



Compendium of modal densities

Part 1: rods, beams and plates

**Associated software: VIEWpac m1108a
VIEWpac m1108b**



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COMPENDIUM OF MODAL DENSITIES PART I: RODS, BEAMS AND PLATES

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COMPENDIUM OF MODAL DENSITIES PART I: RODS, BEAMS and PLATES

1. NOTATION

		Units	
		SI	British
A	cross-sectional area	m^2	in^2
a	panel length in x -direction	m^2	in^2
B	faceplate longitudinal stiffness parameter	Nm	lbf in
b	panel length in y -direction	m	in
C_p	torsional rigidity of plate	Nm^2	lbf in^2
C_r	torsional rigidity of rib	Nm^2	lbf in^2
c_l	longitudinal wave velocity in rod or beam (phase velocity)	m/s	in/s
c_l'	longitudinal wave velocity in plate (phase velocity)	m/s	in/s
c_s	shear (or torsional) wave velocity	m/s	in/s
c_0	velocity of sound in fluid (air)	m/s	in/s
D	bending stiffness of isotropic plate	Nm	lbf in
D_x	bending stiffness of plate in x -direction	Nm	lbf in
D_y	bending stiffness of plate in y -direction	Nm	lbf in
D_{xy}	bending-torsional stiffness of plate about x - and y -axes	Nm	lbf in
D_b	effective bending stiffness of sandwich panel $= E_f h_f h_c^2 (1 + h_f/h_c)^2 / \{2(1 - \nu^2)\}$	Nm	lbf in
d	equivalent thickness (Equation (6.13))	m	in
E	Young's modulus of rod, beam or plate material	Pa	lbf/in^2
E_c	effective Young's modulus of regular honeycomb	Pa	lbf/in^2
E_f	Young's modulus of isotropic faceplates of sandwich panel	Pa	lbf/in^2
E_{f1}, E_{f2}	Young's moduli of isotropic faceplates of sandwich panel	Pa	lbf/in^2

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E_s	Young's modulus of foam material	Pa	lbf/in ²
E_x, E_y	Young's moduli in x - and y -directions of orthotropic plate	Pa	lbf/in ²
E_1	vibrational energy associate with subsystem 1	J	lbf in
E_2	vibrational energy associate with subsystem 2	J	lbf in
F	elliptic integral of first kind		
f	frequency	Hz	Hz
f_1, f_2	functions in modal density expressions of Renji and Nair (1996)		
G	shear modulus for rod or plate	Pa	lbf/in ²
G_c	shear modulus of isotropic core	Pa	lbf/in ²
G_{cx}, G_{cy}	core shear moduli in x - and y -directions, respectively	Pa	lbf/in ²
G_{xy}	shear modulus in xy -plane of orthotropic plate	Pa	lbf/in ²
g	core stiffness parameter	m ⁻²	in ⁻²
H	height of ribs (of ribbed plate)	m	in
h	thickness of beam, plate or length of honeycomb wall in x -direction	m	in
h_c	thickness of sandwich core	m	in
h_e	equivalent thickness of ribbed plate	m	in
h_f	thickness of faceplate	m	in
h_{f1}, h_{f2}	thickness of faceplates	m	in
I	second moment of area	m ⁴	in ⁴
I_m	mass density moment of inertia per unit width about middle surface of panel	kg	*
I'	second moment of area about neutral axis for ribbed plate	m ⁴	in ⁴
K	complete elliptic integral of first kind		
K_1	constant in torsion of rectangular cross-sections		
K_{1p}, K_{1r}	values of K_1 for plate and rib		

k_g	radius of gyration of beam cross-section	m	in
k_b	bending wavenumber for infinite plate	m^{-1}	in^{-1}
k_l	longitudinal wavenumber	m^{-1}	in^{-1}
k_m	plate wavenumber in x -direction	m^{-1}	in^{-1}
k_n	plate wavenumber in y -direction	m^{-1}	in^{-1}
l	length of rod, slant length of honeycomb wall	m	in
M	mass per unit area	kg/m^2	*
M_{ol}	modal overlap		
m	modal number, number of nodal circles of circular plate		
N	number of modes below a certain frequency (mode count)		
N_{total}	number of modes below a certain frequency for multi-beam system (mode count)		
N_x	in-plane force per unit edge length in x -direction, acting on rectangular plate	N/m	lbf/in
N_y	in-plane force per unit edge length in y -direction, acting on rectangular plate	N/m	lbf/in
n	modal number, number of diametral nodes of circular plate		
n_d	modal density	s	s
n_{d1}	modal density of subsystem 1	s	s
n_{d2}	modal density of subsystem 2	s	s
q	number of beam supports		
r	modulus of elliptic integral (orthotropic plate)		
r_d	parameter defined in Equation (6.26)		
r_{eff}	parameter defined in Equation (6.29)		
r_f	parameter defined in Equation (6.24)	m^2	in^2
S	area of beam cross-section	m^2	in^2
S_c	core transverse shear parameter, Equation (6.6)		

* See footnote at end of Notation Section

S_r	transverse shear flexibility	N/m	lbf/in
s	separation between ribs	m	in
t	thickness of ribs, thickness of honeycomb wall	m	in
$\bar{z}$	distance between the neutral axis and the edge of the plate (Sketch B.1)	m	in
β	parameter in Equation (6.30) defined by Equation (6.32)		
γ	integral variable		
γ_g	G_{cx}/G_{cy}		
η	loss factor		
η_1	loss factor of subsystem 1		
η_{12}	coupling loss factor associated with energy flow from subsystem 1 to subsystem 2		
η_{21}	coupling loss factor associated with energy flow from subsystem 2 to subsystem 1		
κ	modulus of elliptic integral (Section 5.1.4)		
κ_0	modulus of elliptic integral (Section 6.6.2)		
λ_b	bending wavelength in plate		
ν_1, ν_2	Poisson's ratios for faceplate material		
ν_{xy}	Poisson's ratio of orthotropic plate for load in x-direction, contraction in y-direction		
ν_{yx}	Poisson's ratio of orthotropic plate for load in y-direction, contraction in x-direction		
τ	geometric parameter for sandwich panel, Equation (6.23)		
ρ	density of material	kg/m ³	*
ρ_c	mass density of core	kg/m ³	*
ρ_{f1}, ρ_{f2}	mass densities of faceplates	kg/m ³	*
ρ_s	density of material used in honeycomb construction	kg/m ³	*
Π_1	input power to subsystem 1	W	lbf in/s

* See footnote at end of Notation Section

Π_{12}	net power transmitted from subsystem 1 to subsystem 2	W	lbf in/s
$\Pi_{1, diss}$	power loss from subsystem 1 due to energy dissipation in subsystem 1	W	lbf in/s
$\Pi_{2, diss}$	power loss from subsystem 2 due to energy dissipation in subsystem 2	W	lbf in/s
θ	angle honeycomb wall makes with x-axis (in x – y plane)		
χ_r	rotary inertia parameter, Equation (6.10)		
ω	circular frequency	rad/s	rad/s
ω_0	fundamental frequency of an infinitely long, simply supported flat panel = $(\pi^2/b^2)\sqrt{(D_b/M)}$	rad/s	rad/s
ω_{c2}	critical frequency of wave coincidence of orthotropic plate (upper value), Equation (5.14)	rad/s	rad/s
Ω	frequency parameter, Equation (6.7)		

* In British units divide lb by 386.4. (The units of mass can be considered to be equivalent to lbf s²/in.)

Abbreviations

CFRP	carbon fibre reinforced plastic
CLF	coupling loss factor
ESA	European Space Agency
GUI	graphical user interface
PVC	polyvinyl chloride
SEA	statistical energy analysis

2. INTRODUCTION

Modal density is a concept associated mostly with Statistical Energy Analysis (SEA). The characteristics of SEA and the principles of its use are described in ESDU Item 99009, as well as numerous publications (Cremer *et al.*, 1988; Lyon and DeJong, 1995; Craik, 1996; ESA, 1996; Fahy, 1997; ESDU, 1999; Norton and Karczub, 2003; Fahy and Gordonio, 2007 and Wijker, 2009).

Medium and high frequency vibrations of complex structures can be analysed by the method of Statistical Energy Analysis. The fundamental requirement for the applicability of SEA is a multimodal structural response which is dominated by resonance modes. The method is especially useful when the structural excitation is random and broadband. It is often applied to vibro-acoustic problems which deal with fluid-structure interactions.

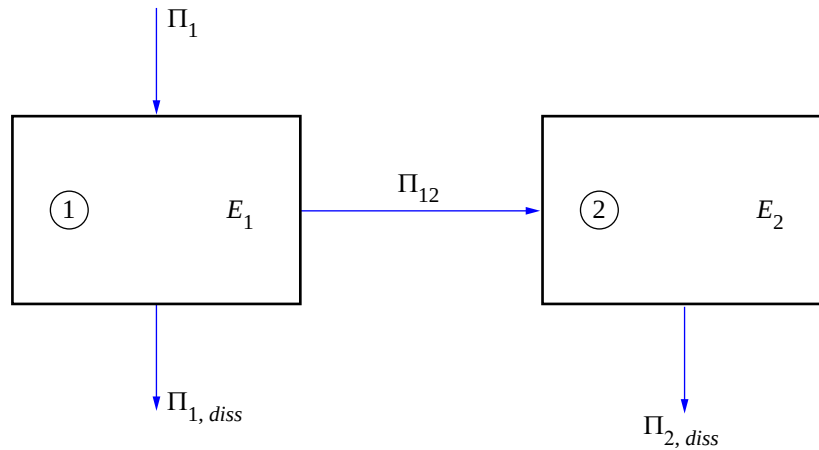
The SEA method, as its name implies, is a statistical approach in which the response of an individual mode is not analysed. Instead, the average response of the structure is considered. The averaging can be performed over space, over an ensemble, and within a frequency band. There are two equivalent approaches in the application of SEA, namely, the modal and the wave methods (Lyon and DeJong, 1995; Fahy, 1997). The name SEA also indicates that energy flows between different parts of the structure or subsystems are analysed. SEA is based on the principle of energy conservation. The structure, with its multi-modal response, is modelled in SEA as a system which consists of many subsystems, each characterised by their average vibrational energy.

A subsystem is associated with a group of response modes, which individually possess a comparable vibrational energy. In other words, the equipartition of vibrational energy between modes of the subsystem within a certain frequency band is assumed. Normally, the subsystems are weakly coupled, but this is not an essential requirement. Additionally, it is assumed that there is modal incoherency, which enables the modal vibrational energies to be summed in the considered subsystem. In a particular case, the subsystem associated with a group of modes can represent a specific component of a structure. Energy is transmitted only between resonant modes of different subsystems within the considered frequency band.

The modal density of the subsystem, $n_d(\omega)$, the subsystem's internal loss factor (or dissipation loss factor) and the coupling loss factor (degree of coupling) are the parameters essential to formulate the energy flows between the subsystems. The modal density is the average number of resonance modes within a unit frequency band and is expressed by the number of modes per rad/s, $n_d(\omega)$, or number of modes per Hz, $n_d(f)$. Since $\omega = 2\pi f$, the modal density $n_d(\omega)$ and $n_d(f)$ are related to each other by

$$n_d(f) = 2\pi n_d(\omega). \quad (2.1)$$

SEA is based on the energy conservation law applied usually to the structure which is vibrating under steady-state conditions. The structure, with its multi-modal response, is modelled in SEA as a system with many subsystems, between which vibrational energy is transmitted with time-averaged rate (time-averaged power) proportional to the difference between the time-averaged modal energies of the subsystems.



Sketch 2.1 Structure with two subsystems

For example, in the case of a structure with two subsystems, shown schematically in Sketch 2.1, the energy conservation law is expressed in terms of the power balance equations as:

$$\Pi_1 = \Pi_{12} + \Pi_{1,diss} \quad (2.2)$$

and

$$\Pi_{12} = \Pi_{2,diss}, \quad (2.3)$$

where Π_1 denotes the power input to the subsystem 1, $\Pi_{1,diss}$ and $\Pi_{2,diss}$ are the power losses associated with energy dissipation in subsystem 1 and subsystem 2, respectively. The averaged energies of both subsystems, E_1 and E_2 , within the frequency band $\Delta\omega$ result from the summation of all time-averaged modal vibrational energies in each of the two subsystems. The term, Π_{12} , denotes net power transmitted from subsystem 1 to subsystem 2. It can be expressed in terms of the modal densities of both subsystems, $n_{d1}(\omega)$ and $n_{d2}(\omega)$, the coupling loss factor between both subsystems, either η_{12} or η_{21} (where η_{12} and η_{21} are the non-dimensional measures of the power transfer between subsystems 1 and 2) and the centre frequency, ω , of the band $\Delta\omega$, as

$$\Pi_{12} = \omega \eta_{12} n_{d1} (E_1/n_{d1} - E_2/n_{d2}). \quad (2.4)$$

Equation (2.4) is a generalization of the equation which describes the time-averaged power transfer between two oscillators. It is applied in SEA to multi-modal weakly coupled subsystems.

The coupling loss factors, η_{12} and η_{21} , are the averages of all CLFs (η_{pq}) between pairs (p, q) of subsystem modes, which are resonant within the considered frequency band, $\Delta\omega$. Hence,

$$\eta_{12} = \frac{1}{N_1} \sum_{p=1}^{N_1} \sum_{q=1}^{N_2} \eta_{pq} \quad (2.5)$$

$$\eta_{21} = \frac{1}{N_2} \sum_{p=1}^{N_1} \sum_{q=1}^{N_2} \eta_{pq}, \quad (2.6)$$

where N_1 and N_2 denote the number of resonance modes in the systems 1 and 2, respectively, within the frequency band $\Delta\omega$. It should be noted that $N_1 = n_{d1}(\omega)\Delta\omega$ and $N_2 = n_{d2}(\omega)\Delta\omega$. The indices, p and q , refer to the modes of the subsystems 1 and 2, respectively. The definitions of η_{12} and η_{21} in Equations (2.5) and (2.6) indicate that

$$\frac{\eta_{12}}{\eta_{21}} = \frac{N_2}{N_1}, \quad (2.7)$$

or

$$\eta_{12} n_{d2} = \eta_{21} n_{d1}, \quad (2.8)$$

which constitute the reciprocity (or consistency) relationships.

The dissipated powers are expressed as

$$\Pi_{1, diss} = \omega \eta_1 E_1 \quad (2.9)$$

and

$$\Pi_{2, diss} = \omega \eta_2 E_2, \quad (2.10)$$

where η_1 and η_2 are the dissipation loss factors for both subsystems. The dissipation loss factors are defined as averages of modal dissipation loss factors in each subsystem, η_{1p} and η_{2p} .

$$\eta_1 = \frac{1}{N_1} \sum_{p=1}^{N_1} \eta_{1p} \quad (2.11)$$

$$\eta_2 = \frac{1}{N_2} \sum_{q=1}^{N_2} \eta_{2q}, \quad (2.12)$$

To formulate Equations (2.9) and (2.10), which are applied to both subsystems 1 and 2, the power dissipation concept for a single oscillator has been extended in SEA to a group of oscillators within a frequency band $\Delta\omega$.

The parameter which estimates the applicability of the SEA method to the considered system is modal overlap factor, defined as

$$M_{ol}(\omega) = \omega \eta n_d(\omega), \quad (2.13)$$

where η is the loss factor. The modal overlap is the ratio of the width of the resonance peak of the response at the half power points, $\eta\omega$, to the average modal spacing. When $M_{ol}(\omega) > 1$, there is an indication that resonance peaks have a tendency to merge together. In such cases, a statistical method (such as SEA) should be applied instead of a deterministic one. In general, the validity of the SEA model depends on the magnitude of the modal overlap, the type of modes and the coupling loss factors (Mace, 2002).

To obtain the modal density for a structure, the equations of motion with imposed boundary conditions are solved to give the natural frequencies. Next, the distribution of resonance modes in relation to frequency is analysed in wave number space (k -space) in order to obtain a modal count $N(\omega)$, where the modal count is the number of modes below a certain frequency, ω (Courant and Hilbert, 1953; Morse and Feshbach, 1953). The mode count and modal density are related according to

$$n_d(\omega) = \frac{\partial N}{\partial \omega}. \quad (2.14)$$

The concept of modal density is applied in relation to the higher order modes in accordance with the SEA model. For higher order of modes, the modal density is practically independent of boundary conditions and, as the mode order increases, $n_d(\omega)$ becomes even less sensitive to them. The majority of modal densities for basic structures, cited in this Item, are obtained for simply-supported structures.

Modal density is important in the estimation of averaged radiation efficiencies of isotropic and orthotropic, rectangular plates (Maidanik, 1962; Leppington *et al.*, 1982; ESDU, 2002; Anderson and Bratos-Anderson, 2005). It is also used in the prediction of the sound transmission loss through plates (Guyader and Leseur, 1978) and in the response of panels to turbulent flow (Ribner, 1965).

In this Item the modal density $n_d(\omega)$ for basic structures are collected together in order to provide a compendium for use in SEA.

3. RODS

This section is concerned with the modal density of an elastic rod in longitudinal and torsional vibration. Rods are examples of one-dimensional systems. The cross-section dimensions (radius or thickness) are small in comparison with the wavelength.

3.1 Longitudinal Vibration

3.1.1 Model assumptions

- (1) Longitudinal vibrations of the rod are considered.
- (2) The rod has a uniform cross-section, which remains plane and normal to the axis during vibration.
- (3) The rod is of length, l , and fixed at both ends.

3.1.2 Natural frequencies and modal density

The natural frequencies for the rod fixed at both ends are

$$\omega_m = \frac{m\pi c_l}{l}, \quad (3.1)$$

where $m = 1, 2, 3, \dots$ and $c_l = \sqrt{E/\rho}$ is the velocity (phase velocity) of the longitudinal wave in the rod (E is Young's modulus for the material of the rod and ρ is the material density). The natural frequencies are distributed at equal intervals of $\pi c_l/l$, thus, the mode count, N , from 0 to ω rad/s for a longitudinal rod

with both ends fixed is given by

$$N = \frac{l\omega}{\pi c_l}. \quad (3.2)$$

With the aid of Equations (2.14) and (3.2) the modal density of the rod in longitudinal vibration is derived:

$$n_d(\omega) = \frac{l}{\pi c_l}, \quad (3.3)$$

or

$$n_d(\omega) = \frac{l}{\pi} \sqrt{\frac{\rho}{E}}. \quad (3.4)$$

The modal density is independent of the boundary conditions at the ends of the rods. Xie *et al.* (2004) have suggested that the mode count is dependent on the end conditions, as considered in Appendix A.

3.2 Torsional Vibration

3.2.1 Model assumptions

- (1) It is assumed that the rod oscillates in torsion about its axis.
- (2) The rod is assumed to have a circular cross-section, which remains plane and normal to the axis during vibratory motion.
- (3) The rod is of length, l , and fixed at both ends.

3.2.2 Natural frequencies and modal density

The natural frequencies are:

$$\omega_m = \frac{m\pi c_s}{l}, \quad (3.5)$$

where $m = 1, 2, 3, \dots$ and $c_s = \sqrt{(G/\rho)}$ is the torsional wave velocity. The shear modulus is G and the density of the rod material is ρ . The modal density of the rod in torsional vibration is:

$$n_d(\omega) = \frac{l}{\pi c_s} \quad (3.6)$$

or

$$n_d(\omega) = \frac{l}{\pi} \sqrt{\frac{\rho}{G}}. \quad (3.7)$$

Equation (3.7) can be used for different end conditions. The modal density is also independent of the geometry of the cross-section of the rod.

4. BEAMS IN BENDING

For the beams described in this Section the Euler-Bernoulli theory applies.

4.1 Model Assumptions

- (1) The beam is thin, simply-supported, of length, l , and thickness, h .
- (2) It is assumed that the beam vibrates in bending.
- (3) The effect of rotary inertia is neglected.
- (4) Displacements due to shear are negligible.
- (5) There are no in-plane forces.

4.2 Natural Frequencies and Modal Density

The natural frequencies for a simply-supported beam in bending are derived from the equation

$$k_b = k_m, \quad (4.1)$$

where $k_m = m\pi/l$ is the mode wavenumber for a beam and

$$k_b = \left(\frac{\omega}{c_l k_g} \right)^{1/2} \quad (4.2)$$

is the bending wavenumber for an infinite beam. The radius of gyration of the beam cross-section ($h/\sqrt{12}$ for a rectangular beam of thickness h) is denoted as k_g and c_l is the propagation velocity (phase velocity) of longitudinal waves in the beam. Equation (4.1) leads to the equation for the natural frequencies of a simply-supported beam in bending,

$$\omega_m = \left(\frac{m\pi}{l} \right)^2 k_g c_l, \quad (4.3)$$

where $m = 1, 2, 3 \dots$

Since the separation between modes is π/l , the number of modes below the frequency ω is $k_b l/\pi$. Hence, the mode count of the simply-supported beam

$$N = \frac{lk_b}{\pi}, \quad (4.4)$$

or,

$$N = \frac{l}{\pi} \sqrt{\frac{\omega}{k_g c_l}}. \quad (4.5)$$

Consequently, the modal density of the beam vibrating in bending is:

$$n_d(\omega) = \frac{l}{2\pi\sqrt{\omega c_l k_g}}, \quad (4.6)$$

or, in terms of the second moment of area, I , ($I = A k_g^2$, A is cross-sectional area of the beam),

$$n_d(\omega) = \frac{l}{2\pi\omega^{1/2}} \left(\frac{\rho A}{EI} \right)^{1/4}. \quad (4.7)$$

The modal density is inversely proportional to the square root of the frequency. In terms of the bending wavenumber, k_b ,

$$n_d(\omega) = \frac{l}{2\pi c_l k_g k_b}. \quad (4.8)$$

The modal density, given by Equations (4.6) to (4.8), is independent of the boundary conditions of the beams. Xie *et al.* (2002; 2004) have discussed the mode count dependence on the boundary conditions, see Appendix A.

5. PLATES

In all cases the plates undergo small deflections and linear theory applies.

5.1 Rectangular Isotropic Plates

5.1.1 Model assumptions

- (1) The thin, rectangular plate vibrates in bending.
- (2) The dimensions of the plate are $a \times b \times h$.
- (3) To fulfil the requirement that the plate is thin, the structural bending wavelength, λ_b , should be at least six times larger than the plate thickness, h .
- (4) Rotary inertia and in-plane forces are neglected.
- (5) Shear deformation is also disregarded, *i.e.*, the condition $k_b h < 1$ must be satisfied, where k_b is the bending wavenumber for the infinite panel. This assumption is applicable only to Section 5.1.2.

5.1.2 Natural frequencies and modal density

For a rectangular, isotropic plate which is simply-supported at all edges, the natural frequencies for modes (m, n) , where $m = 1, 2, 3 \dots$ and $n = 1, 2, 3 \dots$, are:

$$\omega_{mn} = \sqrt{\frac{D}{\rho h}} \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right], \quad (5.1)$$

where

$$D = \frac{Eh^3}{12(1 - \nu^2)} \quad (5.2)$$

is the plate bending stiffness. The product ρh is the mass per unit area. Young's modulus and Poisson's ratio for the material are denoted by E and ν , respectively. The plate wavenumbers in the x - and y -directions, k_m and k_n , respectively, are equal to $m\pi/a$ and $n\pi/b$. The bending wavenumber for the infinite plate, k_b , is $\sqrt{\omega}(\rho h/D)^{0.25}$.

The expression for the modal density of a rectangular, isotropic plate, which is applicable for all boundary conditions at high frequencies, is

$$n_d(\omega) = \frac{ab}{4\pi} \sqrt{\frac{\rho h}{D}}. \quad (5.3)$$

An approximate expression for the number of modes below a frequency ω is

$$N = \frac{ab}{4\pi} \omega \sqrt{\frac{\rho h}{D}}. \quad (5.4)$$

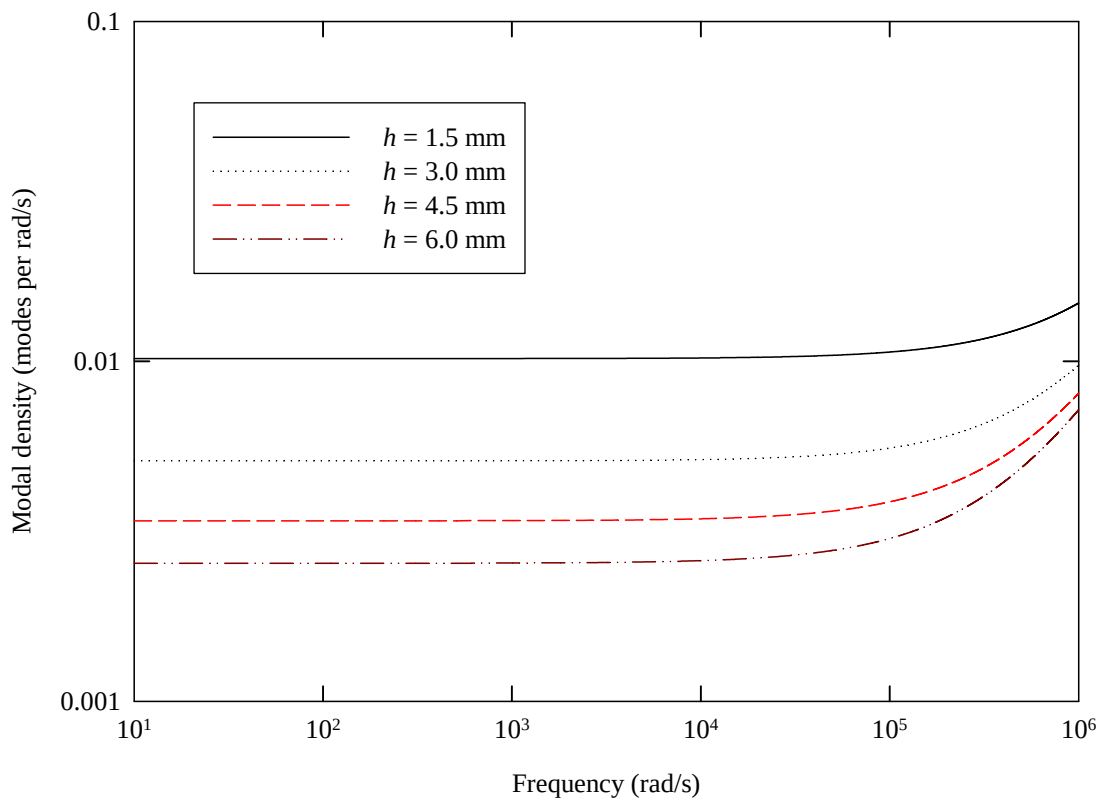
Courant and Hilbert (1953) have claimed that an additional term is required in Equation (5.3) to account for the boundary conditions of the plate, when modes of lower order are considered. Xie *et al.* (2002; 2004) have developed this idea and have established expressions for the modal density of the plate with different boundary conditions. Further information is provided in Appendix A.

5.1.3 Effect of shear on modal density

If shear is taken into account, an extra term is added to the basic expression for modal density in Equation (5.3)

$$n_d(\omega) = \frac{ab}{4\pi} \sqrt{\frac{\rho h}{D}} + \frac{\rho \omega ab}{2\pi G}, \quad (5.5)$$

where G is the shear modulus (Pinnington, 1998). The effect of the shear term is to cause an increase in the modal density at higher frequencies, as shown in Sketch 5.1.



Sketch 5.1 Effect of shear on the modal density of an isotropic plate with different thicknesses

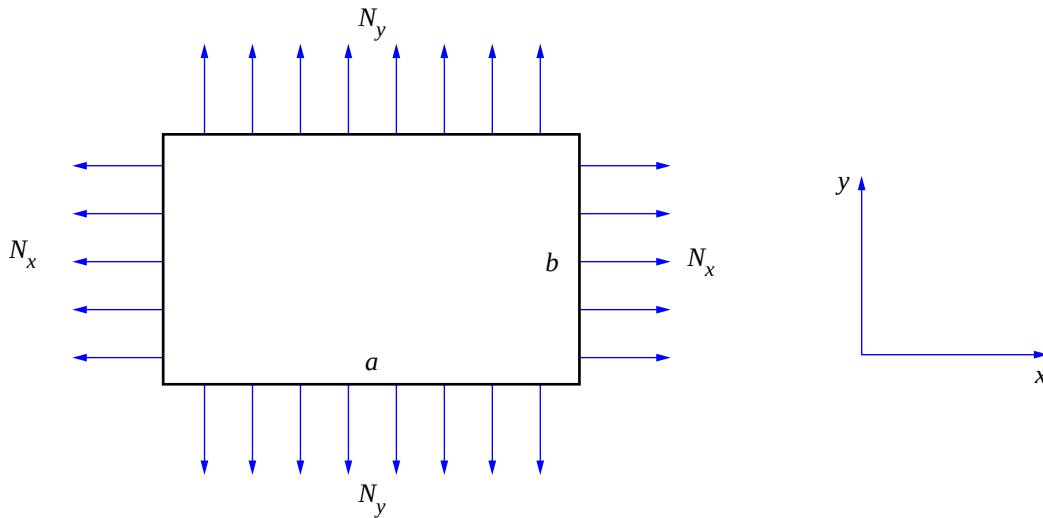
($E = 72 \times 10^9$ Pa, $\rho = 2700$ kg/m³, $\nu = 0.3$, $a = 0.5$ m and $b = 0.6$ m)

The shear term becomes significant when

$$\omega > \frac{c_l}{2h} \sqrt{\frac{3(1-\nu)}{1+\nu}}.$$

5.1.4 Effect of in-plane forces on modal density

The effect of forces which act in the plane of the rectangular plate is considered in this section. The forces, represented by in-plane force intensities expressed in units of N/m, N_x and N_y , are parallel with the x - and y -axes, and are assumed to be uniformly distributed along the plate edges, as shown in Sketch 5.2. To obtain the total force (in N or lbf), acting on the edge with length b , N_x has to be multiplied by the edge length, b . Shear forces acting on the plate are assumed to be zero.



Sketch 5.2 Rectangular plate with in-plane forces, N_x and N_y

An expression for the undamped natural frequencies is given (Wilkinson, 1968; Leissa, 1993) as

$$\omega_{mn} = \sqrt{\frac{D}{\rho h}} \left\{ \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]^2 + \frac{1}{D} \left[N_x \left(\frac{m\pi}{a} \right)^2 + N_y \left(\frac{n\pi}{b} \right)^2 \right] \right\}^{1/2}. \quad (5.6)$$

Equation (5.6) is applicable for all modes when all the edges are simply supported and may be used for higher modes for other boundary conditions. Note that positive values of N_x and N_y indicate tension and negative values compression. With increasing compression the natural frequencies decrease. For compression in one direction a critical buckling load is attained as the natural frequency approaches zero (Leissa, 1993). If N_x and N_y are both zero, Equation (5.6) is reduced to Equation (5.1).

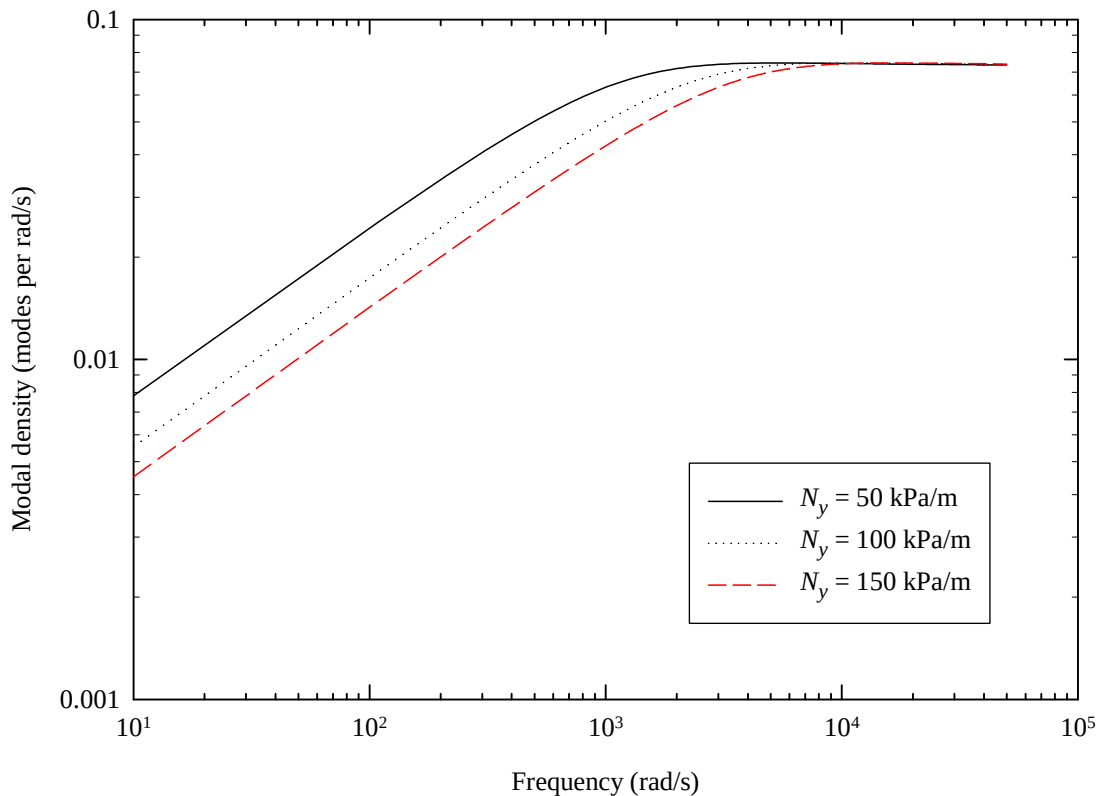
The expression for the modal density may be written as (Wilkinson, 1968)

$$n_d(\omega) = \frac{ab}{\pi^2} \omega \rho h \frac{F(\pi/2, \kappa)}{\left(N_y^2 + 4\omega^2 \rho h D \right)^{1/4} \left(N_x^2 + 4\omega^2 \rho h D \right)^{1/4}}, \quad (5.7)$$

where $F(\pi/2, \kappa)$ is a complete elliptic integral of the first kind and

$$\kappa^2 = \frac{(N_y - N_x)^2 - \left\{ \left(N_y^2 + 4\omega^2 \rho h D \right)^{1/2} - \left(N_x^2 + 4\omega^2 \rho h D \right)^{1/2} \right\}^2}{4 \left(N_y^2 + 4\omega^2 \rho h D \right)^{1/2} \left(N_x^2 + 4\omega^2 \rho h D \right)^{1/2}}. \quad (5.8)$$

When $N_x = N_y = 0$, $\kappa = 0$ and $F(\pi/2, 0) = \pi/2$, thus $n_d(\omega)$ reduces to that given in Equation (5.3). The variation of modal density with frequency is shown in Sketch 5.3 for different values of N_y (with $N_x = 0$).



Sketch 5.3 Variation of modal density with frequency for in-plane forces in y-direction
 ($E = 72 \times 10^9$ Pa, $\rho = 2700$ kg/m³, $\nu = 0.3$, $h = 2$ mm, $a = 2.45$ m and $b = 1.22$ m)

At high frequencies the modal density approaches that for an isotropic plate (see Equation (5.3)). At low frequencies the modal density increases approximately at a rate proportional to $\omega^{1/2}$.

5.2 Rectangular Orthotropic Plates

Glass or carbon fibre reinforced laminates, corrugated or ribbed panels can be regarded as orthotropic. The rectangular plates considered in this section are specially orthotropic, which means that the edges of the plates are aligned with the principal material axes. In the case of laminates made from layers of carbon fibres or glass fibres in a matrix, certain conditions have to be satisfied in relation to the flexural and in-plane stiffnesses (Jones, 1999). Ribbed or corrugated plates may be regarded as 'structurally or technically orthotropic' (Troitsky, 1976; ESDU, 2002) and can be replaced by an equivalent orthotropic plate of constant thickness. The modal density of specially orthotropic plates has been considered by Wilkinson (1968), Hart and Shah (1971) and Elishakoff (1975). Langley (1996) has investigated more complicated anisotropic plates. Natural frequencies of specially orthotropic composite panels and anisotropic laminates have been obtained experimentally and theoretically by Maheri and Adams (2003).

5.2.1 Model assumptions

Assumptions 1 – 5 from Section 5.1, which refers to the rectangular isotropic plate, remain valid.

- (6) The simply-supported plate is specially orthotropic with its edges aligned with the principal material axes.
- (7) The bending stiffness of the orthotropic plate varies for different directions of bending wave propagation.