



Introduction to Damping

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INTRODUCTION TO DAMPING

1. NOTATION

		Units	
		SI	British
A	cross-sectional area of rod	m^2	in^2
A_0	amplitude		
A_1, B_1	constants		
a, b	major and minor lengths of ellipse		
$[C]$	viscous damping constant matrix	Ns/m	lbf s/in
C_1, C_2	constants		
c	viscous damping constant	Ns/m	lbf s/in
c_{crit}	critical viscous damping constant	Ns/m	lbf s/in
c_{eq}	equivalent viscous damping constant	Ns/m	lbf s/in
E_c	complex Young's modulus	Pa	lbf/in ²
E	Young's modulus	Pa	lbf/in ²
F	applied force	N	lbf
F_c	Coulomb friction force	N	lbf
F_h	hysteretic damping force	N	lbf
F_0	amplitude of applied force	N	lbf
$\tilde{F}_0$	dimensionless force amplitude		
f	generalized (modal) force	N	lbf
f_A, f_B, f_C, f_D	frequencies corresponding to points on fitted circle (Sketch 9.3)	Hz	Hz
f_n	undamped natural frequency	Hz	Hz
f_1, f_2	frequencies corresponding to half power points	Hz	Hz
g_1	constant in Rayleigh damping	s^{-1}	s^{-1}

g_2	constant in Rayleigh damping	s	s
h	hysteretic damping constant	N/m	lbf/in
$[K]$	stiffness matrix	N/m	lbf/in
k	spring stiffness	N/m	lbf/in
k_c	complex spring stiffness	N/m	lbf/in
$[k]$	diagonalized spring stiffness matrix	N/m	lbf/in
l	length of rod	m	m
$[M]$	mass matrix	kg	*
M	dynamic magnification factor		
M_{max}	maximum value of dynamic magnification factor		
M_{res}	value of dynamic magnification factor at resonance		
m	mass of rigid body	kg	*
$[m]$	diagonalized mass matrix	kg	*
n	number of degrees of freedom; number of cycles		
$[P]$	modal matrix	m	in
Q	quality factor		
q	generalized (modal) coordinates	m	in
R	ratio of amplitude of response to amplitude of peak (for SDOF system)		
r	$(\omega/\omega_n)^2$		
t	time	s	s
u_1	amplitude of displacement of longitudinal wave at position 1	m	in
u_2	amplitude of displacement of longitudinal wave at position 2	m	in
v_0	initial velocity	m/s	in/s
W	maximum potential energy	J	lbf in
$\tilde{W}$	dimensionless energy		
x	displacement from equilibrium position	m	in

$\tilde{x}$	dimensionless displacement		
x_0	initial displacement	m	in
X_s	static deflection	m	in
X_0	amplitude of displacement response	m	in
$\tilde{X}_0$	dimensionless amplitude		
α	attenuation coefficient or constant	m ⁻¹	in ⁻¹
γ	viscosity coefficient (= cI/A)	Ns/m ²	lbfs/in ²
γ_s	spatial attenuation rate	m ⁻¹	in ⁻¹
ΔW	energy loss per cycle in steady-state vibration	J	lbf in
ΔW_c	energy loss per cycle in steady-state vibration with Coulomb damping	J	lbf in
ΔW_h	energy loss per cycle in steady-state vibration with hysteretic damping	J	lbf in
ΔW_v	energy loss per cycle in steady-state vibration with equivalent viscous damping	J	lbf in
$\Delta \tilde{W}$	dimensionless energy loss per cycle		
$\Delta \omega$	bandwidth	rad/s	rad/s
δ	logarithmic decrement		
δ_s	logarithmic decrement for longitudinal waves		
ϵ	strain		
ϵ_0	amplitude of strain		
$\tilde{\epsilon}$	dimensionless strain		
ζ	viscous damping ratio		
ζ_{eq}	equivalent viscous damping ratio		
η	loss factor		
θ	phase lag of displacement of rigid body with respect to applied force	rad	rad
λ	variable in characteristic equation		
λ_s	wavelength of longitudinal wave	m	in

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ν	Poisson's ratio		
σ	stress	Pa	lbf/in ²
σ_0	amplitude of stress	Pa	lbf/in ²
$\tilde{\sigma}$	dimensionless stress		
φ	phase angle by which response lags input (loss angle)	rad	rad
ϕ	phase angle	rad	rad
ψ	specific damping capacity		
Ω	ω/ω_n		
Ω_{max}	value of Ω corresponding to M_{max}		
ω	circular frequency	rad/s	rad/s
ω_1	circular frequency at point on response curve where response is $1/\sqrt{2}$ of value at resonance ($\omega_1 < \omega_n$)	rad/s	rad/s
ω_2	circular frequency at point on response curve where response is $1/\sqrt{2}$ of value at resonance ($\omega_2 > \omega_n$)	rad/s	rad/s
ω_d	damped natural circular frequency	rad/s	rad/s
ω_{max}	circular frequency at which maximum value of M occurs	rad/s	rad/s
ω_n	undamped natural circular frequency	rad/s	rad/s

* In British units for the system to be coherent the mass in lb must be divided by the constant 386.4

Abbreviations

MDOF	multi-degrees of freedom
SDC	specific damping capacity
SDOF	single degree of freedom

2. INTRODUCTION

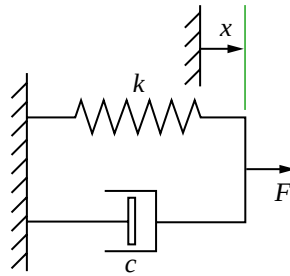
This Item is intended to provide introductory and background information to a series of Items that consider some aspects of damping in more detail. For example, ESDU 07011 (Reference 43) is concerned with radiation damping and ESDU 09006 (Reference 44) presents a description of material damping. Future Items will deal with the damping of composites and the provision of a compendium of damping for built-up structures. In this Item concepts related to different measures applied to damping in materials, as well as to damping in mechanical systems, are introduced. Every structure in vibration can be approximated by a dynamically equivalent mechanical system with a finite number of degrees of freedom. The degrees of freedom refer to the number of independent coordinates necessary to describe the motion of the system. In particular situations a single degree of freedom system can be an adequate model for the real structure. Although the majority of structures cannot be treated as SDOF systems, the measures specifying damping can be defined using a SDOF model without compromising the generality of the definitions.

Damping in structures may be categorised as either intrinsic, *i.e.*, within the material of the structure, or as the damping which depends upon other dissipative factors, such as friction in rivets or bolts, or from lubrication (Reference 22). Damping also occurs in a structure which is radiating sound, as vibrational energy is lost in the surrounding medium rather than by dissipation in the structure. Material and radiation damping can be estimated theoretically more easily than other types of damping; in the case of material damping prediction is based on the material properties and for radiation damping from the geometrical and material properties of the structure. When the damping of a structure is measured in most practical tests, it is the damping of all kinds which is determined, including the material damping and, provided there is a surrounding fluid medium, radiation damping.

The measures of damping which are most often used in dynamic analysis are: the viscous damping ratio (ζ), the specific damping capacity (ψ), the loss factor (η), the inverse quality factor (Q^{-1}), the logarithmic decrement (δ) and, in the case of material damping, the tangent of the phase lag (ϕ). and the ratio of the loss modulus to the Young's modulus. Definitions of the measures are introduced in the Item using examples of linear mechanical models, such as Kelvin-Voigt (with viscous damper) and hysteretic damping model (with hysteretic damper), especially applicable to material damping. In ESDU Items information on damping is mostly provided in terms of the viscous damping ratio. For example, in Data Item No. 72005, 'The estimation of r.m.s. stress in stiffened skin panels subjected to random acoustic loading' (Reference 20), nomographs are used which are based on a viscous damping ratio of 0.017, a value regarded as typical for aerospace structures, and, if a different value is used, a correction factor is applied. The relationships between the different measures of damping are summarised in the Appendix A.

3. KELVIN-VOIGT MODEL

The Kelvin-Voigt model is formed by a combination of a viscous damper and an elastic spring in parallel, as shown in Sketch 3.1 (References 13, 21, 23 and 29). Note that the name Kelvin-Voigt is not universally used. For example, Zener, Kolsky and Nowick (References 3, 9 and 19) refer to it as the Voigt solid, Thomson as the Voigt model (Reference 29), whereas Christensen calls it the Kelvin model (Reference 40). Bishop uses the term viscous damping for the system in Sketch 3.1, without introducing the term Kelvin-Voigt (Reference 4). Originally devised to explain viscoelastic behaviour of materials (References 9 and 40), the model can be extended to provide a basis for the explanation of experimental results for damping in structures as a whole. When combined with a rigid body, it forms a single degree of freedom system which is described in detail in all standard texts on vibration, (References 7, 12, 25, 26, 27, 29 and 32), and in Section 6 of this Item.



Sketch 3.1 Kelvin-Voigt model

In the case of a viscous damper, the damping force is proportional to the relative velocity of vibrating components, as described, for example, in the Item on 'Mobilities and impedances of structures', Reference 42, Section 2.2.3. A viscous damper alone is insufficient to model material damping and, hence, the most elementary model is formed from a viscous damper in parallel with an elastic spring. In reality the damper and spring may be combined in a single structural entity.

If, in Sketch 3.1, the left side of the damper-spring element is fixed and on the right a force, F , is applied, the displacement from the equilibrium position, denoted by x , must fulfil the equation of motion given by

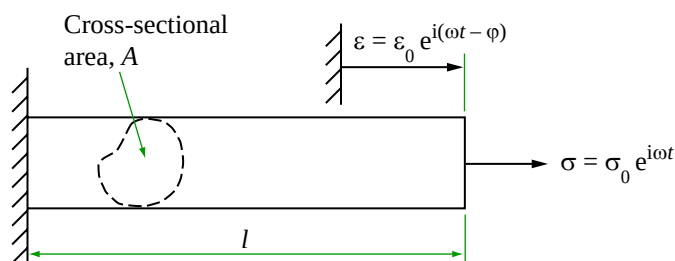
$$c\dot{x} + kx = F, \quad (3.1)$$

where c is the viscous damping constant of the viscous damper and k is the stiffness of the elastic spring element.

To extend the properties of the Kelvin-Voigt model to the behaviour of the material, Equation (3.1), may be written in terms of stress, σ , and strain, ϵ , if the spring and damper are replaced by an equivalent structure, for example, an elastic rod of length l and cross-sectional area A with an equivalent stiffness $k = EA/l$ and viscous damping constant $c = \gamma A/l$, as shown in Sketch 3.2. In this case Equation (3.1) leads to

$$\gamma\dot{\epsilon} + E\epsilon = \sigma, \quad (3.2)$$

where E is the Young's modulus of the material of the rod and γ is a viscosity coefficient. The specimen represents a simple model of an anelastic solid (References 3 and 19).



Sketch 3.2 Resilient element

3.1 Phase Lag (or Loss Angle)

For a sinusoidal applied force of amplitude, F_0 , and applied frequency, ω , the force, F , in Equation (3.1) may be written in the complex exponential form as,

$$F = F_0 e^{i\omega t}. \quad (3.3)$$

Since the system is linear and in steady-state vibration, the displacement response is assumed to be

$$x = X_0 e^{i(\omega t - \varphi)}, \quad (3.4)$$

where X_0 is the displacement amplitude and φ is the phase angle by which the response lags the input and is sometimes called the loss angle. Substitution of F and x from Equations (3.3) and (3.4) into Equation (3.1) leads to the solution for X_0 in terms of F_0 and the phase angle.

$$X_0 = \frac{F_0}{k} \frac{1}{\sqrt{1 + \tan^2 \varphi}}, \quad (3.5)$$

where

$$\tan \varphi = \frac{c\omega}{k}. \quad (3.6)$$

In the context of material damping the Kelvin-Voigt model, applied to a resilient element or structure, results in Equation (3.2). For forced, harmonic excitation, when it is assumed that stress leads strain by the angle φ (loss angle), see Sketch 3.2, the relationship between the amplitudes of stress and strain (σ_0 and ε_0 , respectively) is presented, similarly as in Equation (3.5), by

$$\varepsilon_0 = \frac{\sigma_0}{E} \frac{1}{\sqrt{1 + \tan^2 \varphi}}, \quad (3.7)$$

and the tangent of the loss angle is represented in this case in terms of the viscosity coefficient of Equation (3.2) by

$$\tan \varphi = \frac{\gamma\omega}{E}. \quad (3.8)$$

3.2 Energy Loss Per Cycle

The energy lost per cycle by the viscous damper in steady-state vibration, ΔW , is given by

$$\Delta W = \int_0^{2\pi/\omega} \text{Re}(F) \cdot \text{Re}(\dot{x}) dt, \quad (3.9)$$

where the displacement x lags in phase behind the force F . The integration of the product of the real part of F and the real part of $\dot{x}$, performed over time, where F and $\dot{x}$ are defined by Equations (3.3) and (3.4), respectively, leads to

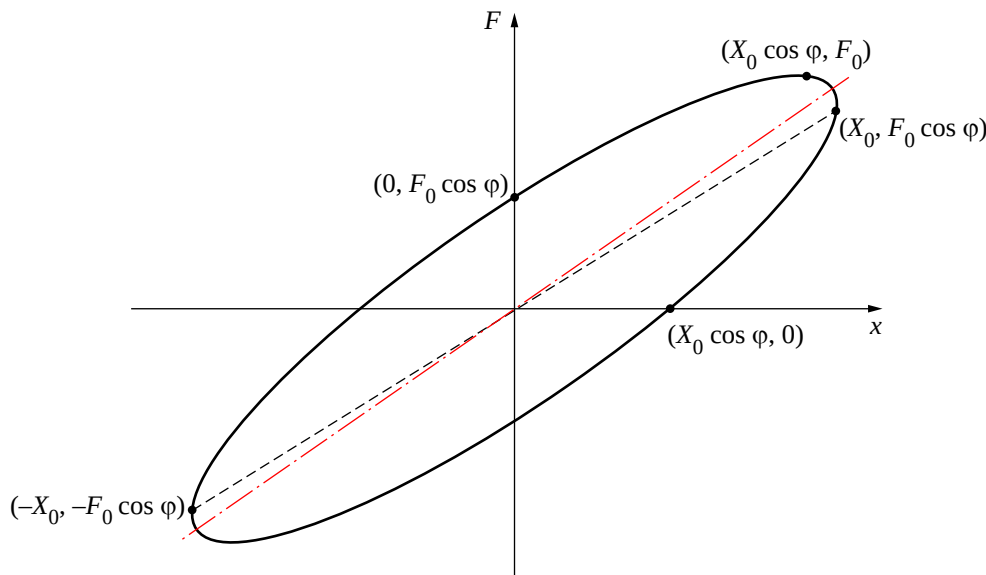
$$\Delta W = \pi X_0^2 k \tan \varphi \quad (3.10)$$

and from Equations (3.6) and (3.10)

$$\Delta W = \pi X_0^2 c \omega. \quad (3.11)$$

The energy loss per cycle of vibration is proportional to the square of the displacement amplitude, the driving frequency and the viscous damping constant.

The energy dissipated by a damper in the Kelvin-Voigt element during a cycle of harmonic vibration can be represented by the area within the so-called hysteresis loop, which in this case is an inclined ellipse in the (x, F) plane, as shown schematically in Sketch 3.3. The energy lost is $\pi X_0^2 c \omega$. The line joining the points with coordinates $(X_0, F_0 \cos \varphi)$ and $(-X_0, -F_0 \cos \varphi)$ has the slope k . The smaller the angle φ , the 'thinner' the ellipse becomes, until with no damping there is a straight line of slope k .



Sketch 3.3 Schematic diagram of elliptical hysteresis loop

It should be noted for any linear system in steady-state harmonic vibration the hysteresis loop is an ellipse (References 7, 34 and 36).

3.3 Specific Damping Capacity and Loss Factor

A frequently used measure of damping, which is defined in terms of the energy loss during a cycle of forced vibration, is the specific damping capacity, ψ , the ratio of the energy loss, ΔW , to the maximum potential

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energy, W , stored in the elastic component of the system during the cycle (References 14 and 28), i.e.,

$$W = \frac{\Delta W}{W}. \quad (3.12)$$

Since,

$$W = \frac{1}{2}kX_0^2 \quad (3.13)$$

and, as the energy loss, ΔW , is specified by Equation (3.10) for the Kelvin-Voigt model, the specific damping capacity (SDC) is given by

$$\psi = 2\pi \tan \phi. \quad (3.14)$$

It should be noted that there are several ways of defining the energy, W ; for example, it can be defined as the total vibrational energy of the system.

Another measure of damping is the loss factor, η , (called also the loss coefficient), which is closely related to the SDC. It is defined as the ratio of the dissipated energy per radian of the cycle, $\Delta W/2\pi$, to W , the maximum potential energy reached during the cycle. Hence,

$$\eta = \frac{\Delta W}{2\pi W}. \quad (3.15)$$

From the definitions of specific damping capacity, Equation (3.12), and loss factor, Equation (3.15)

$$\eta = \frac{\psi}{2\pi} \quad (3.16)$$

and from Equation (3.14)

$$\eta = \tan \phi. \quad (3.17)$$

In the case of the Kelvin-Voigt model the loss factor is specified, Equation (3.6), as,

$$\eta = \frac{c\omega}{k}. \quad (3.18)$$

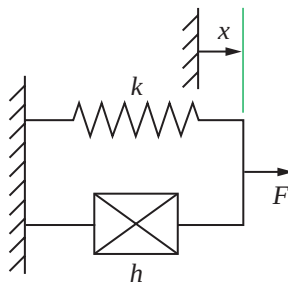
Similarly, for material representation, see Equation (3.8), the loss factor can be expressed as,

$$\eta = \frac{\gamma\omega}{E}. \quad (3.19)$$

The dependence of the loss factor on frequency or, more generally the dissipated energy on frequency, exists for certain viscoelastic materials, *e.g.*, rubber, within a limited range of frequencies (Reference 15). However, for the majority of commonly used materials, frequency independence is observed (References 1, 9, 14 and 34).

4. HYSTERETIC DAMPING

In the previous section damping mechanisms associated with viscous damping were discussed and it was mentioned that for most common materials the energy loss per cycle was frequency independent. Hence, there is limited use for the Kelvin-Voigt element, where the energy loss per cycle is proportional to frequency, as a model for material damping. Consequently, the concept of the hysteretic damper was introduced. It replaces the viscous damper in the Kelvin-Voigt element, as shown in Sketch 4.1. The representation of the damper in Sketch 4.1 was introduced by Bishop and Johnson (Reference 7).



Sketch 4.1 Hysteretic damper with damping coefficient h in parallel with a spring of stiffness k

The hysteretic damper opposes the relative motion between its ends with a force F_h , which is proportional to displacement x , but is in phase with velocity. The condition required by the definition is achieved, if

$$F_h = \frac{h}{\omega} \dot{x}, \quad (4.1)$$

where h , the hysteretic damping coefficient, is a constant. The hysteretic model, which applies only to harmonic excitation, leads to the equation

$$\frac{h}{\omega} \dot{x} + kx = F, \quad (4.2)$$

which is compared with Equation (3.1). The steady-state solution for displacement, x , in Equation (4.2) for forced, harmonic vibration have the same form as the solution for displacement occurring in Equation (3.1), i.e., similar to Equation (3.4) and the associated Equation (3.5). However, the loss angle, as introduced in Equation (3.4) and specified by Equation (3.6), in this case is independent of frequency and is given by

$$\tan \phi = \frac{h}{k}. \quad (4.3)$$

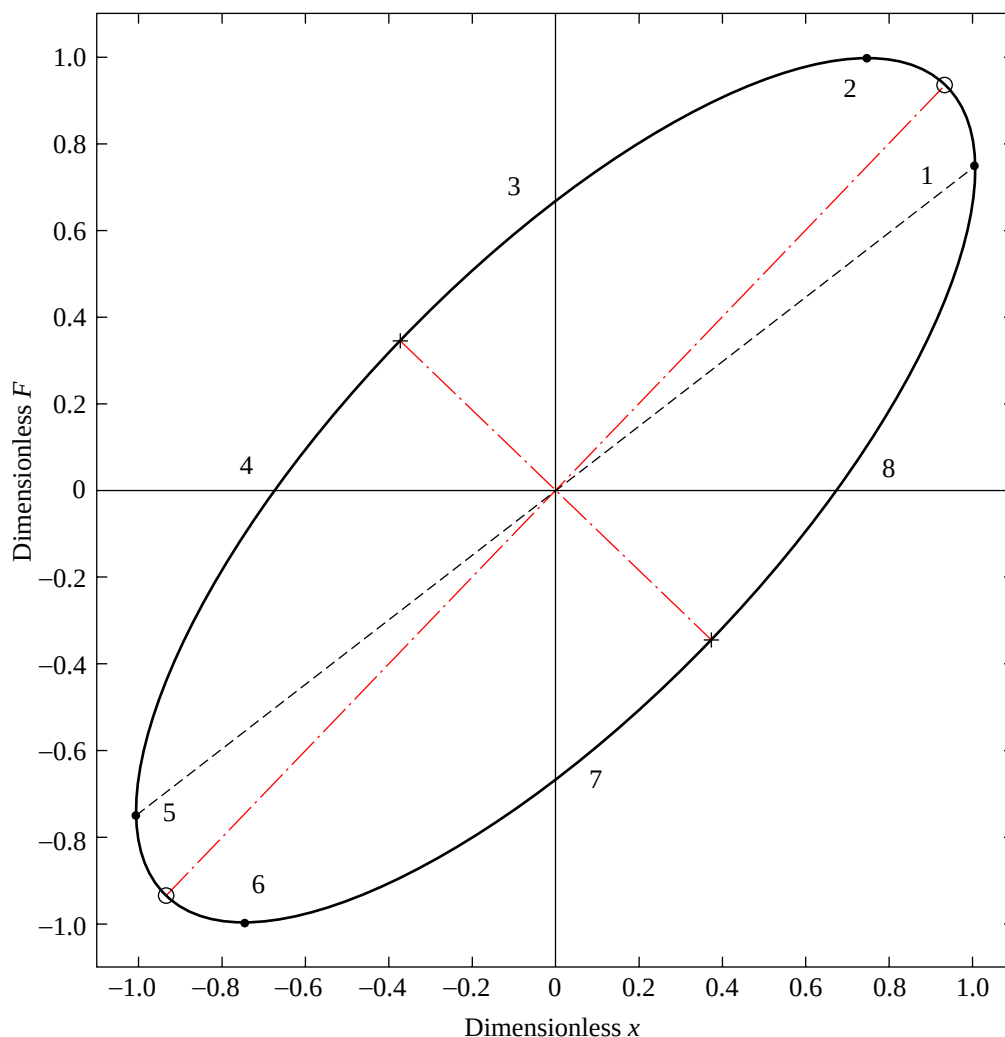
As $\eta = \tan \phi$, it follows that

$$\eta = h/k. \quad (4.4)$$

This type of damper-spring system describes more adequately the behaviour of real materials, the loss factors of which are either frequency independent or are weakly dependent on frequency (References 33 and 34).

4.1 Hysteresis Loops

The energy loss per cycle is represented by the area enclosed in the (x, F) plane as displacement changes with the applied force. The hysteresis loop formed is an ellipse for any linear system in steady-state harmonic vibration, see Section 3.1. It is convenient to introduce a dimensionless displacement-force $(\tilde{x}, \tilde{F})$ coordinate system (or strain-stress $(\tilde{\epsilon}, \tilde{\sigma})$ dimensionless coordinate system) which standardizes the appearance of the hysteresis loops as functions of the loss factor.



Sketch 4.2 Dimensionless representation of energy loss per cycle

The dimensionless coordinates $(\tilde{x}, \tilde{F})$ (alternatively, $(\tilde{\epsilon}, \tilde{\sigma})$) are defined as $\tilde{x} = x/X_0$, $\tilde{F} = F/F_0$ (or $\tilde{\epsilon} = \epsilon/\epsilon_0$, $\tilde{\sigma} = \sigma/\sigma_0$), where X_0 and F_0 denote the amplitudes of the displacement and force, respectively, and ϵ_0 and σ_0 are the amplitudes of the strain and stress, respectively. The area of the ellipse, shown in Sketch 4.2, represents the dissipated energy per cycle of harmonic vibration in a dimensionless form, i.e., $\Delta \tilde{W} = \Delta W/(F_0 X_0)$. For example, the energy dissipated by materials with hysteretic damping, as well as those with viscous damping, is expressed, by means of Equations (3.5) and (3.10), as,

$$\Delta \tilde{W} = \pi \sin \phi, \quad (4.5)$$

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or, as $\eta = \tan \phi$,

$$\Delta \tilde{W} = \pi \frac{\eta}{\sqrt{1 + \eta^2}}. \quad (4.6)$$

In the case of materials with viscous damping incorporated into the model, the loss factor, η , is dependent on the frequency of the forced vibrations. Finally, the maximum potential energy reached during the cycle (in dimensionless form, $\tilde{W} = W/F_0 X_0$) is expressed by means of Equations (3.5) and (3.13) as,

$$\tilde{W} = \frac{1}{2\sqrt{1 + \eta^2}}. \quad (4.7)$$

The major axis of length $2a$ is inclined to the displacement coordinate by an angle $\pi/4$ and has a length given by

$$2a = 2\sqrt{1 + \cos \phi}. \quad (4.8)$$

The minor axis of the ellipse is

$$2b = 2\sqrt{1 - \cos \phi}. \quad (4.9)$$

The area under the ellipse is $\Delta \tilde{W}$ and is given by πab , which from Equations (4.8) and (4.9) leads to $\pi \sin \phi$ (Equation (4.5)).

Apart from the major and minor axes, some significant points on the ellipse are indicated in Sketch 4.2. At the points denoted by 1 with coordinates $(1, \cos \phi)$ and 5 with coordinates $(-1, -\cos \phi)$, $d\tilde{F}/d\tilde{x} = \infty$, whilst at points indicated by 2 and 6, $d\tilde{F}/d\tilde{x} = 0$. Numbers 3 and 7 mark the intersections of the ellipse with the coordinate axis $\tilde{F}$ and have the coordinates $(0, \sin \phi)$ and $(0, -\sin \phi)$, respectively. The line joining point 1 with coordinates $(1, \cos \phi)$ and the centre of the coordinate system $(0, 0)$ is inclined to the displacement coordinate axis by the angle

$$\alpha = \arctan \left(\frac{1}{\sqrt{1 + \eta^2}} \right). \quad (4.10)$$

In the dimensional (x, F) representation the slope of this line refers to the stiffness of the system, see Sketch 3.1.

For nonlinear systems the hysteretic loops can have various, non-elliptical shapes (References 14 and 37), usually with pointed joints at the loading and unloading branches of the loop.