha + B \sin \alpha \equiv \left(A^2 + B^2\right)^{1/2} \cos\left(\alpha - \tan^{-1}\left(\frac{A}{B}\right)\right),$$

Equation (3.6a) can be re-written in the form

$$\xi = \left(\delta_{st}^2 + \left(v_0 / \omega_n\right)^2\right)^{1/2} \cos\left(\omega_n t - \tan^{-1}\left(\frac{-v_0}{\omega_n \delta_{st}}\right)\right). \quad (3.6b)$$

The maximum displacement, ξ_{max} , occurs when

$$\cos\left(\omega_n t - \tan^{-1}\left(\frac{-v_0}{\omega_n \delta_{st}}\right)\right) = 1, \text{ or } \omega_n t - \tan^{-1}\left(\frac{-v_0}{\omega_n \delta_{st}}\right) = 0.$$

Hence

$$\xi_{max} = \left(\delta_{st}^2 + \left(v_0/\omega_n\right)^2\right)^{1/2}. \quad (3.7)$$

The maximum displacement occurs at time, t_1 , given by

$$t_1 = \frac{1}{\omega_n} \tan^{-1}\left(\frac{-v_0}{\omega_n \delta_{st}}\right) = \frac{1}{\omega_n} \left(\frac{\pi}{2} + \tan^{-1}\left(\frac{\omega_n \delta_{st}}{v_0}\right)\right). \quad (3.8)$$

The total displacement, ξ_{total} , is simply the sum of the static, δ_{st} , and the dynamic, ξ_{max} , displacements and so

$$\xi_{total} = \xi_{st} + \xi_{max} = \xi_{st} + \left(\xi_{st}^2 + \left(v_0/\omega_n\right)^2\right)^{1/2}. \quad (3.9a)$$

The corresponding maximum spring force is

$$(F_z)_{max} = \xi_{total} k = G + \left(G^2 + \left(\frac{v_0}{\omega_n} k\right)^2\right)^{1/2}, \quad (3.10a)$$

in which $G (= \delta_{st} k)$ is the force due to the weight of the mass m .

The total duration of the impact process is

$$t_2 = 2 t_1 = \frac{2}{\omega_n} \left(\frac{\pi}{2} + \tan^{-1}\left(\frac{\omega_n \delta_{st}}{v_0}\right)\right). \quad (3.11)$$

The total dynamic displacement and the corresponding maximum force, given by Equations (3.9a) and (3.10a), can also be derived from energy considerations for vertical impact. The maximum potential energy, U_{max} , stored in the spring is equal to the sum of the kinetic energy of the mass, T_0 , just before the impact and the work done by its gravitational force over the total travel, *i.e.*

$$U_{max} = T_0 + G \xi_{total}. \quad (3.12)$$

But the maximum potential energy of the spring can also be expressed in terms of the total displacement and the spring stiffness, given by

$$U_{max} = \frac{1}{2} \xi_{total}^2 k.$$

Inserting the above quantity into Equation (3.12) and noting that the gravitational force, $G (= \delta_{st} k)$, the following quadratic equation is obtained

$$\xi_{total}^2 - 2 \delta_{st} \xi_{total} - \frac{2 T_0}{k} = 0.$$

The positive root of this equation provides the desired result for the total displacement

$$\xi_{total} = \delta_{st} + \left(\delta_{st}^2 + \frac{2 T_0}{k} \right)^{1/2} \quad (3.9b)$$

and the maximum force, $(F_z)_{max}$, is then

$$(F_z)_{max} = \xi_{total} k = \delta_{st} k + \left((\delta_{st} k)^2 + 2 T_0 k \right)^{1/2}. \quad (3.10b)$$