

The equations of motion of a rigid aircraft

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THE EQUATIONS OF MOTION OF A RIGID AIRCRAFT

1. NOTATION

For notation see Item No. 67002^[1].

2. INTRODUCTION

The equations of motion are required before any analysis can be started. In many problems the earth can be regarded as ideally flat and non-rotating. In the main this Data Item makes these assumptions and the equations for a rigid aircraft, first in a general form, and then in a linearised perturbation form, are presented. The latter form is in widespread use for investigating the small disturbance of an aircraft from some datum flight condition, which is usually taken as a steady state defined by constant linear and angular velocities.

The development of the equations in dimensional and non-dimensional form brings out the salient features of the new notation system (see Item Nos. 67001^[2] and 67002^[1]) adopted for these Data Items. It should be noted that some symbols, in particular those for aerodynamic and concise derivatives, have been used with different meanings in the UK and USA, so that care must be taken in interpreting other literature.

After a general discussion of the various axis systems that may be adopted the attitude angles are defined. This discussion (Sections 3 and 4) is not restricted to a flat earth. Kinematic equations are then given that express the relationships between the angular velocity components and the rates of change of the attitude angles. It is also convenient at this stage to write down the gravitational components that will appear in the equations of motion.

The general equations of motion and a linearised form are first presented with the forces and moments unspecified, but it is then assumed that these are essentially aerodynamic and linearised expansions are introduced in terms of aerodynamic derivatives. From these follow the definitions of concise derivatives, which enable the equations of motion to be written in a compact way.

In practice aerodynamic derivatives are most often put into non-dimensional form, and the equations of motion are very often treated non-dimensionally. The two processes usually adopted are described: the aero-normalised system is appropriate for the aerodynamic derivatives, and the dynamic-normalised system for the equations of motion and concise derivatives.

Examples of the completely linearised equations are given for the disturbance of a symmetric aircraft from symmetric equilibrium flight. In this case the equations separate into a longitudinal group and a lateral group. The possibility of separating the equations is discussed for more general conditions in the final section, which also considers the choice of axis system and the validity of the concept of constant derivatives.

3. AXIS SYSTEMS

3.1 Body Axis Systems

The general equations of motion are derived for body axes fixed in the aircraft. A right-handed Cartesian system Oxyz is chosen whose origin is at the centre of gravity, with the x-axis in the plane of symmetry and directed towards the nose and the y-axis to starboard.

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Two types of such body axes are in common use and their relative merits will be discussed in Section 9.1.

- (a) *Principal body axes.* These axes are so orientated that they are coincident with the principal axes of inertia.
- (b) *Aerodynamic-body axes*[†]. These axes are so orientated that the x -axis is in the direction of the projection of the tangent to the flight path, in some datum condition, on to a plane of symmetry.

Positive forces and velocities are in the directions of the axes; angular velocities and moments are positive when they tend to rotate the aircraft in a positive sense, which is clockwise looking along the appropriate axis from the origin.

3.2 Earth Axis Systems

Since the body axes move in space, a complete description of the motion requires a reference axis system to be defined. The usual choice is an earth axis system in which the directions of the axes are specified by some property of the earth, such as the vertical. The definition of the vertical depends on the assumptions that have to be made in the particular application, for example, on whether the earth can be taken as flat or round^[3], or on whether geometric or gravitational properties^[4] are the more important. The origin may be either fixed or moving. The particular axis system adopted here is one in which the z_0 -axis is the downward vertical because this is usually most convenient. The x_0 -axis is taken in the vertical plane containing the aircraft's datum direction of motion, positive forward, and the y_0 -axis forms a right-handed system with the other two.

The position taken up by the origin of the earth axis system is dictated by the nature of the problem under consideration. In problems dealing with the trajectory of a vehicle relative to the earth the origin can conveniently be taken as the point of launch. On the other hand it may prove more convenient to take as origin the touch-down point as, for example, in calculations dealing with blind landings. Again it may be a point at a fixed height above ground level as in the calculation of a dive recovery path.

To relate certain vectors such as gravity, or terms arising when considering flight under navigational instruments, to the body axis system it is necessary to define the aircraft's attitude, or the relative orientation of the body and earth axes.

4. ATTITUDE

The position occupied by an aircraft in space may be specified by the position of a point (O), fixed in the aircraft, relative to the earth and by the aircraft's attitude. The latter may be defined in a number of ways. For instance the direction cosines of the body axes with respect to the earth axis system $Ox_0y_0z_0$ (having coincident origin) may be used. Another scheme is to define three independent angles through which one set of axes must be rotated successively to bring it into alignment with the other set. The most commonly used trio is bank angle, Φ , inclination angle, Θ , and azimuth angle, Ψ . If body axes and earth axes are adopted, these angles may be defined as in Table 4.1.

[†] In the UK these have been referred to as wind axes, and more recently as wind-body axes. In the USA they are usually referred to as stability axes. The use of these other terms is to be deprecated.

TABLE 4.1

<i>Symbol</i>	<i>Definition</i>
Ψ	Angle between Ox_0 and the projection of Ox on the horizontal plane Ox_0y_0 , positive clockwise from Ox_0 about Oz_0 .
Θ	Angle between Ox and the horizontal plane Ox_0y_0 , positive when Ox is above this plane.
Φ	Angle between Oz and the vertical plane containing Ox , positive when Oy is below the horizontal plane Ox_0y_0 .

Figure 1 illustrates these attitude angles. For clarity, a plane parallel to the horizontal plane through O is used and a line parallel to Ox_0 is drawn. Clearly, these angles are solely defined by the instantaneous orientation of the aircraft and are in no way dependent on the motion that brought about this attitude. It may help in an appreciation of these angles to consider the rotations that have to be applied to a body axis system initially coincident with the earth axes $Ox_0y_0z_0$ in order to align it with the final attitude of the body axes $Oxyz$, both the earth axes and the body axes having the same origin. To define the orientation completely the order in which these rotations are applied must also be specified.

The usual sequence of rotations, which is illustrated in Figure 1, is

- (i) a rotation through Ψ about the initial z -axis (Oz_0),
- (ii) a rotation through Θ about the intermediate or “carried” y -axis, *i.e.* the position of the y -axis after operation (i) has been performed, and
- (iii) a rotation through Φ about the final x -axis.

Another sequence of rotations that involves angles of the same magnitude and brings about an identical attitude consists of rotations Φ , Θ and Ψ about the earth axes Ox_0 , Oy_0 and Oz_0 in turn.

It may be more convenient to visualise the sequence of rotations taking place in the reverse sense starting from the attitude of the body axis system $Oxyz$ to be defined. For the first of the sequences considered this implies a rotation $-\Phi$ about the x -axis followed by a rotation $-\Theta$ about the carried y -axis. This brings the aircraft to a level attitude (with Oz vertically downwards) and a final rotation $-\Psi$ about the carried z -axis brings the aircraft into the datum attitude $Ox_0y_0z_0$.

It should be noted that these attitude angles, which are sensible when an aircraft is flying more or less horizontally, become inconvenient when, for example, it is flying more or less vertically. Further sets of angles that may be preferred for certain problems are not considered here but are discussed in detail in Reference 5.

As already mentioned, the attitude may be defined by the direction cosines of one axis system with respect to the other and it may be noted that the matrix for these in terms of the angles Φ , Θ and Ψ is as given in Table 4.2.

TABLE 4.2

	x_0	y_0	z_0
x	$l_1 = \cos \Theta \cos \Psi$	$l_2 = \cos \Theta \sin \Psi$	$l_3 = -\sin \Theta$
y	$m_1 = \sin \Phi \sin \Theta \cos \Psi - \cos \Phi \sin \Psi$	$m_2 = \sin \Phi \sin \Theta \sin \Psi + \cos \Phi \cos \Psi$	$m_3 = \sin \Phi \cos \Theta$
z	$n_1 = \cos \Phi \sin \Theta \cos \Psi + \sin \Phi \sin \Psi$	$n_2 = \cos \Phi \sin \Theta \sin \Psi - \sin \Phi \cos \Psi$	$n_3 = \cos \Phi \cos \Theta$

5. KINEMATIC RELATIONSHIPS

The following relationships can be shown to hold between the attitude angles, their rates of change and the angular velocities of the aircraft in the body axis system.

$$p = \dot{\Phi} - \Psi \sin \Theta, \quad (5.1a)$$

$$q = \dot{\Theta} \cos \Phi + \dot{\Psi} \sin \Phi \cos \Theta, \quad (5.1b)$$

$$r = -\dot{\Theta} \sin \Phi + \dot{\Psi} \cos \Phi \cos \Theta. \quad (5.1c)$$

These may be inverted giving

$$\Phi = p + q \sin \Phi \tan \Theta + r \cos \Phi \tan \Theta, \quad (5.2a)$$

$$\Theta = q \cos \Phi - r \sin \Phi, \quad (5.2b)$$

$$\Psi = q \sin \Phi \sec \Theta + r \cos \Phi \sec \Theta. \quad (5.2c)$$

The components of velocity along the earth axes (u_0, v_0, w_0) are related to the components along the body axes (u, v, w) according to the equations

$$u_0 = l_1 u + m_1 v + n_1 w, \quad (5.3a)$$

$$v_0 = l_2 u + m_2 v + n_2 w, \quad (5.3b)$$

$$w_0 = l_3 u + m_3 v + n_3 w, \quad (5.3c)$$

where l_1, m_1, n_1 , etc. are the elements of the direction-cosine matrix in Table 4.2.

It is sometimes convenient to express the attitude of an aircraft in terms of its orientation relative to some datum attitude Φ_e, Θ_e, Ψ_e . This is normally done by defining attitude deviation angles ϕ, θ, ψ with respect to the datum attitude, the associated sequence of rotations being in the same order as described for Φ, Θ and Ψ . When the angular velocity is zero in the datum condition, the components p, q, r are given in terms of these deviation angles by relations similar to those expressed by Equations (5.1a) to (5.1c):

$$p = \dot{\phi} - \dot{\psi} \sin \theta, \quad (5.4a)$$

$$q = \dot{\theta} \cos \phi + \dot{\psi} \sin \phi \cos \theta, \quad (5.4b)$$

$$r = -\dot{\theta} \sin \phi + \dot{\psi} \cos \phi \cos \theta. \quad (5.4c)$$

When ϕ, θ, ψ are small, the components p, q, r are therefore nearly equal to $\dot{\phi}, \dot{\theta}, \dot{\psi}$, respectively.

The attitude may also be expressed in the usual increment notation by defining the attitude angles relative to the earth axis system, $\Phi = \Phi_e + \Phi'$, $\Theta = \Theta_e + \Theta'$ and $\Psi = \Psi_e + \Psi'$. The increments Φ', Θ', Ψ' are not, in general, equal to ϕ, θ, ψ . For small perturbations simple relations exist between the two sets. Writing Equations (5.1a) to (5.1c) in terms of the incremental deviation attitude angles $\Phi_e + \Phi'$ etc. and substituting $\phi = p\delta t$ and $\Phi' = \Phi\delta t$, $\theta = q\delta t$ and $\Theta' = \Theta\delta t$ etc., yields

$$\phi = \Phi' - \Psi' \sin \Theta_e, \quad (5.5a)$$

$$\theta = \Theta' \cos \Phi_e + \Psi' \sin \Phi_e \cos \Theta_e, \quad (5.5b)$$

$$\psi = -\Theta' \sin \Phi_e + \Psi' \cos \Phi_e \cos \Theta_e. \quad (5.5c)$$

The inverse relations are

$$\Phi' = \phi + \theta \sin \Phi_e \tan \Theta_e + \psi \cos \Phi_e \tan \Theta_e, \quad (5.6a)$$

$$\Theta' = \theta \cos \Phi_e - \psi \sin \Phi_e, \quad (5.6b)$$

$$\Psi' = \theta \sin \Phi_e \sec \Theta_e + \psi \cos \Phi_e \sec \Theta_e. \quad (5.6c)$$

6. GRAVITATIONAL TERMS

If g denotes the acceleration due to gravity, its components g_x, g_y, g_z along body axes are given by

$$g_x = gl_3 = -g \sin \Theta, \quad (6.1a)$$

$$g_y = gm_3 = g \sin \Phi \cos \Theta, \quad (6.1b)$$

$$g_z = gn_3 = g \cos \Phi \cos \Theta. \quad (6.1c)$$

When small perturbations in attitude occur from a datum attitude Φ_e, Θ_e, Ψ_e , the corresponding first-order increments in the gravity components are obtained directly by differentiation:

$$g'_x = -\Theta'g \cos \Theta_e,$$

$$g'_y = \Phi'g \cos \Phi_e \cos \Theta_e - \Theta'g \sin \Phi_e \sin \Theta_e,$$

$$g'_z = -\Phi'g \sin \Phi_e \cos \Theta_e - \Theta'g \cos \Phi_e \sin \Theta_e.$$

Alternative expressions in terms of the attitude deviation angles ϕ, θ, ψ are obtained by substitution for Φ' and Θ' from Equations (5.6a) to (5.6c) and a compact form of all the results is as follows:

$$g'_x = -g_1\Theta' = -g_3\theta + g_5\psi, \quad (6.2a)$$

$$g'_y = g_3\Phi' - g_6\Theta' = g_3\phi + g_2\psi, \quad (6.2b)$$

$$g'_z = -g_5\Phi' - g_4\Theta' = -g_5\phi - g_2\theta, \quad (6.2c)$$

where

$$\begin{aligned} g_1 &= g \cos \Theta_e, & g_2 &= g \sin \Theta_e, \\ g_3 &= g \cos \Phi_e \cos \Theta_e, & g_4 &= g \cos \Phi_e \sin \Theta_e, \\ g_5 &= g \sin \Phi_e \cos \Theta_e, & g_6 &= g \sin \Phi_e \sin \Theta_e. \end{aligned}$$

When Φ_e is zero only g_1 and g_2 are required, since $g_3 = g_1$, $g_4 = g_2$ and g_5, g_6 are zero.

7. EQUATIONS OF MOTION

To obtain the equations of motion in the form given in this Data Item two assumptions are required. These are as follows.

- (i) The aircraft is a rigid body (any rotors attached to it being also rigid) with an axis system passing through its centre of gravity.
- (ii) The rotor axes are fixed relative to the body axes and the rotors rotate with constant angular velocity about their own axes.

The equations of motion in still air referred to any system of body axes are then as follows (see, for example, Reference 6 for a complete derivation):

$$m(\dot{u} + qw - rv) = X + mg_x, \quad (7.1a)$$

$$m(\dot{v} + ru - pw) = Y + mg_y, \quad (7.1b)$$

$$m(\dot{w} + pv - qu) = Z + mg_z, \quad (7.1c)$$

$$I_x \dot{p} - I_{yz}(q^2 - r^2) - I_{zx}(\dot{r} + pq) - I_{xy}(\dot{q} - rp) - (I_y - I_z)qr + J_z q - J_y r = L, \quad (7.1d)$$

$$I_y \dot{q} - I_{zx}(r^2 - p^2) - I_{xy}(\dot{p} + qr) - I_{yz}(\dot{r} - pq) - (I_z - I_x)rp + J_x r - J_z p = M, \quad (7.1e)$$

$$I_z \dot{r} - I_{xy}(p^2 - q^2) - I_{yz}(\dot{q} + rp) - I_{zx}(\dot{p} - qr) - (I_x - I_y)pq + J_y p - J_x q = N, \quad (7.1f)$$

where mg_x, mg_y, mg_z are the components of the weight mg along the axes and g_x, g_y and g_z can be written in terms of the attitude angles as indicated by Equations (6.1a) to (6.1c).

The equations are more conveniently written:

$$\dot{u} + qw - rv - g_x = X/m, \quad (7.2a)$$

$$\dot{v} + ru - pw - g_y = Y/m, \quad (7.2b)$$

$$\dot{w} + pv - qu - g_z = Z/m, \quad (7.2c)$$

$$\dot{p} + d_x(q^2 - r^2) + e_x(\dot{r} + pq) + f_x(\dot{q} - rp) + b_x qr + c_{zx}q - c_{yx}r = L/I_x, \quad (7.2d)$$

$$\dot{q} + e_y(r^2 - p^2) + f_y(\dot{p} + qr) + d_y(\dot{r} - pq) + b_y rp + c_{xy}r - c_{zy}p = M/I_y, \quad (7.2e)$$

$$\dot{r} + f_z(p^2 - q^2) + d_z(\dot{q} + rp) + e_z(\dot{p} - qr) + b_z pq + c_{yz}p - c_{xz}q = N/I_z. \quad (7.2f)$$