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Elastic stress concentration factors. Geometric discontinuities in flat bars or strips of isotropic material

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ELASTIC STRESS CONCENTRATION FACTORS. GEOMETRIC DISCONTINUITIES IN FLAT BARS OR STRIPS OF ISOTROPIC MATERIAL

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ELASTIC STRESS CONCENTRATION FACTORS. GEOMETRIC DISCONTINUITIES IN FLAT BARS OR STRIPS OF ISOTROPIC MATERIAL

1. INTRODUCTION

This Item presents data on stress concentration effects that occur in flat bars or strips when they are loaded in tension or bending. A stress concentration factor is defined here as the ratio of the highest principal stress to a reference stress calculable from simple two-dimensional theory. Throughout this Item, the term 'bars or strips' applies to bodies having solid, thin rectangular sections.

The data apply only to isotropic materials that obey Hooke's Law. However, over the practical working range most engineering materials conform substantially in these respects and for these the data may be used without significant error. If the stress concentration is such that the maximum stress is above the limit of proportionality of the material, stress is redistributed and creates residual stress on unloading.

The various geometries for which data are presented are listed on Table 1.1. The gross stress concentration factor, K , is based on a reference stress found using the gross cross-section of the bar or strip (ignoring the discontinuity). The symbol K' indicates that the stress concentration factor is based on the net cross-section at the discontinuity. Table 1.1 also indicates the methods used to obtain stress concentration factors in the various derivations. The term 'analytical' implies that the relevant elasticity equations were solved directly for the range of parameters indicated. For a bar or strip of any thickness, loaded in tension and containing a hole, the stress concentration is a maximum at the mid-plane of the bore of the hole (and is greatest for thick bar or strips when the ratio of the hole diameter, d , to the thickness, t , of the bar or strip, d/t , is greater than 0.5) and is a minimum at the free surface: an effect that becomes more pronounced when $d/t < 2.0$, see ESDU 93030*.

The term 'closed-form equation' is defined in this Item as an explicit expression for the stress concentration factor. The equations presented are based in some cases on empirical curve-fitting to finite element, photoelastic or analytically based data, and in the other cases, designated 'based on crack solutions', the equations have a partially analytical basis. Curves from closed-form equations are presented for Figures 3, 8a, 8b, 8c, 11a, 11b, 11c, 12a, 12b, 12c, 15a and 15b (see Derivations 34, 31, 32, 33 and 35) which deal with a centrally placed hole or variously shaped notches and fillets that lie within specified bounds.

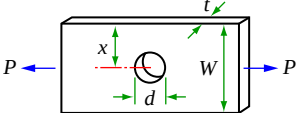
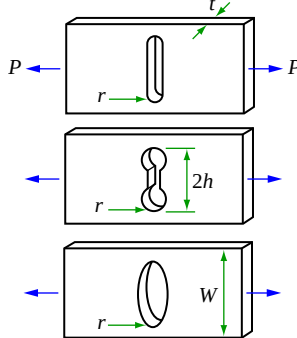
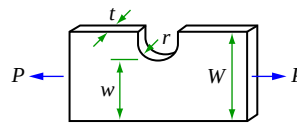
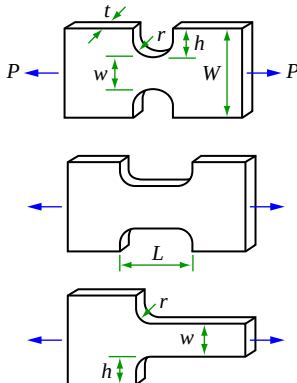
The fracture mechanics approach underlying the 'closed-form equations based on crack solutions' is a special case of the 'analytical' method, as it is based on elastic solutions for sharp-ended ellipses, cracks or sharp corners. If the radius of the notch involved is small, so that the region of concentrated stress is small compared to the other in-plane dimensions of the geometry, including the notch depth, a solution based on results for a crack with rounded ends will be accurate, becoming exact in the limit of a mathematically sharp crack. This method can therefore be applied to any sharp stress concentration for which a corresponding sharp-crack Linear Elastic Fracture Mechanics (LEFM) solution is available (see Derivation 8 and ESDU 80036†).

* ESDU 93030 'Three-dimensional elastic stress concentration factors. Plain or countersunk hole in a wide plate subjected to tension, bending or pin loading'.

† ESDU 80036 'Introduction to the use of linear elastic fracture mechanics in estimating fatigue crack growth rates and residual strength of components'.

Reference 36 discusses the use of photoelasticity for the determination of stress concentration factors at sharp notches and concludes that the method tends to underestimate stress concentration factors unless the thickness of the test specimen is reduced in step with the notch tip radius. It demonstrated that the thickness should be less than 1.5 times the radius to ensure accurate data and that this may be impractical for very sharp notches. The earlier photoelastic results used in this Item (see Figures 1 to 7 and 13 to 15a) tend to give stress concentration factors up to 10 per cent lower than those given by finite element analysis. Recent photoelastic data used in Figure 11a agree completely with the finite element data for fillets in tension, for a ratio of maximum to minimum width, W/w of 2.0 and r/w greater than 0.1, where w is the minimum width of the bar or strip and r is the radius of the fillet (see Derivations 29 and 30). For geometries outside this range, and for other cases presented in Figures 8a to 12a, the agreement between the results decreases as r/w decreases, *i.e.* as the notch becomes sharper. These photoelastic stress concentration factors can be up to 17 per cent lower than those from the finite element determinations for r/w between 0.002 and 0.2. Given that the finite element results in this sharp notch regime align with the analytical solutions, the presentations are based on finite element data, where available, and the photoelastic results are then used to validate the data in regions where the sharp notch theory is less applicable.

TABLE 1.1 Discontinuities and Loading Considered

Form of discontinuity (shown in tension)		Section number	Loading	Position	Figure	Method(s) used to obtain data
Hole		2	Tension	Off-centre hole	1	Photoelastic and analytical methods
			Out-of-plane bending	Central hole	2	Photoelastic methods only
			Tension	Central hole	3	Photoelastic and analytical methods and closed-form equations
			In-plane bending	Central hole	4	Photoelastic and analytical methods
Transverse slot		3	Tension	Central slot	5	Photoelastic and analytical methods
Notch on one edge		4	Tension	U-notch	6	Photoelastic and analytical methods
			In-plane bending		7	Photoelastic and analytical methods
Notches and fillets on both edges		5	Tension	U-notch	8a 8b 8c	Finite element methods and closed-form equations
			Tension	Notch with $L/w = 0.25$	9	Finite element and analytical methods
			Tension	Notch with $L/w = 1.0$	10	Finite element and analytical methods
			Tension	Fillet	11a 11b 11c	Finite element and photoelastic methods and closed-form equations
			In-plane bending	U-notch	12a 12b 12c	Finite element methods and closed-form equations
			In-plane bending	Notch with $L/w = 0.25$	13	Photoelastic and analytical methods
			In-plane bending	Notch with $L/w = 1.0$	14	Photoelastic and analytical methods
			In-plane bending	Fillet	15a 15b	Photoelastic and analytical methods and closed-form equations

1.1 Notation

Both SI and British units are quoted but any coherent system of units may be used.

d	diameter of hole	m	in
f_{max}	maximum direct stress in bar or strip	N/m ²	lbf/in ²
f_{ref}	reference stress in bar or strip	N/m ²	lbf/in ²
f_z	stress at site z	N/m ²	lbf/in ²
h	dimension of notch, see Table 1.1	m	in
K	stress concentration factor based on gross cross-sectional stress		
K_1	stress concentration factor based on gross cross-sectional stress at position 1		
K_2	stress concentration factor based on gross cross-sectional stress at position 2		
K'_d	stress concentration factor based on net cross-sectional stress for a sharp, deep notch		
K'_s	stress concentration factor based on net cross-sectional stress for a shallow, elliptical notch		
K'_t	stress concentration factor based on net cross-sectional stress for modified Neuber formula (see Equation (5.6))		
K_z	stress concentration factor based on gross cross-sectional stress at site z defined by $f_z = K_z f_{ref}$ where:		
	$f_{ref} = P/Wt$ (for Figures 1 and 3)	N/m ²	lbf/in ²
	$f_{ref} = 6M/Wt^2$ (for Figure 2)	N/m ²	lbf/in ²
	$f_{ref} = 6M/W^2t$ (for Figure 4)	N/m ²	lbf/in ²
K'	stress concentration factor based on net cross-sectional stress defined by $f_{max} = K' f_{ref}$ where:		
	$f_{ref} = \frac{P}{(W-2h)t}$ (for Figure 5)	N/m ²	lbf/in ²
	$f_{ref} = \frac{6M}{w^2t}$ (for Figures 7 and 12a to 15b)	N/m ²	lbf/in ²
	$f_{ref} = \frac{P}{wt}$ (for Figures 6 and 8a to 11c)	N/m ²	lbf/in ²
L	dimension of notch, see Table 1.1	m	in
M	bending moment in bar or strip	N m	lbf in

m	constant in Equation (5.6)		
P	direct load in bar or strip	N	lbf
r	radius of end of slot, fillet or bottom of notch	m	in
t	thickness of bar or strip	m	in
W	maximum width of bar or strip	m	in
w	minimum width of bar or strip	m	in
Y	crack configuration factor (α in ESDU 80036)		
Y_n	notch configuration factor		
Y_c	square-corner configuration factor		
x	distance of centre of hole from nearer edge of bar or strip	m	in
α	$= 2\left(1 - \frac{r}{w}\right) = 2 - \frac{1}{\beta}$		
β	$= \frac{w}{2r} = \frac{h}{r}\left(\frac{1}{2h/W} - 1\right)$		

2. CIRCULAR HOLES

2.1 Notes

In Figure 1, values of K_1 are plotted against d/W for various values of x/W for a circular hole in a flat bar or strip loaded in tension. The maximum stress occurs at site 1 on the edge of the hole at the point nearest to the edge of the bar or strip.

In Figure 2, values of K_1 are plotted against d/W for various values of t/W for a hole centrally placed in a flat bar or strip subjected to out-of-plane bending. The maximum stress occurs at sites 1 on the edge of the hole at the points nearest to the edge of the bar or strip.

In Figure 3, values of K_1 and K_2 are plotted against d/W for a hole centrally placed in a flat bar or strip loaded in tension. The curve for K_1 is the $x/W = 0.5$ curve of Figure 1 and is re-plotted here to facilitate its use in conjunction with the curve for K_1 in Figure 4 in cases of combined loading. Similarly the curve for K_2 is presented for use with that for K_2 in Figure 4. The maximum stress occurs at sites 1 on the edge of the hole at the points nearest to the edge of the bar or strip.

In Figure 4, values of K_1 and K_2 are plotted against d/W for a hole centrally placed in a flat bar or strip subjected to in-plane bending. The maximum stress occurs at sites 1 shown in the diagram for d/W values above 0.46 and at sites 2 for lower values of d/W .

The data in Figures 1 to 4 do not apply to holes when they are filled with any rigid or elastic inclusion (*e.g.* a bolt, a rivet or a pin).

See Section 1 for notes on the accuracy of Figures 1 to 4.

TABLE 2.1 Index to data for circular holes

<i>Hole position</i>	<i>Loading</i>	<i>Figure number</i>	<i>Figure based upon Derivation</i>	<i>Closed-form equation based upon Derivation</i>
Off-centre	Tension	1	5 and 17	–
Centre	Out-of-plane bending	2	14 and 15	–
Centre	Tension	3	2, 10, 11, 19 and 24	34
Centre	In-plane bending	4	3, 13 and 19	–

2.2 Closed-form Equation for a Central, Circular Hole in a Bar or Strip Loaded in Tension

In Derivation 34, an approximate analytical method is used to find the gross stress concentration factor at the edge of the hole, K_1 , as:

$$K_1 = \frac{3}{1 - 1.25d^2/W^2 + 0.25d^4/W^4}. \quad (2.1)$$

Equation (2.1) is plotted on Figure 3.

3. TRANSVERSE SLOTS

3.1 Notes

In Figure 5, values of K' are plotted against $r/2h$ for various values of $2h/W$ for a transverse slot centrally placed in a bar or strip loaded in tension. The curves may be applied with sufficient accuracy for practical purposes to any of the three types of slot shown in the diagram, *i.e.* a parallel-sided slot with semi-circular ends, a narrow slot with enlarged circular ends and an elliptical slot. In each case, the maximum stress occurs at sites 1 at the ends of the slot.

The data in Figure 5 do not apply to holes when they are filled with any rigid or elastic inclusion (*e.g.* a bolt, a rivet or a pin).

Figure 5 is based on Derivations 1, 6, 18 and 27.

See Section 1 for notes on the accuracy of Figure 5.

4. U-NOTCH ON ONE EDGE

4.1 Notes

Values of K' are plotted against r/w for various values of W/w for a bar or strip with a single notch loaded in tension in Figure 6 and for a bar or strip with a single notch under in-plane bending in Figure 7.

In both cases the maximum stress occurs at the bottom of the notch and therefore f_{max} values due to combined tensile loading and in-plane bending may be obtained by direct addition.

See Section 1 for notes on the accuracy of Figures 6 and 7.

TABLE 4.1 Index to data for a U-notch on one edge

<i>Class of notch</i>	<i>Loading</i>	<i>Figure number</i>	<i>Figure based upon Derivation</i>
U-notch	Tension	6	4, 23, 26 and 28
U-notch	In-plane bending	7	4 and 23

5. NOTCHES AND FILLETS ON BOTH EDGES

5.1 Notes

Figures 8a to 11c give values of K' plotted against r/w for various values of W/w for a bar or strip with notches or fillets on both edges loaded in tension. Figures 12a to 15b give values of K' plotted against r/w for various values of W/w for a bar or strip with notches and fillets on both edges under in-plane bending.

TABLE 5.1 Index to data for notches or fillets on both edges

<i>Class of notch or fillet</i>	<i>Loading</i>	<i>Figure number</i>	<i>Figure based upon Derivation</i>	<i>Closed-form equation based upon Derivation</i>
U-notch	Tension	8a, 8b, 8c	30	31 and 33
Notch with $L/w = 0.25$		9	1, 4, 7 and 30	–
Notch with $L/w = 1.0$		10	1, 4, 7 and 30	–
Fillet		11a, 11b, 11c	29 and 30	32 and 35
U-notch	In-plane bending	12a, 12b, 12c	30	31 and 33
Notch with $L/w = 0.25$		13	4, 7, 9, 12 and 21	–
Notch with $L/w = 1.0$		14	4, 7, 9, 12 and 21	–
Fillet		15a, 15b	7, 9, 12 and 21	35

The maximum stresses due to tension and in-plane bending occur at sites 1 at the bottom of the U-notch in Figures 8a and 12a and in the fillet-radii – close to their junction with the straight edge of the reduced section – in the remainder of the figures.

5.2 Closed-form Equations for U-notches in a Bar or Strip Loaded in Tension

5.2.1 Based on crack solutions (Figure 8b)

In Derivation 31, finite element methods are used to determine stress concentration factors in planar notched bars or strips for a range of notch geometries. An LEFM crack configuration factor, Y , for a double edge crack in tension is then used to modify the linear elastic solution for an elliptical hole of finite tip radius in an infinite plate to give a gross stress concentration factor, K , of:

$$K = 1 + 2Y\sqrt{h/r}, \quad (5.1)$$

where
$$Y = \frac{1.122 - 0.561(2h/W) - 0.205(2h/W)^2 + 0.471(2h/W)^3 - 0.19(2h/W)^4}{\sqrt{1 - (2h/W)}}. \quad (5.2)$$

(Note that here Y is synonymous to α , the geometrical factor in the stress intensity factor, used in ESDU 80036.)

As stated in the introduction, this approach is only valid if the notch is relatively sharp, *i.e.* h/r is large. However, if this is not the case, the crack configuration factor Y can be replaced by a 'notch configuration factor', Y_n , identical to Equation (5.2) except that the initial term 1.122 in Equation (5.2) is replaced by:

$$1 + 0.122 \tanh \left[0.796 \sqrt{h/r} - 0.3832 \right].$$

The resulting notch configuration factor is thus dependent on h/r and when applied in Equation (5.1) it modifies the elliptical hole expression in a way that reflects the interaction of the stress concentrated region with the boundaries of the notched geometry.

The hyperbolic tangent approaches a limiting value of unity as $\sqrt{h/r}$ tends to a large value corresponding to a crack-like notch. Equation (5.1) is applicable over the range $0.025 \leq h/r \leq 100$ and $0.01 \leq 2h/W \leq 0.5$ but cannot be used if $r/w > 2Y_n^2/(W/w - 1)$.

From Equation (5.1), K is used to obtain a net stress concentration factor, K' :

$$K' = K(1 - 2h/W). \quad (5.3)$$

5.2.2 Based on curve-fitting (Figure 8c)

By fitting smooth curves to stress concentration factors derived for shallow and blunt or deep and sharp notches in a bar or strip, a formula, Equation (5.7), is found from which stress concentration factors in tension, K' , may be calculated. In most cases, these agree to within 1 per cent when compared to numerical results obtained using the body force method (see Derivation 33).

The stress concentration factor, K'_s , for a shallow notch loaded in tension when $K' \rightarrow K'_s$ and $w, W \rightarrow \infty$ and when $0 \leq \sqrt{h/r} \leq 1$ is:

$$K'_s = \left[1 - 0.127\sqrt{h/r} + 0.2908(\sqrt{h/r})^2 - 0.142(\sqrt{h/r})^3 \right] (1 + 2\sqrt{h/r}), \quad (5.4a)$$