



Effect of link length tolerances on the performance of four-bar linkages

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THE PREPARATION OF THIS DATA ITEM

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The Mechanisms Committee, which first met in 1975, is continuing the work that previously was the responsibility of the former Gearing and Mechanisms Committee, which first met in November 1964. Like the earlier Committee it has the cooperation of many engineers and scientists in industry, research establishments and universities from whom much assistance and information is being received.

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EFFECT OF LINK LENGTH TOLERANCES ON THE PERFORMANCE OF FOUR-BAR LINKAGES

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1. NOTATION AND UNITS

Units are given in SI. However any coherent system of units may be used.

		Units
ABCD	four-bar chain or crank-rocker mechanism in which AB is the crank, BC is the coupler, CD is the rocker and AD is the base (see Sketch 1.1)	–
a_1, b_1	co-ordinates of point P_1 relative to moving axis system $X_B Y_B$ (see Sketch 1.1)	m
a_2, b_2	co-ordinates of point P_2 relative to moving axis system $X_D Y_D$ (see Sketch 1.1)	m
c_{axle}	distance between front and rear axles (see Sketch 4.9)	m
c_{link}	distance GF (see Sketch 4.8)	m
d	distances AA' and DD' (see Sketch 4.8)	m
E	centre point of rear axle (see Sketch 4.9)	–
e	distance GH (see Sketch 4.9)	m
F	point between E and G defining length c_{link}	–
G	centre point of front axle	–
H	point on line joining mid points of front and rear axles (see Sketch 4.9)	–
k_1, k_2, k_3	intermediate non-dimensional calculation quantities (see Equation (3.5))	–
k_4, k_5	intermediate non-dimensional calculation quantities (see Equation (4.16))	–
L_1	distance between fixed centres of crank and rocker (= AD)	m
L_2	length of crank (= AB)	m
L_3	length of coupler or connecting rod (= BC)	m
L_4	length of rocker (= CD)	m
N	number of measurements of link length (see Equation (3.27))	–
O	centre of turn of vehicle (see Sketch 4.9)	–
P_1, P_2	point offset from, and fixed in relation to, coupler and rocker, respectively (see Sketch 1.1)	–

$\Pr[\]$	probability of event in $[\]$	–
p, q, r	intermediate calculation variables (see Equations (3.7))	m^2
p_1, q_1, r_1	intermediate calculation variables (see Equations (3.15))	m^2
R_i, R_o	wheel turning radii of inner and outer front wheels, respectively (see Sketch 4.9)	m
R_t	radius of turn of a vehicle (see Sketch 4.9)	m
X, Y	global orthogonal x and y axes fixed relative to mechanism frame with origin at A and x -axis passing through D	–
X_B, Y_B	local orthogonal x and y axes rotating relative to mechanism frame with origin at B and x -axis passing through C	–
X_D, Y_D	local orthogonal x and y axes rotating relative to mechanism frame with origin at D and x -axis passing through C	–
x, y	x and y co-ordinates within axis system XY	m
x_B, y_B	x and y co-ordinates of point B relative to axis system XY	m
x_C, y_C	x and y co-ordinates of point C relative to axis system XY	m
x_1, y_1	x and y co-ordinates of point P_1 relative to axis system XY	m
x_2, y_2	x and y co-ordinates of point P_2 relative to axis system XY	m
Δ, δ	increments	–
$\varepsilon_i, \varepsilon_o$	steer angles of inner and outer wheels, respectively (see Sketch 4.9)	degree
ζ	$= \varepsilon_i - \varepsilon_o$	degree
θ	angle between link and line of centres (positive in the sense illustrated in Sketch 1.1)	degree
θ_I	crank angle in the straight-ahead, non-turning, configuration	degree
σ	standard deviation	–
ϕ_1, ϕ_2	kinematic constraint parameters (see Appendix A)	m

Subscripts

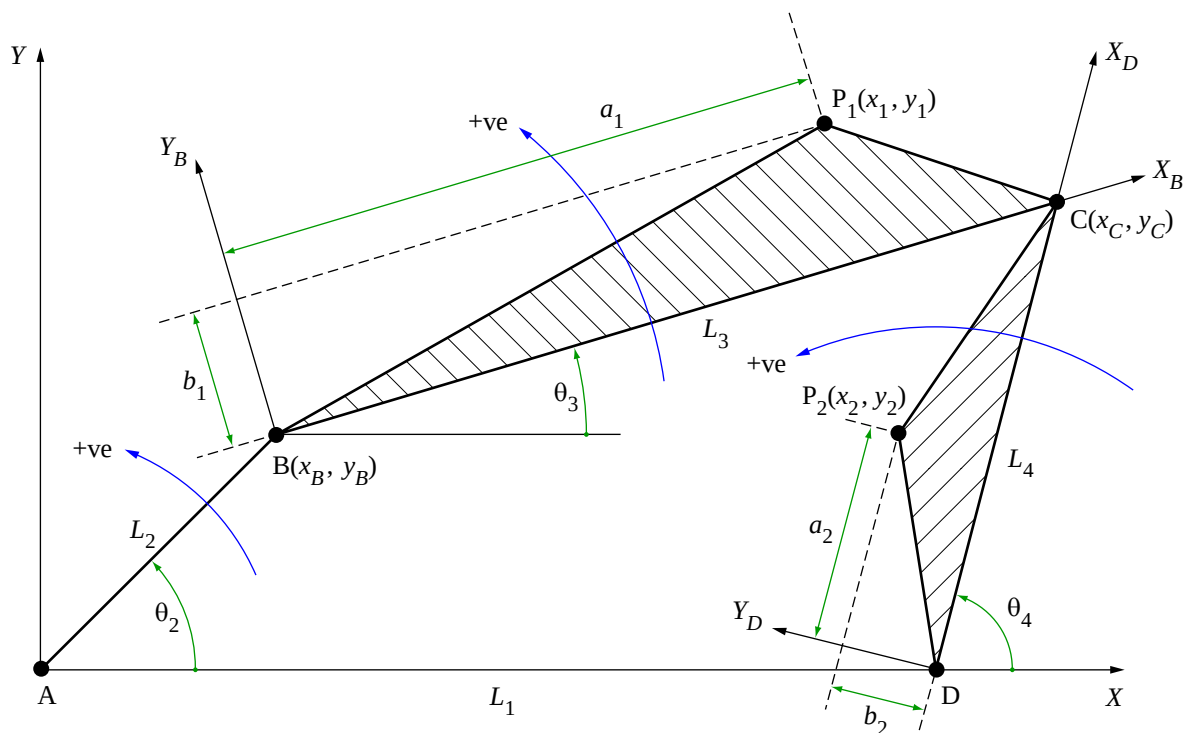
m	used to represent a mechanical tolerance
s	used to represent a statistical tolerance
1, 2, 3, 4	relate to frame, input crank, coupler and rocker, respectively

Superscripts

T denotes the transform of a matrix

-1 denotes the inverse of a matrix

A bar below a quantity indicates a matrix.



Sketch 1.1 Four bar chain linkage with associated notation

2. INTRODUCTION

Mechanisms, especially the four-bar chain and its variants, are employed in a range of applications which include packaging, diggers and motor vehicles to name but three.

In some applications, such as the compaction of hay or refuse, precision may not be essential. However, even in this case it is necessary to deduce what magnitude of tolerance would be allowable. In other situations, for example vehicle steering (the Ackermann linkage, see Example 2 in Section 4.2), the effect of tolerance has to be carefully studied.

The four-bar chain is possibly the most commonly-used linkage because of the diversity of outputs that can be generated by suitable sizing of links. This Item presents an analysis of the four-bar chain mechanism, which includes derivation of:

the coupler and rocker angles in terms of the link lengths,

errors in the link angles due to changes in the link lengths from their nominal values (the effects of bearing clearances are not considered in this Item)

and position errors of the offsets from the coupler and rocker.

An example of a simple four-bar crank-rocker linkage illustrates the analysis process, together with the error in the coupler and rocker angles resulting from small changes in the link lengths. The error in the position of the coupler point from small changes in the link lengths only is also shown. A second example illustrates the steering linkage of a vehicle, known as the Ackermann linkage, which is driven by the action of a four-bar linkage.

In an Appendix the equations of kinematic constraint are derived. This approach is an alternative to that of Section 3.

3. THE FOUR-BAR CHAIN

A four-bar planar mechanism or chain, ABCD, together with the notation employed, is presented in Sketch 1.1. The mechanism rotates in a global, or fixed, orthogonal axis system XY , with the origin located at point A. The angles θ_2 , θ_3 and θ_4 are the crank angle, the coupler angle and the rocker angle respectively. The lengths L_2 , L_3 and L_4 are the crank length, the coupler length and the rocker length, respectively. The distance between the fixed centres of the crank and the rocker is the length L_1 .

A coupler plate is given by the triangle BP_1C and a rocker plate by the triangle DP_2C . The positions of points P_1 and P_2 , relative to the global or fixed axes are given by the co-ordinates (x_1, y_1) and (x_2, y_2) , respectively.

The orthogonal axis systems $X_B Y_B$ and $X_D Y_D$, with origins located at B and D, respectively, are moving axes with respect to the frame or axis system XY . The co-ordinates (a_1, b_1) and (a_2, b_2) of the points P_1 and P_2 are distances relative to the axis systems $X_B Y_B$ and $X_D Y_D$, respectively.

The crank angle, θ_2 , is assumed in the analysis as the input angle, *i.e.* it is assumed prescribed.

Closure of the mechanism gives the following two equations

$$L_2 \cos \theta_2 + L_3 \cos \theta_3 - L_4 \cos \theta_4 = L_1 \quad (3.1)$$

$$\text{and} \quad L_2 \sin \theta_2 + L_3 \sin \theta_3 = L_4 \sin \theta_4. \quad (3.2)$$

3.1 Derivation of Rocker Angle and Errors

3.1.1 Rocker angle

To find the rocker angle, θ_4 , the coupler angle, θ_3 , can be eliminated from Equations (3.1) and (3.2) to give

$$\begin{aligned} L_3^2 &= (L_4 \cos \theta_4 + L_1 - L_2 \cos \theta_2)^2 + (L_4 \sin \theta_4 - L_2 \sin \theta_2)^2 \\ &= L_4^2 + L_1^2 + L_2^2 + 2L_4 L_1 \cos \theta_4 - 2L_2 L_1 \cos \theta_2 \\ &\quad - 2L_2 L_4 \cos(\theta_2 - \theta_4), \end{aligned} \quad (3.3)$$

which can be written in the form

$$k_1 \cos \theta_2 - k_2 \cos \theta_4 - k_3 + \cos(\theta_2 - \theta_4) = 0, \quad (3.4)$$

$$\text{in which} \quad k_1 = L_1/L_4,$$

$$k_2 = L_1/L_2$$

$$\text{and} \quad k_3 = (L_2^2 - L_3^2 + L_4^2 + L_1^2) / (2L_2 L_4).$$

$$(3.5)$$

The well known Freudenstein equation (see Reference 1) uses an alternative presentation, in which $(\theta_4 + \pi)$ is used in place of θ_4 [†].

[†] Note that this relationship is purely geometrical and hence is independent of the scale of the mechanism.

It is more useful to express Equation (3.4) in the following form, in which the crank angle, θ_2 , is contained solely within the coefficients p , q and r ,

$$p \sin \theta_4 + q \cos \theta_4 = r, \quad (3.6)$$

$$\left. \begin{aligned} \text{in which } p &= 2L_2L_4\sin\theta_2, \\ q &= 2L_2L_4\cos\theta_2 - 2L_4L_1 \\ \text{and } r &= L_2^2 - L_3^2 + L_4^2 + L_1^2 - 2L_2L_1\cos\theta_2. \end{aligned} \right\} \quad (3.7)$$

With use of the relationships

$$\sin\theta_4 = \frac{2\tan(\theta_4/2)}{1 + \tan^2(\theta_4/2)} \quad \text{and} \quad \cos\theta_4 = \frac{1 - \tan^2(\theta_4/2)}{1 + \tan^2(\theta_4/2)},$$

Equation (3.6) may be directly solved for the rocker angle, θ_4 , which is given by

$$\tan(\theta_4/2) = \frac{p \pm (p^2 - r^2 + q^2)^{1/2}}{r + q}, \quad (3.8)$$

in which the coefficients p , q and r are seen from Equations (3.7) to be solely functions of the link lengths and the crank angle θ_2 . Once the rocker angle is known the coupler angle may be evaluated from Equation (3.1) or (3.2).

3.1.2 Errors in rocker angle due to changes in link lengths

The rocker angle error, $\Delta\theta_4$, due to small changes in the link lengths ΔL_2 , ΔL_3 , ΔL_4 , and ΔL_1 can be found from the differential form of Equation (3.6) given by

$$\Delta p \sin \theta_4 + p \Delta \theta_4 \cos \theta_4 + \Delta q \cos \theta_4 - q \Delta \theta_4 \sin \theta_4 = \Delta r, \quad (3.9)$$

which, assuming that θ_2 is specified, may be expressed in the form

$$\Delta \theta_4 = \frac{\Delta r - \Delta p \sin \theta_4 - \Delta q \cos \theta_4}{p \cos \theta_4 - q \sin \theta_4}, \quad (3.10)$$

$$\begin{aligned} \text{in which } \Delta p &= \frac{\partial p}{\partial L_2} \Delta L_2 + \frac{\partial p}{\partial L_4} \Delta L_4 \\ &= 2 \left(\Delta L_2 L_4 + \Delta L_4 L_2 \right) \sin \theta_2, \end{aligned} \quad (3.11a)$$

$$\begin{aligned} \Delta q &= \frac{\partial q}{\partial L_1} \Delta L_1 + \frac{\partial q}{\partial L_2} \Delta L_2 + \frac{\partial q}{\partial L_4} \Delta L_4 \\ &= 2 \left(\Delta L_2 L_4 + \Delta L_4 L_2 \right) \cos \theta_2 - 2 \left(\Delta L_4 L_1 + \Delta L_1 L_4 \right) \end{aligned} \quad (3.11b)$$

and

$$\begin{aligned}\Delta r &= \frac{\partial r}{\partial L_1} \Delta L_1 + \frac{\partial r}{\partial L_2} \Delta L_2 + \frac{\partial r}{\partial L_3} \Delta L_3 + \frac{\partial r}{\partial L_4} \Delta L_4 \\ &= 2 \left(\Delta L_2 L_2 - \Delta L_3 L_3 + \Delta L_4 L_4 + \Delta L_1 L_1 \right) \\ &\quad - 2 \left(\Delta L_2 L_1 + \Delta L_1 L_2 \right) \cos \theta_2.\end{aligned}\quad (3.11c)$$

With use of Equations (3.10), (3.11a), (3.11b), and (3.11c), the following four expressions for the partial differential of the angle θ_4 with respect to the four link lengths are

$$\frac{\partial \theta_4}{\partial L_2} = 2 \left(\frac{L_2 - L_1 \cos \theta_2 - \sin \theta_4 L_4 \sin \theta_2 - \cos \theta_4 L_4 \cos \theta_2}{p \cos \theta_4 - q \sin \theta_4} \right), \quad (3.12a)$$

$$\frac{\partial \theta_4}{\partial L_3} = 2 \left(\frac{-L_3}{p \cos \theta_4 - q \sin \theta_4} \right), \quad (3.12b)$$

$$\frac{\partial \theta_4}{\partial L_4} = 2 \left(\frac{L_4 - \sin \theta_4 L_2 \sin \theta_2 - \cos \theta_4 L_2 \cos \theta_2 + \cos \theta_4 L_1}{p \cos \theta_4 - q \sin \theta_4} \right) \quad (3.12c)$$

and

$$\frac{\partial \theta_4}{\partial L_1} = 2 \left(\frac{L_1 - L_2 \cos \theta_2 + L_4 \cos \theta_4}{p \cos \theta_4 - q \sin \theta_4} \right). \quad (3.12d)$$

3.2 Derivation of Coupler Angle and Errors

3.2.1 Coupler angle

In a manner similar to the evaluation of θ_4 , Equations (3.1) and (3.2) can be re-arranged, eliminating θ_4 , to give

$$\begin{aligned}L_4^2 &= \left(L_2 \cos \theta_2 - L_1 + L_3 \cos \theta_3 \right)^2 + \left(L_2 \sin \theta_2 + L_3 \sin \theta_3 \right)^2 \\ &= L_1^2 + L_2^2 + L_3^2 - 2L_1 L_2 \cos \theta_2 + \left(2L_2 L_3 \sin \theta_2 \right) \sin \theta_3 \\ &\quad + \left(2L_2 L_3 \cos \theta_2 - 2L_1 L_3 \right) \cos \theta_3.\end{aligned}\quad (3.13)$$

This last equation may be written in a form similar to that of Equation (3.6) to give

$$p_1 \sin \theta_3 + q_1 \cos \theta_3 = r_1, \quad (3.14)$$

in which

$$\left. \begin{aligned}p_1 &= 2L_2 L_3 \sin \theta_2, \\ q_1 &= 2L_2 L_3 \cos \theta_2 - 2L_1 L_3 \\ \text{and } r_1 &= L_4^2 - L_1^2 - L_2^2 - L_3^2 + 2L_1 L_2 \cos \theta_2.\end{aligned} \right\} \quad (3.15)$$

In a manner similar to that of the rocker angle, Equation (3.14) may be directly solved for the coupler angle, θ_3 , which is given by

$$\tan(\theta_3/2) = \frac{p_1 \pm (p_1^2 - r_1^2 + q_1^2)^{1/2}}{r_1 + q_1}, \quad (3.16)$$

in which, the coefficients p_1 , q_1 and r_1 are functions solely of the link length and the crank angle θ_2 .

3.2.2 Errors in coupler angle due to changes in link lengths

The coupler angle error, $\Delta\theta_3$, due to small changes in the link lengths, is found in a manner identical to that of $\Delta\theta_4$ obtained earlier, and is given by

$$\Delta\theta_3 = \frac{\Delta r_1 - \Delta p_1 \sin \theta_3 - \Delta q_1 \cos \theta_3}{p_1 \cos \theta_3 - q_1 \sin \theta_3}, \quad (3.17)$$

in which $\Delta p_1 = 2 (\Delta L_3 L_2 + \Delta L_2 L_3) \sin \theta_2$, (3.18a)

$$\Delta q_1 = 2 (\Delta L_3 L_2 + \Delta L_2 L_3) \cos \theta_2 - 2 (\Delta L_1 L_3 + \Delta L_3 L_1) \quad (3.18b)$$

and $\Delta r_1 = 2 (\Delta L_4 L_4 - \Delta L_2 L_2 - \Delta L_3 L_3 - \Delta L_1 L_1) + 2 (\Delta L_2 L_1 + \Delta L_1 L_2) \cos \theta_2$. (3.18c)

With use of Equations (3.17), (3.18a), (3.18b) and (3.18c), the following four expressions for the partial differential of the angle θ_3 with respect to the four link lengths are

$$\frac{\partial \theta_3}{\partial L_2} = 2 \left(\frac{-L_2 + L_1 \cos \theta_2 - \sin \theta_3 L_3 \sin \theta_2 - \cos \theta_3 L_3 \cos \theta_2}{p_1 \cos \theta_3 - q_1 \sin \theta_3} \right), \quad (3.19a)$$

$$\frac{\partial \theta_3}{\partial L_3} = 2 \left(\frac{-L_3 - \sin \theta_3 L_2 \sin \theta_2 - \cos \theta_3 L_2 \cos \theta_2 + \cos \theta_3 L_1}{p_1 \cos \theta_3 - q_1 \sin \theta_3} \right), \quad (3.19b)$$

$$\frac{\partial \theta_3}{\partial L_4} = 2 \left(\frac{L_4}{p_1 \cos \theta_3 - q_1 \sin \theta_3} \right) \quad (3.19c)$$

and $\frac{\partial \theta_3}{\partial L_1} = 2 \left(\frac{-L_1 + L_2 \cos \theta_2 + \cos \theta_3 L_3}{p_1 \cos \theta_3 - q_1 \sin \theta_3} \right)$. (3.19d)