



Compendium of modal densities

Part II: shell structures

Associated software: m1109a



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COMPENDIUM OF MODAL DENSITIES. PART II: SHELL STRUCTURES

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COMPENDIUM OF MODAL DENSITIES. PART II: SHELL STRUCTURES

1. NOTATION

		Units	
		SI	British
A	cross-sectional area	m^2	in^2
a_n	cone parameter, $n\pi/(1 - \alpha_c)$		
a	axial length of curved panel, cylinder radius (Szechenyi), length of side of spherical panel	m	in
b	circumferential width of cylindrical panel, cylinder circumference, length of side of spherical panel	m	in
B	faceplate longitudinal stiffness parameter	Nm	lbf in
C	curvature parameter $= (1/R) (b^2/\pi^2) \sqrt{E_{ext}/D_b}$		
c_l	longitudinal wave velocity in rod or beam (phase velocity) $= \sqrt{E/\rho}$	m/s	in/s
c_l'	longitudinal wave velocity in plate or shell (phase velocity) $= \sqrt{E/\{\rho(1 - \nu^2)\}}$	m/s	in/s
D	bending stiffness of shell $= Eh^3/\{12(1 - \nu^2)\}$	Nm	lbf in
D_b	bending stiffness of shell (Erickson) $= E_f h_f h_c^2 (1 + h_f/h_c)^2 / \{2(1 - \nu^2)\}$	Nm	lbf in
E	Young's modulus of rod, beam or shell material	Pa	lbf/in ²
E_{ext}	extensional stiffness $= 2h_f E_f$	N/ m	lbf/in
E_f	Young's modulus of isotropic faceplate material of sandwich panel	Pa	lbf/in ²
F	complete elliptic integral of first kind		
F	bandwidth factor		
f	frequency	Hz	Hz
f_1	dimensionless frequency function		
G	shear modulus	Pa	lbf/in ²
G_c	shear modulus of isotropic core	Pa	lbf/in ²
G_{cx}, G_{cy}	core shear moduli in x- and y-directions, respectively	Pa	lbf/in ²
H	Bolotin's function for modal density of curved shells		
h	thickness of curved plate or cylinder	m	in

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h_c	thickness of sandwich core	m	in
h_f	thickness of faceplate	m	in
I_m	mass density moment of inertia per unit width about middle surface of panel	kg	*
K	complete elliptic integral of first kind		
L	slant length of cone	m	in
L_T	slant length defined in Sketch 3.5	in	
l	length of cylinder	m	in
M	mass per unit area of panel	kg/m ²	*
m	modal number		
n	modal number		
n_d	modal density	s	s
$\tilde{n}_d$	dimensionless modal density of paraboloidal shells		
p	focal length of parabola	m	in
Q	factor in expression for modal density of paraboloidal shells	s	s
R	radius of cylinder	m	in
R_l, R_u	lower and upper radii of cone	m	in
R_1	principal radius of curvature	m	in
R_2	principal radius of curvature	m	in
r_d	bending stiffness and shear parameter		
r_{eff}	bending stiffness and shear parameter for orthotropic core		
r_{max}	parameter used in doubly-curved sandwich panels	m ⁻²	in ⁻²
S_c	dimensionless core shear parameter		
α	frequency ratio (ω_1/ω), Equation (3.9)		
α_c	truncation ratio of cone		
α_1	parameter for curved panel, Equation (4.9)		
γ_R	R_1/R_2		
ε	$m\pi R/a$		
ζ	substitution symbol in Bolotin's theory		

* See footnote at end of Notation Section.

$\eta_{1,2}$	substitution symbol in Bolotin's theory		
θ	variable in modal density integration		
θ_1	lower limit of integration, Equations (3.4), (4.10) and (4.21)		
θ_2	upper limit of integration, Equations (3.4) and (4.24)		
ν	Poisson's ratio for faceplate or shell material		
ξ	$= \sin^2 \theta$ (Equation (3.8))		
ρ	density of shell	kg/m ³	*
ρ_c	mass density of core	kg/m ³	*
ρ_f	mass density of faceplate	kg/m ³	*
φ_0	limiting angle of parabola	rad	rad
χ	rotary inertia parameter $= \pi^2 I_m / (b^2 M)$		
χ_r	$= \chi / r_d$		
ψ	one-half cone angle at apex	rad	rad
Ω	dimensionless frequency function		
Ω_l, Ω_u	dimensionless frequency function for lower and upper ring frequencies		
ω	circular frequency	rad/s	rad/s
$\tilde{\omega}$	ω / ω_{max}		
ω_0	fundamental frequency of an infinitely long, simply supported flat panel $= (\pi^2 / b^2) \sqrt{D_b / M}$	rad/s	rad/s
ω_1	ring frequency (plane stress)	rad/s	rad/s
ω_{1l}	lower ring frequency of conical shell	rad/s	rad/s
ω_{1u}	upper ring frequency of conical shell	rad/s	rad/s
ω_c	centre frequency of band	rad/s	rad/s
ω_{max}	frequency at which modal density of paraboloidal shells is maximum	rad/s	rad/s
ω_{mn}	natural frequencies	rad/s	rad/s
ω_R	ring frequency (plane strain)	rad/s	rad/s
ω_s	ring frequency for spherical sandwich panel	rad/s	rad/s

* In British units divide lb by 386.4.

2. INTRODUCTION

This Item is the second part of a Compendium of modal densities. The first part considered the modal densities of rods, beams and plates (ESDU, 2011). The second part deals with shell structures which are both plain cylinders and curved panels, as well as shells which have a sandwich core.

The modal density of thin, elastic shell structures undergoing small displacements, are only considered in the Item. The thickness of the shell, understood as the distance between two shell surfaces, measured along the normal to the middle surface, should be small in comparison with the characteristic dimension of the structure, such as the smallest radius of curvature. In the Item the expressions “shallow shell” or “curved panel”, are used. They should be understood as shells with small changes of slope along the undeformed middle surface (Leissa, 1973).

The modal density of shells, such as cylinders, and shallow shells in the form of curved panels is important in statistical energy analysis of structures. The natural frequencies of cylinders depend on the number of circumferential nodes and the number of nodes in the axial direction. Theoretical and experimental work for the accurate determination of natural frequencies was undertaken by Arnold and Warburton (1949). Approximate formulae for natural frequencies have been quoted by Heckl (1962) and Szechenyi (1971). Heckl derived expressions for the modal density of thin-walled cylinders and compared his results with some measurements. His expressions tended to underestimate the values of modal density. Based on Heckl, Szechenyi was able to propose semi-empirical formulae for the prediction of the modal densities of cylinder in one-third octaves or octave bands (Szechenyi, 1971). The most substantial work on the modal density of cylinders was carried out by Bolotin (1963, 2010). In his study Courant’s concept of modal density, derived from k-space analysis, is adopted (Courant and Hilbert, 1953). Bolotin’s results are applicable to a wide range of isotropic shells, including cylinders, spheres and panels with both cylindrical and spherical surfaces.

In the approach of Heckl the modal densities are reduced by a factor of two. Due to the symmetry of the problem and the degeneracy of the modes on the complete cylindrical shell, the number of modes is twice the number of resonance frequencies (Heckl, 1962). However, only circumferential waves with mode numbers, $m = 2, 4, 6, \dots$, are permissible, which introduces a factor of $\frac{1}{2}$ into the formulae for modal density. Finally, both factors cancel themselves (Langley, 1994), leading to results similar to those of Bolotin. Experimentally, the circumferential modes are responsible for considerable fluctuations in the modal density at low frequencies (Rennison and Bull, 1977).

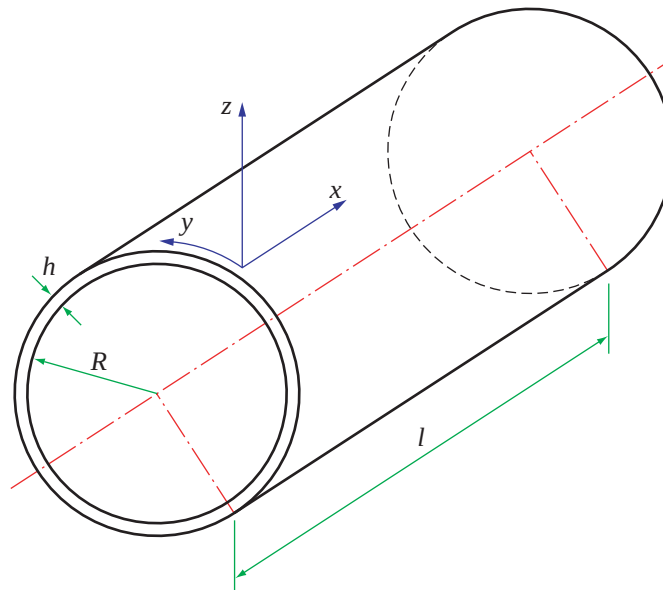
Bolotin’s work has been extended by others (Miller and Hart, 1969) and applied to sandwich structures (Wilkinson, 1968; Erickson, 1969; Ferguson and Clarkson, 1986).

Most of the results for modal density in the Item are presented in terms of the angular frequency, ω , but can also be written as a function of frequency in Hz, as

$$n_d(f) = 2\pi n_d(\omega). \quad (2.1)$$

3. ISOTROPIC SHELLS

The main method for this Section is based on Bolotin (1963 and 2010). It applies to both closed shells and sections of shells. The coordinate system for a cylindrical shell is shown in Sketch 3.1



Sketch 3.1 Cylinder with coordinate system

Some formulae supplied by Heckl and Szechenyi are also presented in this Section.

3.1 Thin-walled Cylinder (Heckl approach)

This section is based on the work of Heckl (Heckl, 1962) and refers to the cylinder shown in Sketch 3.1.

3.1.1 Model assumptions

- (1) A thin-walled cylinder is considered ($h \ll R$).
- (2) The dimensions of the cylinder are: length l , radius R and thickness of shell, h .
- (3) The cylinder is simply-supported.
- (4) The out-of-plane flexural vibrations of the cylinder walls are only considered.

3.1.2 Modal density

For $\omega R/c_l' < 1$, where c_l' is the velocity of the longitudinal waves in the shell,

$$n_d(\omega) = \frac{\sqrt{3}lR}{2\pi h c_l'} \left[\frac{\pi}{2} + \arcsin(2\omega R/c_l' - 1) \right] \quad (3.1)$$

For $\omega R/c_l' > 1$,

$$n_d(\omega) = \frac{\sqrt{3}lR}{2h c_l'}. \quad (3.2)$$

Equation (3.2) for the modal density is equivalent to half that for a flat, isotropic plate (Heckl, 1962).

3.1.3 Applicability

Due to the symmetry of the problem and the degeneracy of the modes on the complete cylindrical shell, the number of modes is twice the number of resonance frequencies (Heckl, 1962). However, only circumferential waves with mode numbers, $m = 2, 4, 6, \dots$, are permissible, which introduces a factor of $\frac{1}{2}$ into the formulae for modal density. Finally, according to Langley both factors cancel themselves (Langley, 1994). Heckl has provided experimental results which give reasonable agreement with his theory (Heckl, 1962).

3.2 Curved Isotropic Panel (Bolotin)

Bolotin devised a method for establishing the modal density of a doubly-curved isotropic shell (Bolotin, 1963). His method is based on the Donnell-Vlasov equation for shallow shells. Bolotin's approach can be extended to include a cylinder and a spherical shell. Furthermore, it can take into account shallow shells in the form of panels with cylindrical or spherical shapes.

3.2.1 Model assumptions

- (1) A thin, curved elastic shell with dimensions $a \times b \times h$ is considered.
- (2) The orthogonal, curvilinear coordinates, x and y , coincide with the lines of principal curvature; R_1 and R_2 denote the principal radii of curvature.
- (3) The shell is isotropic with material density ρ , modulus of elasticity E and panel stiffness D , where $D = Eh^3/12(1-\nu^2)$.
- (4) Tangential inertia forces are neglected.
- (5) Flexural vibrations which are out-of-plane are considered.

3.2.2 Natural frequencies and modal densities

Natural frequencies are approximately estimated by the expression

$$\omega_{mn}^2 = \frac{D}{\rho h} \left\{ \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]^2 + \frac{Eh}{DR_1^2} \frac{\left[\gamma_R \left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]^2}{\left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]^2} \right\}, \quad (3.3)$$

where $\gamma_R = R_1/R_2$. (The boundaries are of the shear diaphragm type, which are equivalent to simple supports.) The modal density is expressed by

$$n_d(\omega) = \frac{ab}{2\pi^2} \sqrt{\frac{\rho h}{D}} \int_{\theta_1(\omega)}^{\theta_2(\omega)} \frac{d\theta}{\left[\omega^2 - \omega_1^2 (\gamma_R \cos^2 \theta + \sin^2 \theta) \right]^{1/2}}, \quad (3.4)$$

where ω_1 is the ring frequency and the integration with respect to the variable θ is carried out over that part of the quadrant, $0 \leq \theta \leq \pi/2$, for which the integrand is positive. The expression for the ring frequency used in Equation (3.4) is often called the plain stress ring frequency (Langley, 1994) and is given by:

$$\omega_1 = \frac{1}{R} \sqrt{\frac{E}{\rho}}. \quad (3.5)$$

Note that the plain strain ring frequency is $\omega_R = (1/R) \sqrt{E/\rho(1-\nu^2)}$ (Langley, 1994).

Introducing the substitutions:

$$\eta_{1,2} = \frac{1 \pm \alpha \gamma_R}{\alpha(1 - \gamma_R)}, \quad (3.6)$$

$$\varsigma = \sqrt{\frac{2\alpha(\gamma_R - 1)}{(1 - \alpha)(1 + \alpha\gamma_R)}}, \quad (3.7)$$

$$\xi = \sin^2 \theta, \quad (3.8)$$

$$\alpha = \omega_1 / \omega \quad (3.9)$$

and, finally,

$$H(\alpha, \gamma_R) = \frac{1}{\alpha\pi|1 - \gamma_R|} \int_{\xi_1}^{\xi_2} \frac{d\xi}{[\xi(1 - \xi)(\eta_1 + \xi)(\eta_2 + \xi)]^{1/2}}, \quad (3.10)$$

the modal density has been evaluated as (Boltin, 1963)

$$n_d(\omega) = \frac{ab}{4\pi} \sqrt{\frac{\rho h}{D}} H(\alpha, \gamma_R). \quad (3.11)$$

The integral in Equation (3.10) can be expressed by the complete elliptic integral of the first kind K (Gradshteyn and Ryzhik, 2000; Whittaker and Watson, 1927).

The integration over ξ , in Equation (3.10), is carried out over the part of the interval $0 \leq \xi \leq 1$ for which the integrand is positive.

Note that the factor $(ab/4\pi) \sqrt{\rho h/D}$ in Equation (3.11) is the asymptotic modal density of an isotropic flat plate (ESDU, 2011).

For $\gamma_R = R_1/R_2 > 1$:

(1) When $\alpha\gamma_R < 1$ and $\alpha < 1$

$$H(\alpha, \gamma_R) = \frac{2}{\pi \sqrt{(1 - \alpha)(1 + \alpha\gamma_R)}} K(\varsigma). \quad (3.12)$$

(2) When $\alpha\gamma_R > 1$ and $\alpha < 1$

$$H(\alpha, \gamma_R) = \frac{\sqrt{2}}{\pi \sqrt{\alpha(\gamma_R - 1)}} K\left(\xi^{-1}\right). \quad (3.13)$$

(3) When $\alpha\gamma_R > 1$ and $\alpha > 1$

$$H(\alpha, \gamma_R) = 0. \quad (3.14)$$

3.2.3 Spherical Shell

For the special case of a spherical shell $\gamma_R = 1 (R_1 = R_2)$ and integral $H(\alpha, \gamma_R)$ is given by

$$H(\alpha, 1) = 0 \text{ for } \alpha > 1 \quad (3.15)$$

and

$$H(\alpha, 1) = 1/\sqrt{1-\alpha^2} \text{ for } \alpha < 1. \quad (3.16)$$

A singularity exists at the ring frequency, $\alpha = 1$

3.2.4 Cylinder

The other special case of the model is that of a cylindrical shell when $\gamma_R = 0 (R_2 = \infty)$. In this case

$$H(\alpha, 0) = \frac{2}{\pi \sqrt{1+\alpha}} K\left(\sqrt{\frac{2\alpha}{1+\alpha}}\right) \quad (3.17)$$

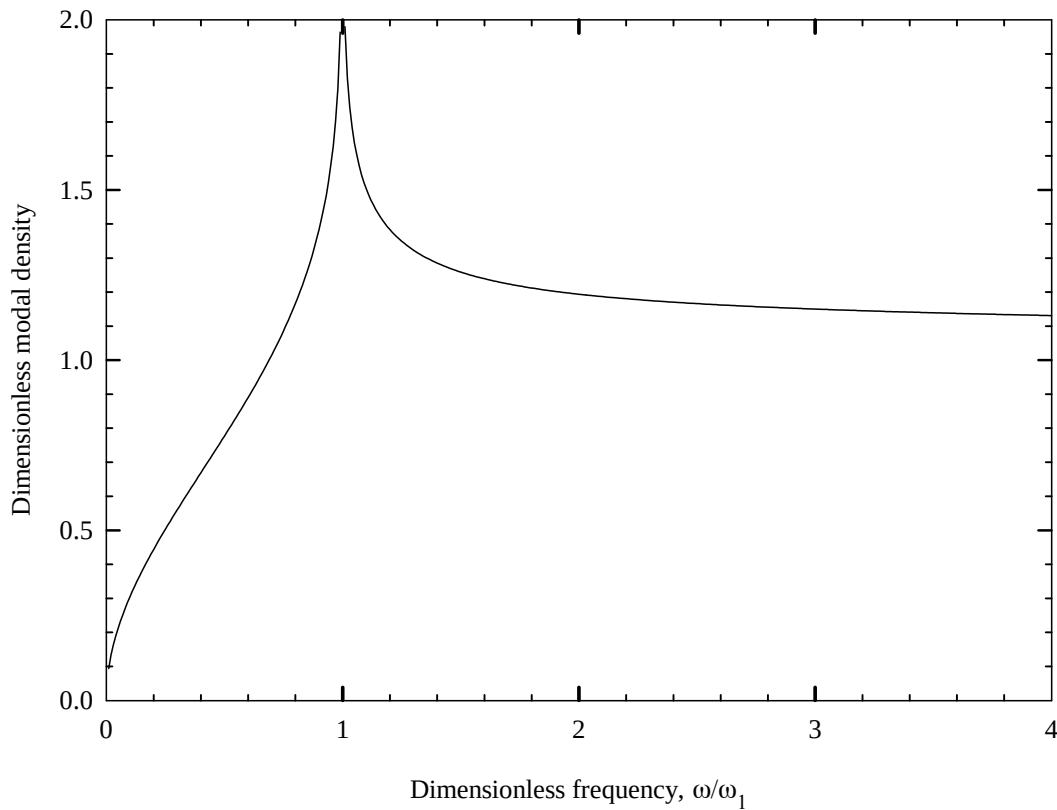
for $\alpha < 1$ and

$$H(\alpha, 0) = \frac{\sqrt{2}}{\pi \sqrt{\alpha}} K\left(\sqrt{\frac{1+\alpha}{2\alpha}}\right) \quad (3.18)$$

for $\alpha > 1$.

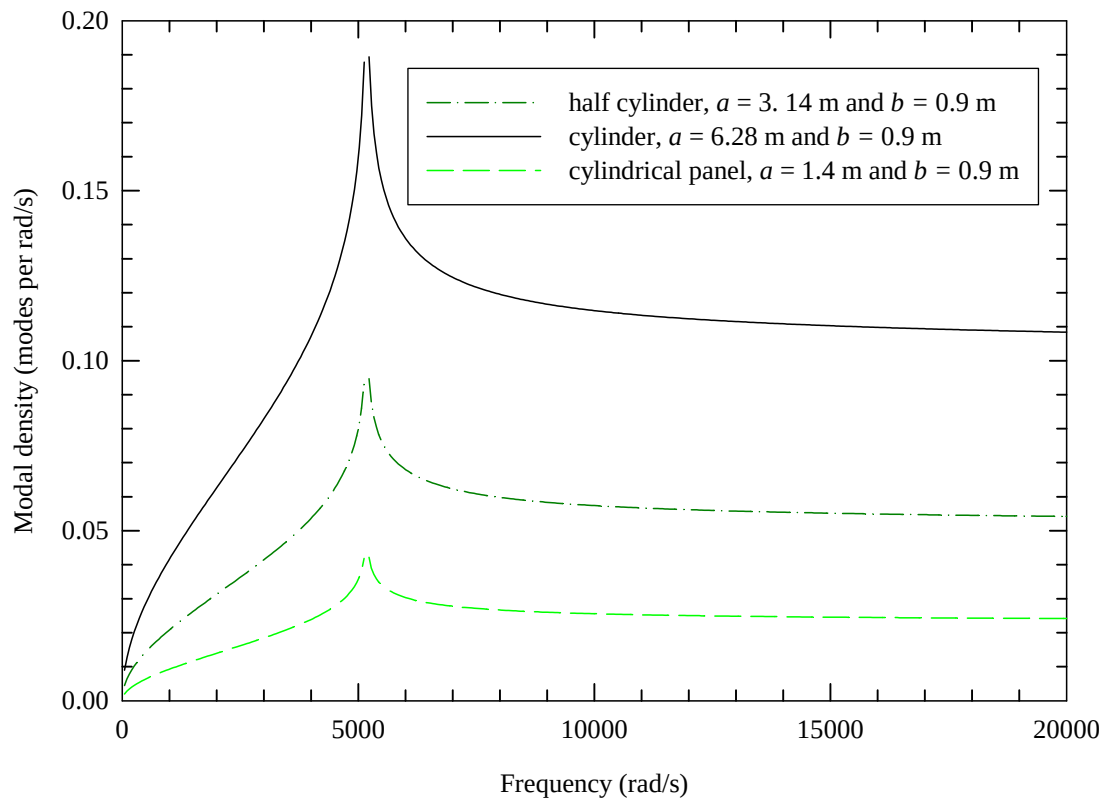
3.2.5 Results

Dimensionless modal density, $H(\alpha, 0)$, for a cylinder and cylindrical surfaces has been plotted as a function of dimensionless frequency, ω/ω_1 , in Sketch 3.2. There is a singularity at $\omega/\omega_1 = 1$ and as ω/ω_1 tends to infinity the dimensionless modal density approaches unity, $H(0, 0) = 1$.



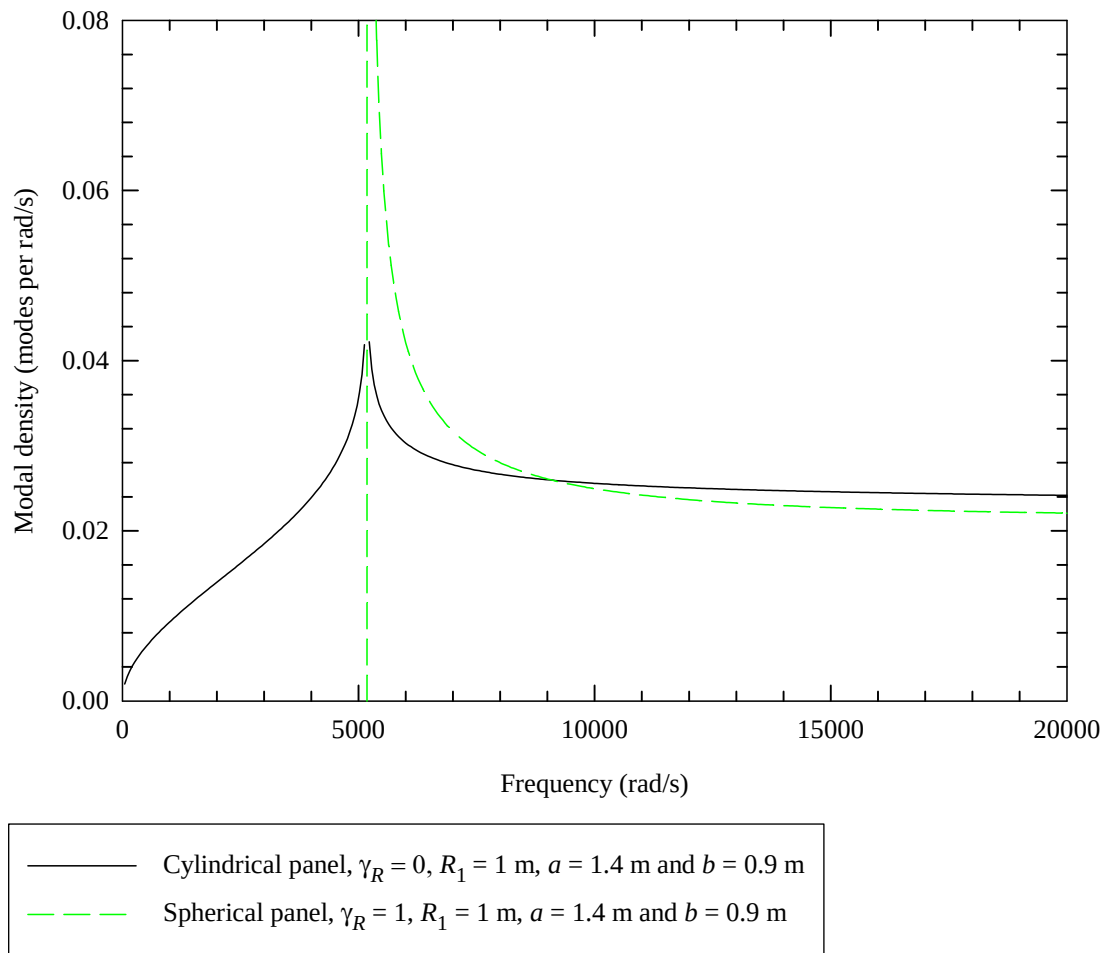
Sketch 3.2 Dimensionless modal density of a cylinder as a function of frequency ratio ω/ω_1

Some results for the modal density as a function of frequency are shown in Sketch 3.3 for a cylinder and sections of a cylinder. The shells were made from aluminium alloy. At high frequencies the modal density tends to that of a flat isotropic plate, *i.e.*, H in Equation (3.17) approaches unity.



Sketch 3.3 Modal density of cylinder and cylindrical surfaces, $R_1 = 1$ m, $h = 3$ mm, $E = 72$ GPa, $\rho = 2700$ kg/m³ and $\nu = 0.3$

In Sketch 3.4 the modal densities of a cylindrical panel and a spherical panel with the same dimensions a and b are compared. For the cylindrical panel $R_1 = 1$ m and $R_2 = \infty$, whilst for the spherical panel $R_1 = R_2 = 1$ m. In the case of the spherical panel there are no modes below the ring frequency. At higher frequencies the modal densities converge on that of the flat isotropic plate in both cases. Bolotin (2010) has obtained a distribution of natural frequencies within frequency bands as part of a numerical experiment and has been able to confirm numerically the general shape of the curves of modal density against frequency.



Sketch 3.4 Modal densities of cylindrical and spherical panels, $h = 3$ mm, $E = 72$ GPa, $\rho = 2700$ kg/m³, $\nu = 0.3$

In the case of a doubly-curved shell with $\gamma_R > 1$ the dimensionless modal density, $\eta_d(\omega)/\{(ab/4\pi)\sqrt{\rho h/D}\}$, is shown in Sketch 3.5 in terms of the dimensionless frequency, ω/ω_1 . In the example shown it is seen that a singularity occurs at $\omega/\omega_1 = \gamma_R$. The doubly-curved shell has two ring frequencies, the lower frequency is associated with R_1 ($\omega_{1l} = (1/R_1)\sqrt{E/\rho}$) and the upper ring frequency with R_2 ($\omega_{1u} = (1/R_2)\sqrt{E/\rho}$). Below $\omega/\omega_1 = 1.0$ there are no modes and the modal density is zero. The lower ring frequency, ω_{1l} , is 2589 rad/s (Equation (3.5)). At high frequencies the dimensionless modal density approaches a value of one, i.e., the modal density tends to $(ab/4\pi)\sqrt{\rho h/D}$, the value for a flat plate.