ESDU Copyright material. For current status contact ESDU.

# ESDU 13003

Issued November 2014
Re-issued March 2016

# Modal density of acoustic spaces

**Associated software: ESDUpac m1303a**

Endorsed by
The Royal Aeronautical Society

Documents
Document Sharing Hub

## ESDU DATA ITEMS

Data Items provide validated information in engineering design and analysis for use by, or under the supervision of, professionally qualified engineers. The data are founded on an evaluation of all the relevant information, both published and unpublished, and are invariably supported by original work of ESDU staff engineers or consultants. The whole process is subject to independent review for which crucial support is provided by industrial companies, government research laboratories, universities and others from around the world through the participation of some of their leading experts on ESDU Technical Committees. This process ensures that the results of much valuable work (theoretical, experimental and operational), which may not be widely available or in a readily usable form, can be communicated concisely and accurately to the engineering community.

We are constantly striving to develop new work and review data already issued. Any comments arising out of your use of our data, or any suggestions for new topics or information that might lead to improvements, will help us to provide a better service.

## THE PREPARATION OF THIS DATA ITEM

The work on this particular Data Item was monitored and guided by the Vibration and Acousitc Fatigue Committee. This Committee first met in 1966 and now has the following membership:

Chairman
Prof. R.D. Adams — University of Bristol

Members
Dr N.S. Ferguson — University of Southampton
Mr S. Georgiou — SELEX Sensors and Airborne Systems Ltd, Luton
Mr E. Green — Rolls-Royce, Derby
Dr P. Green* — BAE Systems, Preston, Lancashire
Dr A. Grillenbeck* — IABG Space Test Center, Germany
Mr A.R. Harris* — Northrop Grumman, USA
Dr D.M.A. Millar* — Bombardier Aerospace Short Brothers plc, Belfast
Mr C.D. Munger* — The Boeing Company, USA
Dr M. Nash — Royal Aeronautical Society, London
Dr S. Patsias — Rolls-Royce, Derby
Dr J. Santiago Prowald* — European Space Agency (ESTEC), The Netherlands
Mr E. Smyth — Bombardier Aerospace Short Brothers plc, Belfast
Prof. R.G. White — University of Southampton (Member Emeritus)
Ir. J.J. Wijker — Twente University, Netherlands.

* Corresponding Member

The construction and subsequent development of the Data Item were undertaken by

Dr J. Anderson — Senior Engineer
Dr M. Bratos-Anderson — Consultant

The person with overall responsibility for the work in this subject area is Dr C.B. Chinoy, Head of the Aircraft Noise and Structural Dynamics Group.

ESDU Copyright material. For current status contact ESDU.

Documents
Document Sharing Hub

# MODAL DENSITY OF ACOUSTIC SPACES

## CONTENTS


| | | Page |
|---|---|---|
| 1. | NOTATION | 1 |
| 2. | INTRODUCTION | 3 |
| 3. | ONE-DIMENSIONAL ACOUSTIC SPACE | 4 |
| 4. | RECTANGULAR ACOUSTIC ENCLOSURES | 5 |
| | 4.1 Representation of Normal Modes in k-space | 6 |
| | 4.2 Axial, Tangential and Oblique Modes | 7 |
| 5. | APPROXIMATE MODELS FOR 2-D AND 3-D RECTANGULAR ACOUSTIC SPACES | 8 |
| | 5.1 Maa's Model | 9 |
| | 5.1.1 Application of Maa's model and comparison with other approximate models | 11 |
| | 5.2 Roe Model | 14 |
| | 5.3 Dimensionless Forms of Formulae in Approximate Models | 15 |
| 6. | EXACT NUMBER OF MODES FOR RECTANGULAR ACOUSTIC SPACE | 19 |
| | 6.1 Application of Blevins Model; Comparison with Approximate Models | 21 |
| | 6.2 Summary and Conclusions | 32 |
| 7. | COMPUTER PROGRAM | 33 |
| | 7.1 Program for Estimation of Number of Modes and Modal Density within Bands for Rectangular Enclosure | 33 |
| | 7.2 Example | 34 |
| 8. | REFERENCES | 42 |


ESDU Copyright material. For current status contact ESDU.



Documents
Document Sharing Hub

Full document details: ESDU 13003-R1 2016

# MODAL DENSITY OF ACOUSTIC SPACES

## 1. NOTATION

| | | Units | |
|---|---|---|---|
| | | *SI* | *British* |
| $A$ | $L_x \times L_y$, area of plate (two-dimensional enclosure) | $m^2$ | $in^2$ |
| $a$ | radius of cylinder or sphere | m | in |
| $c_0$ | velocity of sound | m/s | in/s |
| $F$ | $f/f_{111}$ | | |
| $f$ | frequency | Hz | Hz |
| $f_n$ | natural frequencies for one-dimensional acoustic enclosure | Hz | Hz |
| $f_{111}$ | natural frequency of first oblique mode (1, 1, 1) | Hz | Hz |
| $f_{n_x n_y n_z}$ | natural frequency of mode $(n_x, n_y, n_z)$ in three-dimensional enclosure | Hz | Hz |
| $\vec{k}$ | wavenumber vector | $m^{-1}$ | $in^{-1}$ |
| $k_{n_x}$, $k_{n_y}$, $k_{n_z}$ | components of wave number vector $\vec{k}_{n_x n_y n_z}$ | $m^{-1}$ | $in^{-1}$ |
| $\vec{k}_{n_x n_y n_z}$ | $[k_{n_x}, k_{n_y}, k_{n_z}] = [\pi n_x/L_x, \pi n_y/L_y, \pi n_z/L_z]$ wavenumber vector of mode $(n_x, n_y, n_z)$ | $m^{-1}$ | $in^{-1}$ |
| $k_x$, $k_y$, $k_z$ | axes in $k$-space | $m^{-1}$ | $in^{-1}$ |
| $L$ | sum of lengths of all edges of rectangular enclosure | m | in |
| $L_x$, $L_y$, $L_z$ | dimensions of edges of rectangular enclosure in $x$, $y$ and $z$ directions | m | in |
| $l$ | length of duct, length of cylinder | m | in |
| $N$ | cumulative number of modes below frequency $f$ | | |
| $N_c$ | cumulative number of modes below frequency $f$ (asymptotic approximation of Courant) | | |
| $N_{cube}$ | cumulative number of modes below frequency $F$ (cube) | | |
| $N_{closed}$ | cumulative number of modes below frequency $f$ for closed rectangular enclosure | | |

Issued November 2014 – 42 pages
Re-issued March 2016 – 42 pages



ESDU Copyright material. For current status contact ESDU.
Documents
Document Sharing Hub
Full document details: ESDU 13003-R1 2016

| | | | |
|---|---|---|---|
| $N_{open}$ | cumulative number of modes below frequency $f$ for rectangular enclosure with open boundary | | |
| $N_x$, $N_y$, $N_z$ | number of modes at each index axis (Blevins) | | |
| $N^{1D}$ | cumulative number of modes in one-dimensional acoustic enclosure | | |
| $N^{2D}$ | cumulative number of modes in two-dimensional acoustic enclosure | | |
| $n$ | integer denoting mode in one-dimensional enclosure | | |
| $\vec{n}$ | normal vector to enclosure wall | | |
| $n_d$ | modal density | s | s |
| $n_x$, $n_y$, $n_z$ | integers defining modes in rectangular enclosure | | |
| $n_{x\,max}$, $n_{y\,max}$, $n_{z\,max}$ | $= 2L_x f/c_0, 2L_y f/c_0, 2L_z f/c_0$ | | |
| $P$ | perimeter of two-dimensional enclosure | m | in |
| $p$ | sound pressure | Pa | lbf/in$^2$ |
| $r$ | radius | m | in |
| $r_{zx}$ | rectangular enclosure shape ratio $L_x/L_z$ | | |
| $r_{zy}$ | rectangular enclosure shape ratio $L_y/L_z$ | | |
| $S$ | surface area of rectangular enclosure | m$^2$ | in$^2$ |
| $\vec{u}$ | sound particle velocity | m/s | in/s |
| $V$ | volume of rectangular enclosure | m$^3$ | in$^3$ |
| $V^{\mathrm{I}}$, $V^{\mathrm{II}}$ | different volumes of rectangular enclosures with the same shape ratios | m$^3$ | in$^3$ |
| $\delta N_f$ | number of modes in frequency band $\delta f$ | | |
| $\lambda$ | wavelength | m | in |
| $\omega$ | circular frequency | rad/s | rad/s |

Any coherent system of units may be used.

**Abbreviation**

| | |
|---|---|
| SEA | statistical energy analysis |



ESDU Copyright material. For current status contact ESDU.

Documents
Document Sharing Hub

Full document details: ESDU 13003-R1 2016

## 2. INTRODUCTION

An important application of Statistical Energy Analysis is the prediction of the noise in a space, such as an aircraft interior, where the noise is caused by structural vibration. The aircraft interior is treated as an acoustic space, or enclosure, in which there are fluctuations in fluid pressure. The modal density of the acoustic spaces is required in SEA estimations.

In this Item acoustic spaces are considered in the context of their cumulative number of modes as well as their modal densities. In general, acoustic space is formed by a three-dimensional enclosure which is filled with a homogeneous, inviscid and compressible fluid. In such a medium sound waves are propagated and, if the walls of the enclosure are reflective, a pattern of standing waves is created. If a standing wave vibrates in simple harmonic motion, it can be referred to as a normal mode of vibration.

The modal density $n_d(f)$ is defined as the number of natural frequencies per Hz and modal density $n_d(\omega)$ is the number of natural frequencies per rad/s. Another definition of modal density is the inverse of the locally averaged (in the frequency domain) interval between neighbouring natural frequencies,

$$n_d(f) = \frac{1}{\langle \Delta f \rangle_f} \tag{2.1}$$

and similarly,

$$n_d(\omega) = \frac{1}{\langle \Delta \omega \rangle_\omega} . \tag{2.2}$$

Consequently,

$$n_d(f) = 2\pi n_d(\omega) . \tag{2.3}$$

Modal density is a statistical concept which is especially useful in the frequency range of natural frequencies of an enclosure for higher orders of modes. For such situations the cumulative number of modes $N(f)$ can be treated as a continuous function of frequency and the relationship between the modal density $n_d(f)$ and the cumulative number of modes $N(f)$ can be expressed as:

$$n_d(f) = \frac{\mathrm{d}N}{\mathrm{d}f} , \tag{2.4}$$

or, in terms of the circular frequency $\omega$,

$$n_d(f) = 2\pi \frac{\mathrm{d}N(\omega)}{\mathrm{d}\omega} . \tag{2.5}$$

In previous ESDU Items compendia of asymptotic modal densities have been provided for (a) rods, beams and plates (ESDU a, 2011) and (b) shell structures (ESDU b, 2011).

Modal densities of acoustic spaces are provided by exact, as well as by approximate formulae. In general, the determination of the modal density for acoustic spaces is equivalent to a boundary value problem. In other words the acoustic wave equation with the appropriate boundary conditions should be solved and the natural frequencies of the acoustic space and their distribution in $k$-space established.



ESDU Copyright material. For current status contact ESDU.

Documents
Document Sharing Hub

Full document details: ESDU 13003-R1 2016

## 3. ONE-DIMENSIONAL ACOUSTIC SPACE

A typical example of a one-dimensional acoustic space is a duct, in which the transverse dimension is small compared with the wavelength of the sound. Usually, such an acoustic space is considered as an air column within a rigid enclosure. The natural frequencies of a duct of length $l$, closed at both ends or open at the ends, are:

$$f_n = n\left(\frac{c_0}{2l}\right), \tag{3.1}$$

where $n = 1\ 2, 3$, *etc*. Since the interval between the neighbouring natural frequencies is in this case

$$\Delta f = \frac{c_0}{2l}, \tag{3.2}$$

the modal density of a duct of length $l$ is constant,

$$n_d(f) = \frac{2l}{c_0}. \tag{3.3}$$

In terms of circular frequency, $\omega$, the modal density is

$$n_d(\omega) = \frac{l}{\pi c_0}, \tag{3.4}$$

as indicated by Equation (2.3).

For a tube which is open at one end and closed at the other, the natural frequencies are given as

$$f_n = n\left(\frac{c_0}{4l}\right), \tag{3.5}$$

where $n = 1, 3, 5$, *etc*. However, Equations (3.2) to (3.4) still apply. Finally, $N$, the number of natural frequencies below a frequency $f$ Hz, is given by

$$N = \mathrm{Int}\left(\frac{2l}{c_0}f\right), \tag{3.6}$$

where Int( ) is an integer function which rounds down the argument to the nearest integer.

In the considered case of the one-dimensional acoustic space, the modal density, as well as the cumulative number of modes, is independent of the end conditions.



ESDU Copyright material. For current status contact ESDU.

Documents
Document Sharing Hub

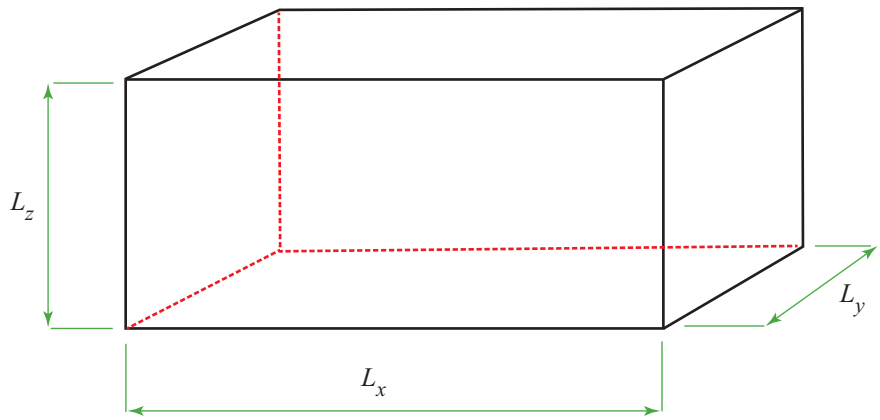
Full document details: ESDU 13003-R1 2016

## 4. RECTANGULAR ACOUSTIC ENCLOSURES

For a rectangular enclosure (more accurately a right parallelepiped or a right cuboid) with rigid sides of lengths $L_x$, $L_y$ and $L_z$, as shown in Sketch 4.1, the natural frequencies, which refer to the appropriate modes of vibration, are given by

$$f_{n_x n_y n_z} = \frac{c_0}{2}\sqrt{\left(\frac{n_x}{L_x}\right)^2 + \left(\frac{n_y}{L_y}\right)^2 + \left(\frac{n_z}{L_z}\right)^2}\,, \tag{4.1}$$

where $n_x$, $n_y$ and $n_z$ are integers: $n_x = 0, 1, 2, 3..$; $n_y = 0, 1, 2, 3..$; $n_z = 0, 1, 2, 3..$




**Sketch 4.1 Three-dimensional enclosure**

Equation (4.1) results from solving the acoustic wave equation

$$\nabla^2 p + (\omega/c_0)^2 p = 0 \tag{4.2}$$

in conjunction with the boundary conditions at the rigid walls of the enclosure (closed enclosure) (Kinsler *et al*, 1982; Morse, 1948; Morse and Feshbach, 1953) . The boundary conditions are formulated for the normal gradient of sound pressure

$$\vec{n} \cdot \nabla p = 0 \tag{4.3}$$

or the sound particle velocity $\vec{u}$

$$\vec{n} \cdot \vec{u} = 0\,, \tag{4.4}$$

where $\vec{n}$ is the normal to the enclosure wall.

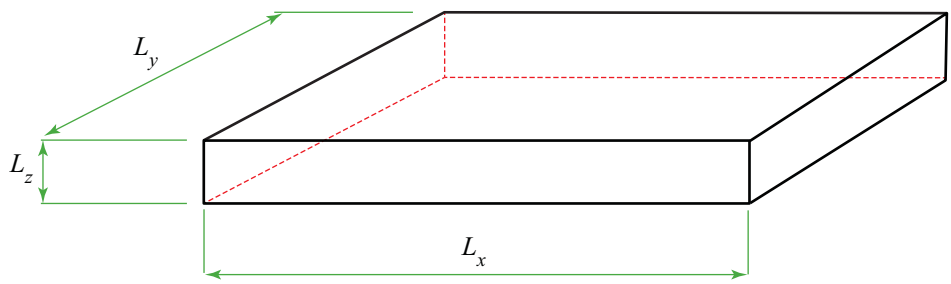
In the case when the rectangular enclosure is open, the acoustic wave Equation (4.2) is considered together with the condition $p = 0$ at the boundary of the enclosure. The natural frequencies are given also by Equation (4.1), but in this case: $n_x = 1, 2, 3..$; $n_y = 1, 2, 3..$; $n_z = 1, 2, 3..$

The application of the boundary conditions restricts the wave equation solutions to a series of discrete functions called normal modes or eigenfunctions, each of which is associated with the appropriate natural frequency. The distribution of the natural frequencies within the frequency band provides the concept of the modal density $n_d(f)$, which is defined as the number of natural frequencies per unit frequency band.



ESDU Copyright material. For current status contact ESDU.
Documents
Document Sharing Hub
Full document details: ESDU 13003-R1 2016

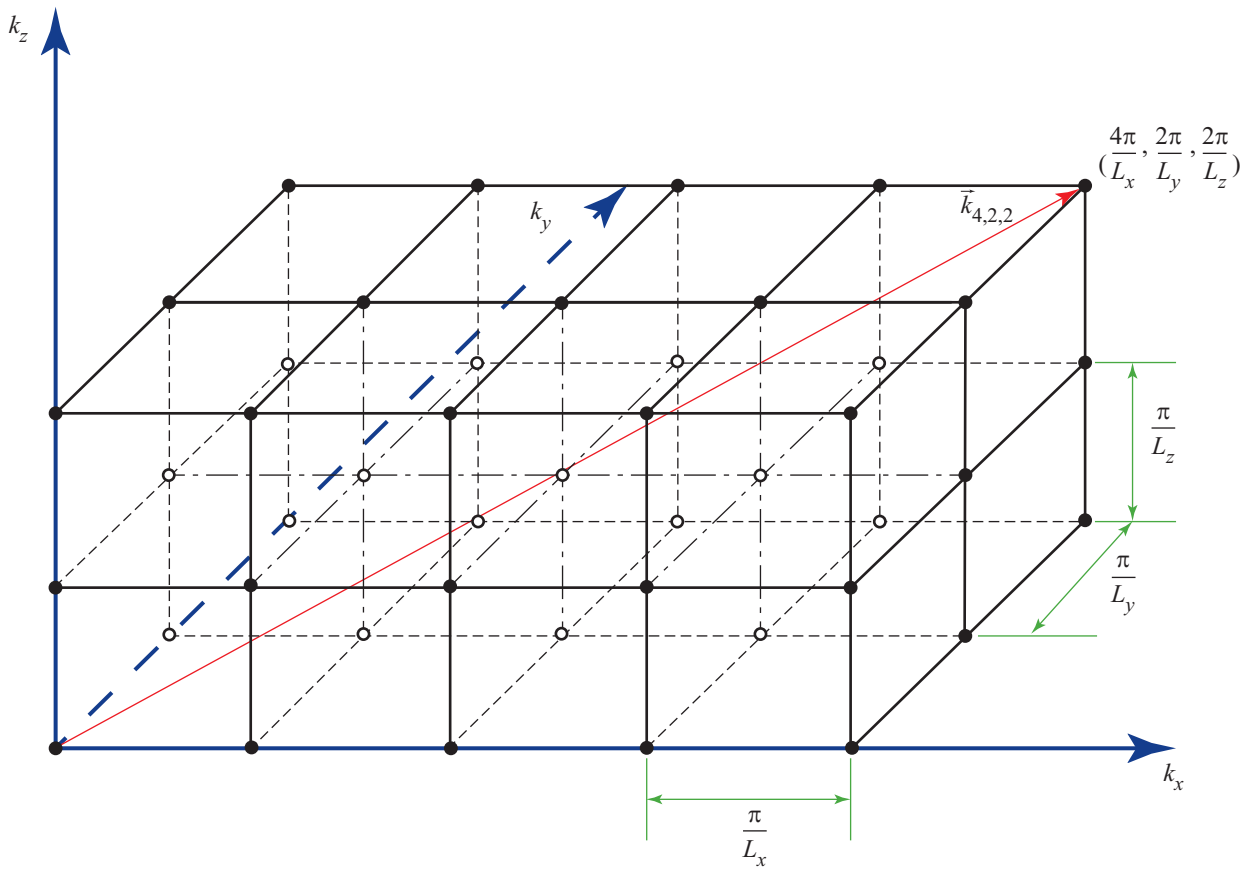
Acoustic space can be treated as a two-dimensional enclosure, see Sketch 4.2, when sound travels between two rigid, rectangular walls, which are in parallel and the separation of the plates is small compared with the sound wavelength $\lambda$ and the dimensions of the plate: $L_x$, $L_y$ ($L_z < \lambda << L_x, L_y$). The boundary conditions are applied in this case to the edges (Blevins, 2006).




**Sketch 4.2 Two-dimensional enclosure**

## 4.1 Representation of Normal Modes in *k*-space

The normal modes of vibration for rectangular enclosures with rigid walls are represented by points in $k$-space ($k$-wavenumber space, where $k = \omega/c_0 = 2\pi f/c_0$), see Sketch 4.3. The axes of the coordinate system are $k_x$, $k_y$ and $k_z$.




**Sketch 4.3 Mode lattice for rectangular enclosure in *k*-space**



ESDU Copyright material. For current status contact ESDU.
Documents
Document Sharing Hub
Full document details: ESDU 13003-R1 2016

The normal modes with values $(n_x, n_y, n_z)$ and with natural frequency $f_{n_x n_y n_z}$, given by Equation (4.1), can be generated by a plane wave which propagates in the direction of the vector, $\vec{k} = \vec{k}_{n_x n_y n_z} = [\pi n_x/L_x, \pi n_y/L_y, \pi n_z/L_z]$ and interacts with its reflections from the enclosure walls, forming a standing wave. The vector from the coordinate origin to the considered point $(\pi n_x/L_x, \pi n_y/L_y, \pi n_z/L_z)$ is the wavenumber vector $\vec{k}_{n_x n_y n_z}$ of the mode $(n_x, n_y, n_z)$. The length of the vector is given by

$$k_{n_x n_y n_z} = \pi \sqrt{\left(\frac{n_x}{L_x}\right)^2 + \left(\frac{n_y}{L_y}\right)^2 + \left(\frac{n_z}{L_z}\right)^2} \,. \tag{4.5}$$

It relates to the frequency of the mode (or standing wave frequency) $f_{n_x n_y n_z}$, as

$$k_{n_x n_y n_z} = \frac{2\pi}{c_0} f_{n_x n_y n_z} \,. \tag{4.6}$$

The mode shape describes the relative displacement of the elements of fluid within the enclosure, as they vibrate in simple harmonic motion with frequency $f_{n_x n_y n_z}$.

## 4.2 Axial, Tangential and Oblique Modes

Normal modes of vibration in a rectangular enclosure may be classified as axial, tangential or oblique. With axial modes the component plane waves travel along one axis, parallel to two pairs of walls. The tangential modes result from component waves which move in parallel with any pair of walls. Finally, oblique modes refer to the case in which the directions of travel of the component waves are parallel to none of the three pairs of walls.

In $k$-space, Sketch 4.3, each point $(\pi n_x/L_x, \pi n_y/L_y, \pi n_z/L_z)$ refers to a single normal mode of vibration. The triplets of integers $n_x, n_y, n_z$ specify modes. The integers have values: $n_x = 0, 1, 2, \ldots$; $n_y = 0, 1, 2, \ldots$; $n_z = 0, 1, 2, \ldots$ for the closed (rigid) boundaries of the enclosure and $n_x = 1, 2, 3, \ldots$; $n_y = 1, 2, 3, \ldots$; $n_z = 1, 2, 3 \ldots$, when the open boundaries are considered. The axial modes (x-axis, y-axis and z-axis modes) have a pair of integers of the $(n_x, n_y, n_z)$ triplets equal to zero. The tangential modes, ($xy$, $xz$ and $yz$-tangential modes) have one integer of $(n_x, n_y, n_z)$ triplets equal to zero. The axial and tangential modes constitute six classes of modes (Morse, 1948; Morse and Ingard, 1968). The seventh class of modes refer to oblique modes for which none of $(n_x, n_y, n_z)$ integers is zero. In the case of oblique modes the component plane waves are oblique to all three pairs of walls. The first oblique mode refers to the triplet (1, 1, 1) and mode frequency $f_{111}$. Modes with frequencies greater than $f_{111}$ are predominantly oblique.

It may be noted that oblique modes are associated with maximum damping. Axial modes have the least damping (Brüel, 1978).



ESDU Copyright material. For current status contact ESDU.

Documents
Document Sharing Hub

Full document details: ESDU 13003-R1 2016

## 5. APPROXIMATE MODELS FOR 2-D and 3-D RECTANGULAR ACOUSTIC SPACES

The asymptotic value $(f \to \infty)$ of the cumulative number of modes below a certain frequency, as well as the asymptotic modal density has been provided by Courant (Courant, 1922; Courant and Hilbert, 1953). The asymptotic estimation of $N_c$, the cumulative number of modes below frequency $f$ for an acoustic space of volume $V$, is

$$N_c = \lim_{f \to \infty} N = \frac{4\pi V f^3}{3c_0^3}, \tag{5.1}$$

and for acoustic spaces, which can be treated as two-dimensional, see Sketch 4.2,

$$N_c = \lim_{f \to \infty} N = \frac{\pi A f^2}{c_0^2}. \tag{5.2}$$

In the case of a rectangular two-dimensional enclosure, see Sketch 4.2, $A = L_x L_y$, and $L_z < \lambda << L_x$, $L_y$. Equations (5.1) and (5.2) can be applied only for the acoustic spaces when the limiting wavelength, $\lambda_{max}$, is much smaller than the dimensions of the considered space. However, Equations (5.1) and (5.2) can be used for any shape of acoustic space and for any boundary conditions.

For rectangular acoustic enclosures there exist approximate estimates of the cumulative modes below the limiting frequency $f$, introduced by Maa (Maa, 1939), independently by Bolt (Bolt, 1939) and followed by other authors (Hunt *et al*, 1939; Roe, 1941 ; Morse, 1948 ; Morse and Ingard, 1968). The formulae for these estimates can be applied within a wider range of limiting frequencies in comparison with the asymptotic formula of Courant, valid for high frequencies. Essentially, all the approaches (Maa, 1939; Bolt, 1939: Hunt *et al*, 1939; Morse, 1948; Morse and Ingard, 1968) lead to similar results in the form of the averaged or smoothed-out cumulative number of modes up to a frequency $f$.

The average of the cumulative number of modes can be estimated by considering the representation of modes in $k$-space. A point in $k$-space, as shown in Sketch 4.3, refers to a normal mode and is associated with a single unit cell. Since an individual lattice cell, see Sketch 4.3, has on average a volume of $\pi^3/(L_x L_y L_z)$ or $\pi^3/V$ in the $k$-space, its inverse $(V/\pi^3)$ provides the number of cells per unit volume of $k$-space. Hence, the average number of normal modes having a wavenumber vector length equal or less than $k$ is:

$$N = \frac{1}{8} \cdot \frac{4}{3}\pi k^3 \cdot \frac{V}{\pi^3}. \tag{5.3}$$

Finally,

$$N = \frac{V k^3}{6\pi^2}, \tag{5.4}$$

since $(1/8) \cdot (4\pi k^3/3)$ represents the volume of an eighth of a sphere of radius $k$.



ESDU Copyright material. For current status contact ESDU.

Documents
Document Sharing Hub

Full document details: ESDU 13003-R1 2016

The magnitude of the wave vector (wavenumber vector) relates to the frequency as $k = 2\pi f/c_0$, see also Equation (4.6). Hence, the average cumulative number of modes below frequency $f$, Equation (5.4), can be expressed in terms of frequency as

$$N = \frac{4\pi}{3} V \left(\frac{f}{c_0}\right)^3 . \tag{5.5}$$

The presented approach to the estimation of the averaged cumulative number of modes below the frequency $f$ leads to Equation (5.5), which has the form of Courant's asymptotic formula, Equation (5.1). An expression for the averaged modal density can be obtained from Equation (5.5) with the aid of Equation (2.4) and is:

$$n_d = \frac{4\pi V f^2}{c_0^3} . \tag{5.6}$$

An important feature of Courant's asymptotic formula, Equation (5.1), is that it can be applied to any shape of enclosure (Kuttruff, 1991).

## 5.1 Maa's Model

The introduction of corrections associated with the presence of axial and tangential modes results in an expression for the cumulative number of modes below $f$ in the form,

$$N_{closed} = \frac{4\pi}{3} V \left(\frac{f}{c_0}\right)^3 + \frac{\pi}{4} S \left(\frac{f}{c_0}\right)^2 + \frac{L}{8}\left(\frac{f}{c_0}\right) \tag{5.7}$$

for the case of the closed rectangular enclosure (with rigid walls) (Maa, 1939). The corresponding expression for the averaged modal density is:

$$n_d = \frac{4\pi V f^2}{c_0^3} + \frac{\pi f S}{2c_0^2} + \frac{L}{8c_0} . \tag{5.8}$$

For the case of open rectangular enclosures the number of modes below the frequency $f$ is given by (Morse, 1948; Blevins, 2006)

$$N_{open} = \frac{4\pi}{3} V \left(\frac{f}{c_0}\right)^3 - \frac{\pi}{4} S \left(\frac{f}{c_0}\right)^2 + \frac{L}{8}\left(\frac{f}{c_0}\right) . \tag{5.9}$$

The parameters $V$, $S$ and $L$ are, respectively, the volume of the rectangular enclosure, $V = L_x L_y L_z$, the total surface area of the enclosure, $S = 2(L_x L_y + L_x L_z + L_y L_z)$, and the sum of the lengths of the edges of the enclosure, $L = 4(L_x + L_y + L_z)$.

It should be noted that the right hand side of Equation (5.5) constitutes the first term in the approximate, 'smoothed-out' formulae, Equations (5.7) and (5.9), for the averaged cumulative number of modes with natural frequencies below frequency $f$. When frequency $f$ tends to infinity, $N_{closed} = N_{open} = N_c$.



ESDU Copyright material. For current status contact ESDU.

Documents
Document Sharing Hub

Full document details: ESDU 13003-R1 2016

Approximate formulae for the cumulative number of modes and the averaged modal density are summarised in Table 5.1. The table includes formulae on both closed and open rectangular enclosures, as well as on two-dimensional enclosures, in addition to formulae for the cumulative number of the constituent axial, tangential and oblique modes.

**TABLE 5.1 Rectangular acoustic enclosures: averaged cumulative number of modes below frequency *f* and averaged modal density**

| Type of mode | Type of boundary | $N$ | $n_d$ |
|---|---|---|---|
| All | 3D closed | $\frac{4\pi}{3}V\left(\frac{f}{c_0}\right)^3+\frac{\pi}{4}S\left(\frac{f}{c_0}\right)^2+\frac{L}{8}\left(\frac{f}{c_0}\right)$ | $\frac{4\pi f^2 V}{c_0^3}+\frac{\pi f S}{2c_0^2}+\frac{L}{8c_0}$ |
| All axial | 3D closed | $\frac{Lf}{2c_0}$ | $\frac{L}{2c_0}$ |
| All tangential | 3D closed | $\frac{\pi f^2 S}{2c_0^2}-\frac{fL}{2c_0}$ | $\frac{\pi f S}{c_0^2}-\frac{L}{2c_0}$ |
| Oblique | 3D closed | $\frac{4\pi f^3 V}{3c_0^3}-\frac{\pi f^2 S}{4c_0^2}+\frac{Lf}{8c_0}$ | $\frac{4\pi f^2 V}{c_0^3}-\frac{\pi f S}{2c_0^2}+\frac{L}{8c_0}$ |
| All | 3D open | $\frac{4\pi}{3}V\left(\frac{f}{c_0}\right)^3-\frac{\pi}{4}S\left(\frac{f}{c_0}\right)^2+\frac{L}{8}\left(\frac{f}{c_0}\right)$ | $\frac{4\pi f^2 V}{c_0^3}-\frac{\pi f S}{2c_0^2}+\frac{L}{8c_0}$ |
| All | 2D closed | $\pi A\left(\frac{f}{c_0}\right)^2+\frac{P}{2}\left(\frac{f}{c_0}\right)$ | $\frac{2\pi A f}{c_0^2}+\frac{P}{2c_0}$ |
| All | 2D open | $\pi A\left(\frac{f}{c_0}\right)^2-\frac{P}{2}\left(\frac{f}{c_0}\right)$ | $\frac{2\pi A f}{c_0^2}-\frac{P}{2c_0}$ |
| Notation: $V = L_x L_y L_z$, $S = 2(L_x L_y + L_x L_z + L_y L_z)$, $L = 4(L_x + L_y + L_z)$, $A = L_x L_y$ and $P = 2(L_x + L_y)$ | | | |

In general, Maa's formulae, Equations (5.7) and (5.8), can be used within a range of frequencies, where oblique modes dominate over tangential and axial modes.

The number of natural frequencies within the frequency band from $f$ to $f+\delta f$ is

$$\delta N_f = n_d(f)\,\delta f. \tag{5.10}$$

Hence, for closed enclosures (Maa, 1939)

$$\delta N_f = \left(\frac{4\pi f^2 V}{c_0^3}+\frac{\pi f S}{2c_0^2}+\frac{L}{8c_0}\right)\delta f. \tag{5.11}$$



ESDU Copyright material. For current status contact ESDU.

Documents
Document Sharing Hub

Full document details: ESDU 13003-R1 2016

It is worth mentioning that the average spacing between adjacent natural frequencies, expressed as the reciprocal of the modal density, see Equations (2.1) and (2.2), is

$$\langle \Delta f \rangle_f = \left( \frac{4\pi f^2 V}{c_0^3} + \frac{\pi f S}{2c_0^2} + \frac{L}{8c_0} \right)^{-1} . \tag{5.12}$$

The approximate formulae, which are listed in Table 5.1, have been derived for rectangular acoustic two and three-dimensional enclosures. Hence, there occurs the question of whether the formulae in Table 5.1 can be applied to acoustic enclosures of any shape. Morse and Ingard have given a negative answer, pointing out that the averaged cumulative number of modes below frequency $f$ depends not only on the volume $V$ and area $S$, but also the sum of the lengths of all the enclosure edges $L$, which would be impossible to define for spaces of irregular shape (Morse and Ingard, 1968). However, Morse and Ingard have confirmed that Maa's formula is valid for a cylindrical enclosure with $V = \pi a^2 l$, $S = 2\pi a^2 + 2\pi a l$ and $L = 4\pi a + 4l$, where $a$ is the radius of the cylinder and $l$ is its height. The expression for $L$ (the total length of the edges) is not an obvious choice.

In the case of a rectangular enclosure Maa's formula for the averaged cumulative number of modes below frequency $f$ has been also analysed and confirmed by Roe (Roe, 1941). Roe did not apply the averaging approach of Maa, but assumed that the cumulative number of modes can be expanded into a power series of $f$ and that the first two terms of the series constitute a sufficiently accurate representation of the cumulative number of modes for acoustic enclosures of different shapes and volumes:

$$N(f) = \frac{4\pi}{3} V \left( \frac{f}{c_0} \right)^3 + \frac{\pi}{4} S \left( \frac{f}{c_0} \right)^2 . \tag{5.13}$$

The terms on the left hand side of Equation (5.13) are the first two terms in the formula of Maa. Roe provided confirmation of the usefulness of Equation (5.13) by considering the cumulative number of modes for certain selected acoustic enclosures; rectangular parallelepiped, cylinder, sphere, half-cylinder, hemisphere and prism.

### 5.1.1 Application of Maa's model and comparison with other approximate models

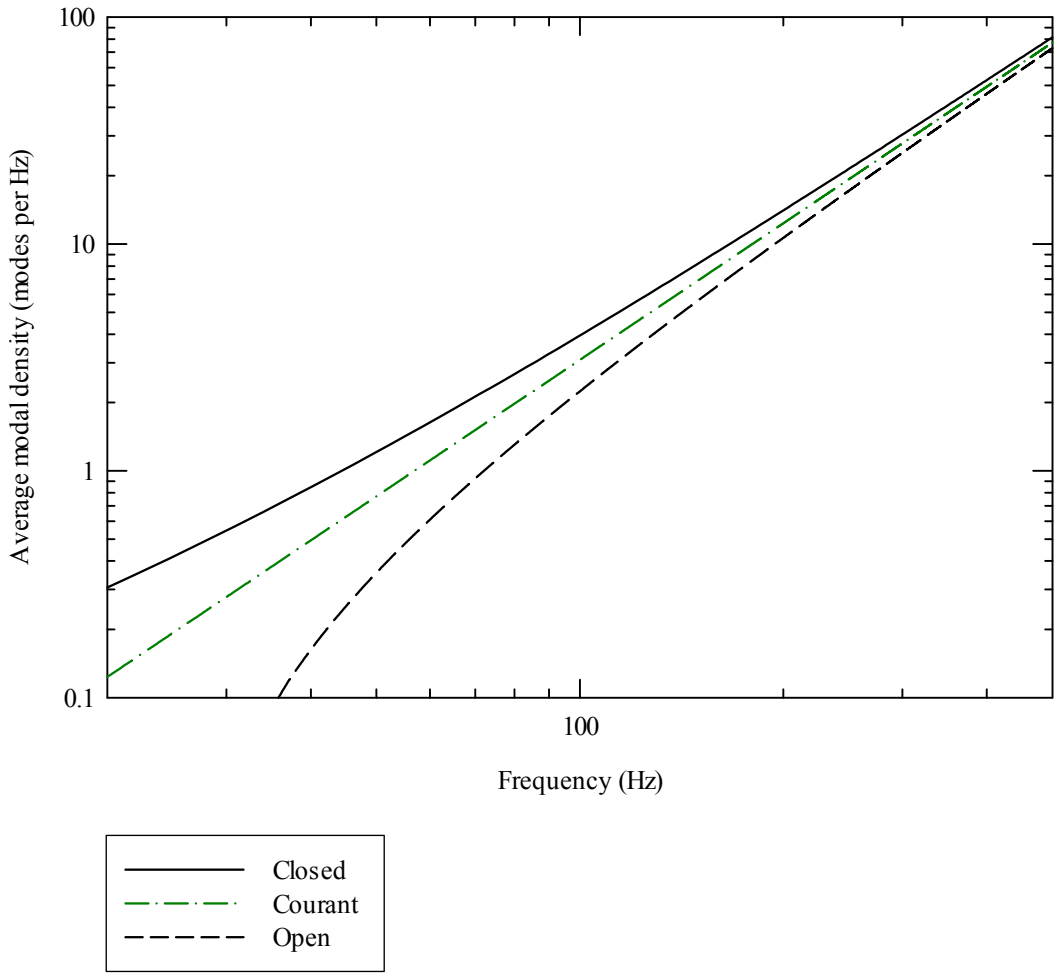
Sketch 5.1 presents a comparison of the averaged modal density as a function of frequency, obtained using the Maa model for closed acoustic space, see Equation (5.8) and Table 5.1, (Maa, 1939) with the averaged modal density for an open acoustic space of the same volume, Table 5.1, (Morse and Ingard, 1968). In the case of the acoustic space described in Sketch 5.1, $c_0$ = 344 m/s, $V$ = 1000 m$^3$, $L_x \simeq 12.6$ m, $L_y$ = 10 m, $L_z \simeq 7.94$ m and the natural frequency of the first oblique mode $f_{111}$ is 30.85 Hz. The asymptotic modal density, based on Courant's approach, Equation (5.1), is shown in Sketch 5.1 for comparison.



ESDU Copyright material. For current status contact ESDU.

Documents
Document Sharing Hub

Full document details: ESDU 13003-R1 2016




**Sketch 5.1 Averaged modal densities for closed and open rectangular acoustic spaces (compared with Courant's expression)**

Difference in the values of the averaged modal density can be observed for frequencies below about 10 $f_{111}$. At higher frequencies for both closed and open spaces, the asymptotic Courant solution is approached.

Comparison of the results obtained by using the different approximate models is provided in Sketch 5.2 for the averaged modal density. The results are for a closed rectangular enclosure with the same dimensions as those described for Sketch 5.1. They are obtained by the method of Courant, Maa (closed space) and Roe. The results are presented for one-third octave bands.



ESDU Copyright material. For current status contact ESDU.
Documents
Document Sharing Hub
Full document details: ESDU 13003-R1 2016