

THE DESIGN OF CROSSED FLEXURE-PIVOTS

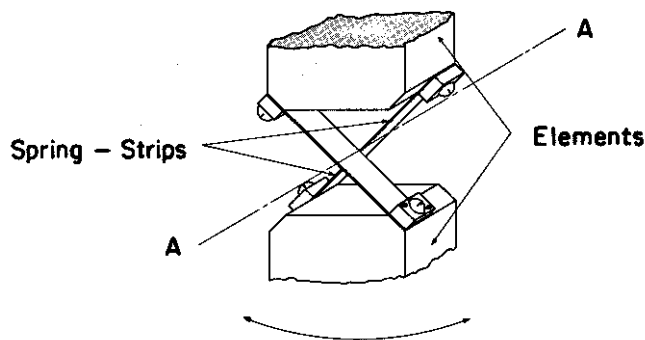
1. INTRODUCTION

This Item describes the capabilities of crossed flexure-pivots. Detailed guidance is given on the design of flexure-pivots whose strips cross midway along their lengths and at right angles to each other. This is the most usual configuration.

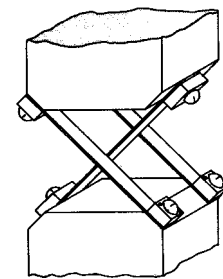
Data are given in Appendix A for those flexure-pivots whose strips do not cross midway along their lengths and at right angles to each other.

1.1 Description

Basically, a crossed flexure-pivot consists of two flat spring-strips, arranged in the form of a cross, that connect two elements of a mechanism as shown in Sketch 1.1. These strips allow one element, the moving end, to move in relation to the other, the fixed end. The stresses in the strips limit the amount of acceptable movement. This movement takes place approximately about axis AA where the undeflected strips cross. The fixed end is analogous to the bearing, and the moving end to the journal, of a conventional sleeve bearing. In practice, for reasons of symmetry, it is more practical to arrange the strips as shown in Sketch 1.2.



Sketch 1.1



Sketch 1.2

Note: The clamping of the spring-strips as shown may not always be desirable. For further information and guidance, see Derivation 4.

Throughout this Item the basic configuration of a flexure-pivot is considered as shown in Sketch 1.4, *i.e.* in the undeflected position.

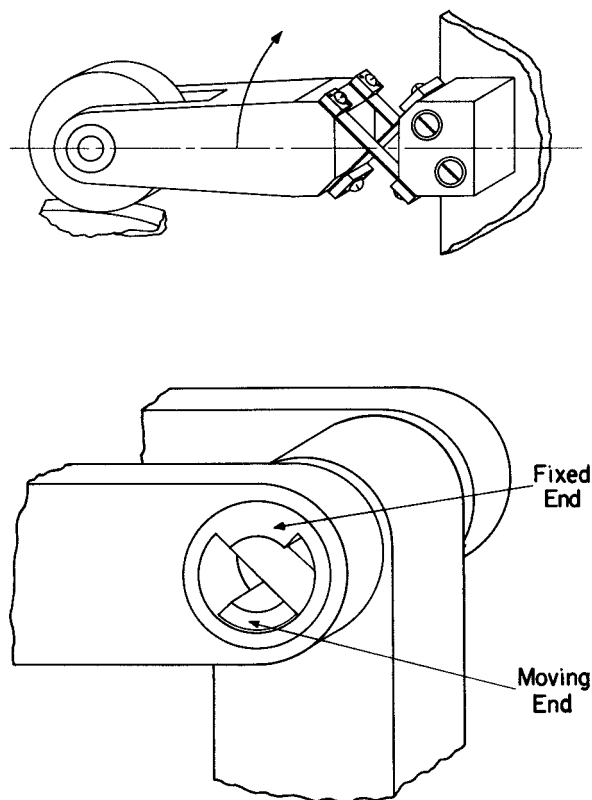
1.2 Advantages, Disadvantages and Applications

There are numerous applications where crossed flexure-pivots could be used with advantage instead of other bearing types that require regular lubrication and maintenance. Because of the absence of wear,

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backlash and the need for lubrication, flexure-pivots will, once designed and installed, continue to operate throughout the life of the equipment with predictable characteristics. However, the manufacturing economics for a flexure-pivot should be considered since these, in certain applications, can be a serious disadvantage. Two possible applications for flexure-pivots are shown in Sketch 1.3.



Sketch 1.3

The general advantages and disadvantages are summarised as follows.

Advantages

- (i) There is no rubbing friction between the two connected elements.
- (ii) The absence of rubbing or rolling surfaces obviates the need for lubrication.
- (iii) The efficiency of operation is not generally affected by dirty conditions.
- (iv) They are suitable for operation in a wide range of environmental conditions, including vacuum, high pressure, low and high temperature and radiation.
- (v) The maintenance costs are negligible – except in the case of delicate instrument pivots which require accurate alignment to maintain precise characteristics.
- (vi) Often, especially when commercially available flexure-pivots are used, the number of parts required is smaller than that necessary for a conventional bearing to perform the same function.

Disadvantages

- (i) It is difficult to design flexure-pivots to provide angular rotations greater than about $\pm 30^\circ$.
- (ii) In some applications the movement, which is generally small, of the 'centre' of the moving end may nevertheless be unacceptable. (See Section 3.5.3 for detailed information.)
- (iii) The angular spring rate varies with the force applied.
- (iv) Very careful assembly is required where it is necessary to obtain a precise angular spring rate, particularly for small pivots.

1.3 Characteristics

Load Capacity

Failure of flexure-pivots can occur by tensile or compressive yielding of the strips, by fatigue, or by buckling under compressive load.

Figure 2 gives guidance on the maximum load per unit area that can be carried by a flexure-pivot, for a given angular rotation using various strip materials and design stresses. The chosen design stress will depend upon individual material specifications, taking into account fatigue considerations. The figure is based upon the values of strip thickness (t) that give the maximum load capacity. If particular values of t are required (for example, to obtain a specified angular spring rate), it may not be possible to obtain the maximum load capacity shown.

Angular Spring Rate

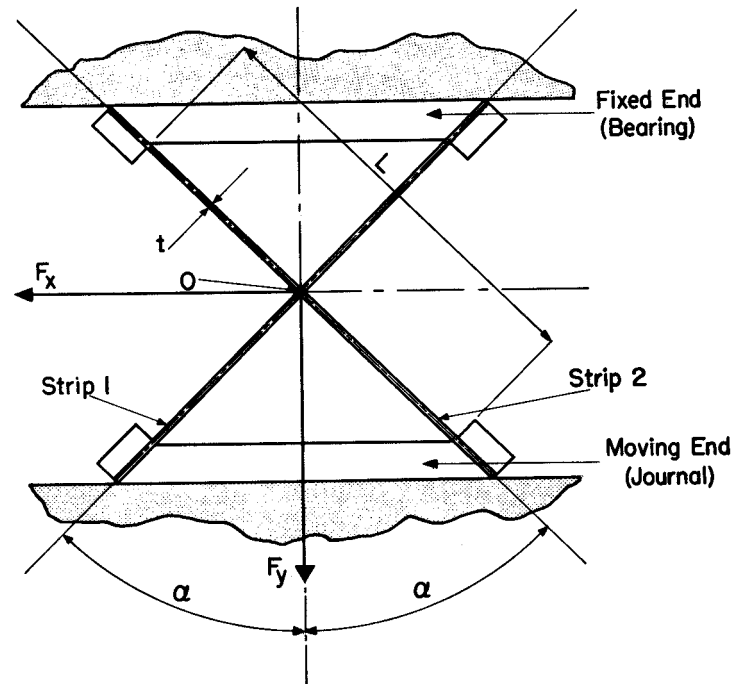
When the flexure-pivot is rotated the bending moments in the strips apply a restoring moment to the moving end that is proportional to the angle of rotation. This restoring moment is expressed as an angular spring rate and, for equilibrium, balances any applied torque (T). Under certain conditions of loading the angular spring rate can be negative, *i.e.* the bending moments in the strips tend to rotate the moving end away from the undeflected position. This condition can be stable provided the applied torque acts in the opposite sense to the rotation and exceeds the (negative) restoring moment.

Figure 3 gives the zero load angular spring rate per unit volume for maximum load capacity flexure-pivots corresponding to those in Figure 2.

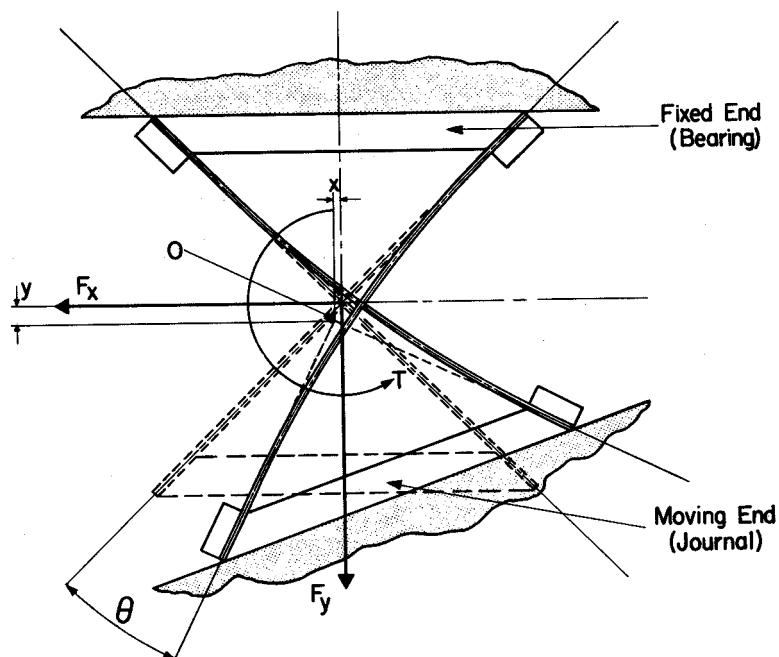
Figure 4 gives the zero load angular spring rate plotted against thickness-length ratio (t/L) for various lengths (L) of steel strip. For other materials see Note on Figure 4.

Movement of the 'Centre'

The movement of the 'centre' of the moving end, which is analogous to the eccentricity in a conventional sleeve bearing, is critical in certain applications. For small angles of rotation the movement is negligible except when buckling conditions are approached. The movement is proportional to the square of the angle of rotation.



Sketch 1.4



Sketch 1.5

2. NOTATION AND UNITS

Two systems of units are given in this Item.

- (i) **Familiar British Units** – this is a non-coherent system. It was chosen as representing the most commonly used units at this time in the United Kingdom.
- (ii) **The International System of Units (SI)** – this is a coherent system. It should be noted that any other coherent system of units can be used in place of SI where the data are given in non-dimensional form.

The configuration and loading of a flexure-pivot, whose strips cross midway along their length and at right angles to each other, are shown in Sketches 1.4 and 1.5.

b	half total width of strips, see Sketch 3.2	m	in
E	Young's modulus of strip material	N/m ²	lbf/in ²
F_x	'horizontal' component (in x -direction) of forces applied to moving end, always taken as positive, see Sketch 1.4	N	lbf
F_y	'vertical' component (in y -direction) of forces applied to moving end, positive when inducing tension in strips, see Sketch 1.4	N	lbf
$ F_y $	absolute value of F_y	N	lbf
f	stress in a strip, tensile positive	N/m ²	lbf/in ²
f_p	permissible stress of strip material	N/m ²	lbf/in ²
I	second moment area of strip, $bt^3/12$ for rectangular sections	m ⁴	in ⁴
K	angular spring rate, m/θ	N m/rad	lbf in/rad
L	length of strip	m	in
M	bending moment in strip	N m	lbf in
m	restoring moment about point where undeflected strips cross	N m	lbf in
O	point fixed in moving end that coincides initially with point at which undeflected strips cross		
P	end load in a strip, tensile positive	in	lbf
T	total applied torque in deflected position about point where undeflected strips cross	N m	lbf in
t	thickness of strip	m	in

x	‘horizontal’ component of movement of O , regarded as moving with moving end (positive when in direction F_x), see Sketch 1.5	m	in
y	‘vertical’ component of movement of O , regarded as moving with moving end (positive when in direction of $+F_y$), see Sketch 1.5	m	in
α	inclination of each strip to vertical (for flexure-pivots crossing at right angles, $\alpha = 45^\circ$)	deg	deg
θ	angular (relative) rotation in radians of two elements, positive as shown in Sketch 1.5, $\pi\phi/180$	rad	rad
ϕ	angular (relative) rotation in degrees of two elements, positive as shown in Sketch 1.5	deg	deg
ρ	dimensionless parameter whose value defines point at which undeflected strips cross, see Appendix A (for flexure-pivots whose strips cross midway along their length $\rho = 0$)		

Subscripts

1	appertaining to strip in which side force F_x is tending to induce tension
2	appertaining to strip in which side force F_x is tending to induce compression
max	indicates maximum

3. DESIGN NOTES

A design procedure and data are given to design a flexure-pivot whose strips cross midway along their length and at right angles to each other. For the purpose of design the flexure-pivot is considered to consist of two strips each of width b . The flexure-pivot can be designed for a given strength at a given maximum angle of rotation and for a given angular spring rate, provided that there are no size limitations, in which case it might not be possible to achieve the required characteristics. Data are also given for checking the resulting stresses and movement of the ‘centre’ of the moving end. The data are based on small-angles-of-rotation theory, but can be used for rotations up to $\pm 30^\circ$ with some loss of accuracy.

To design a flexure-pivot, values for b , L and t have to be derived and a suitable material chosen. A design procedure follows that is illustrated as a flow chart on Figure 1 and followed in Example 1, see Section 5.1. It is based upon knowing initially the loading, angular spring rate required and any space limitations. The initial information can vary from design to design but it is felt that the example given is of a type commonly encountered. The flow chart can be easily modified for other cases.

Firstly the external forces acting on the flexure-pivot are determined and the ‘vertical’ (F_y) and ‘horizontal’ (F_x) components and torque (T) are derived.

3.1 External Forces to be Transmitted by the Flexure-Pivot (F_x and F_y)

The moving end can be subjected to external forces acting in any direction. These forces are resolved into 'vertical' (F_y) and 'horizontal' (F_x) components acting through the point at which the undeflected strips cross, see Sketches 1.4 and 1.5. It is important to note the following.

- (i) A positive 'vertical' force (F_y) is one that tends to induce tension in the strips.
- (ii) A 'horizontal' force (F_x) tends to induce tension in one strip (strip No. 1 as shown in Sketch 1.5) and compression in the other strip (strip No. 2). The overall characteristics of the flexure-pivot are independent of the sign of F_x . Therefore F_x is always taken to be positive.

3.2 Strip Materials

An initial idea of the strip size is obtained from Figure 2 (see Section 1.3) and a possible strip material chosen. Figure 2 can also be used for determining which loading condition is likely to be most critical in cases where the forces vary substantially with angle. (A value of bL is assumed and $(|F_y| + F_x)/bL$ plotted for various values of ϕ in Figure 2. The region of this plot that lies nearest below the curve for the proposed material is likely to give the most critical stress condition.)

A suitable strip material has to be chosen depending upon the following material properties and operating requirements:

- (i) electrical conductivity if the strips are to carry current,
- (ii) fatigue strength,
- (iii) material cost,
- (iv) thermal conductivity if the strips are to be used as a heat path,
- (v) yield strength,
- (vi) Young's modulus of elasticity,
- (vii) operating environment, *e.g.* a corrosive atmosphere,
- (viii) operating temperature.

In general, if a high load capacity at large angles of rotation is required together with low angular spring rate, a material with a high ratio of design stress to Young's modulus is required. If only small rotation is required, and spring rate is unimportant but a high load capacity is needed, a material with high yield strength or fatigue strength is necessary. When load capacity is unimportant but large angles of rotation are required with low spring rate, a low Young's modulus material is appropriate.

3.3 Angular Spring Rate (K)

This is defined as m/θ , where θ is the angular (relative) rotation of the two elements in radians. The angular spring rate for a given flexure-pivot is dependent upon F_x and F_y , and can be positive, negative or zero.

When a constant torque T is applied, the flexure-pivot rotates until the internal restoring moment due to bending equals the applied torque at an equilibrium position where $m = K\theta = T$. If F_x and F_y are such

that K is negative, the applied torque required to provide stable equilibrium must also be negative, *i.e.* it must be in the opposite sense to the rotation. Otherwise the pivot is unstable.

If a torque T is applied that varies with θ , again the flexure-pivot will rotate until the internal restoring moment equals the applied torque. However, in this case it is possible to obtain a stable condition with a negative K , if T acts in the negative (restoring) sense, *e.g.* a pendulum suspended from a flexure-pivot.

Choice of K

Depending upon the particular application it is required either to

- (i) specify a spring rate, which is required at a given load or over a range of load, or
- (ii) specify suitable spring rate characteristics, but without defining precise values, *e.g.* where a conventional bearing is to be replaced and it is required to keep the spring rate to a minimum.

3.4 Choice of Trial Dimensions

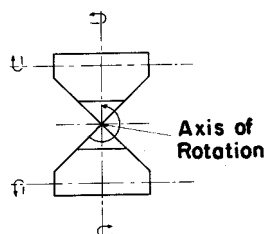
To minimise the number of iterations before the design is finalised, an initial design procedure is given to assist in the choice of trial dimensions.

Firstly the end loads in the strips, P_1 and P_2 , are calculated from the following:

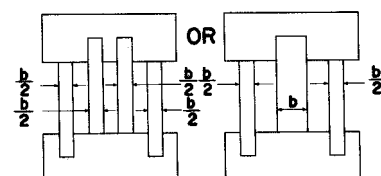
$$P_1 = \frac{1}{2}(F_y \sec \alpha + F_x \operatorname{cosec} \alpha) \quad (3.1)$$

$$\text{and} \quad P_2 = \frac{1}{2}(F_y \sec \alpha - F_x \operatorname{cosec} \alpha) . \quad (3.2)$$

In cases where the flexure-pivot is required to transmit forces that give rise to large moments about axes at right angles to the axis of rotation, see Sketch 3.1, the effect of these moments on the end loads in the strips must be considered. In this case it is desirable, for stability reasons, to use two pairs of strips as shown in Sketch 3.2 and to increase the distance between the strips to reduce the additional forces in the outer strips due to these moments. It should also be arranged, if possible, that force F_x induces tension in the outer strips so that the danger of buckling from compressive forces due to the moments is reduced.



Sketch 3.1



Sketch 3.2

Figures 6, 7, 8, 9 and 10 give guidance on the initial choice of dimensions for values of f_p/E equal to 0.001, 0.0015, 0.002, 0.003 and 0.004, respectively. Figure 5 gives guidance on interpreting these figures.

The value of f_p/E is the ratio of permissible stress to Young's modulus and is the elastic strain at the permissible stress. Guidance on the choice of material and hence f_p/E is given above and in Section 1.3.

For each value of f_p/E the parameter $10^8 P/EbL$, which is a measure of the load per unit area of the flexure-pivot, is plotted in full curves against t/L for various values of ϕ . The numerically larger value of P_1 and P_2 is substituted for P in $10^8 P/EbL$.

The dashed lines for various values of K/EbL^2 may be used to obtain K , the angular spring rate, as follows. When $F_x = 0$, the dashed curves can be used directly to obtain K/EbL^2 and hence K . When $F_x \neq 0$, however, two values of K/EbL^2 must be obtained, corresponding to $10^8 P_1/EbL$ and $10^8 P_2/EbL$, from the mean of which K is calculated.

Two methods of using these curves are illustrated in the Examples. In general, a region *A, B, C et cetera* in which it is desired to operate is chosen with the aid of Figure 5. If maximum values of b and L have been specified it is possible to determine a minimum value of $10^8 P/EbL$ and hence to restrict the area of choice on the diagram. According to the particular requirements of the application and remembering that there is rarely one 'correct' answer but rather a range of possible solutions, it is now necessary to choose (in some cases by trial and error) a particular point on the diagram corresponding to specific values of $10^8 P/EbL$, K/EbL^2 and t/L from which corresponding values of b , L and t can be deduced as illustrated in the examples. If impractical dimensions result a new point is chosen.

3.5 Final Design

3.5.1 Checking Angular Spring Rate

Figure 11 is used to determine K more accurately than is possible with the initial design charts. In this figure, KL/EI is plotted against $(F_y L^2/EI) \sec \alpha$ for various values of $(F_x L^2/EI) \csc \alpha$.

3.5.2 Checking Maximum Stresses and for Failure by Buckling

The maximum stress is calculated for each of the two strips and compared with the yield stress or, more usually, the fatigue stress of the strip material, taking account of (if data are available) the stress ratio (ratio of minimum to maximum stress in the fatigue cycle).

Figure 12 shows $M_{max}L/\theta EI$ plotted against $PL^2/4EI$ for various values of ρ . For flexure-pivots whose strips cross midway along their length $\rho = 0$. The maximum stress in each strip (f_{max}) is given by:

$$f_{1max} = \pm \frac{6M_{1max}}{bt^2} + \frac{P_1}{bt} \quad (3.3)$$

$$\text{and} \quad f_{2max} = \pm \frac{6M_{2max}}{bt^2} + \frac{P_2}{bt} \quad (3.4)$$

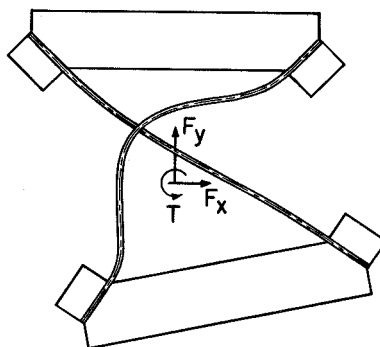
With flexure-pivots the practice of applying safety factors to loads should be treated with caution. The bending moment in the strips depends on, but is not generally directly proportional to, the applied forces. Hence if a load factor is applied to F_y and F_x only, the stresses due to bending (which in many cases predominate) will not be factored proportionally. Accordingly, either suitably reduced design stresses should be used initially or the safety factor should be applied to the stresses given by Equations (3.3) and (3.4).

The steep gradient on the left of each curve for angular spring rate characteristics is associated with the onset of buckling in compression. As the buckling condition is approached the bending moment(s) (M) and the movement of the 'centre' of the moving end increase.

An absolute limit is therefore imposed on the forces to which a flexure-pivot can be subjected without

collapsing.

The condition of buckling is illustrated in Sketch 3.3.



Sketch 3.3

Failure by buckling will occur if for either strip P is compressive (negative) and $PL^2/4EI$ is numerically greater than π^2 .

3.5.3 Movement of the 'Centre' of the Moving End

The 'centre' of the moving end is defined by point O , which is fixed relative to the moving end, and initially coincides with the point at which the undeflected strips cross, see Sketch 1.4.

For infinitesimally small angles of rotation this 'centre' coincides exactly with the point at which the undeflected strips cross. As the angle of rotation increases there is a small movement of the 'centre'. This movement is illustrated in Sketch 1.5 and has horizontal and vertical components x and y .

Figures 13 and 14 give data on the movement of the 'centre' of the moving end away from its initial position. Figures 13 and 14 show, respectively, $x \sin \alpha / \theta^2 L$ and $y \cos \alpha / \theta^2 L$ plotted against $(F_y L^2 / EI) \sec \alpha$ for various values of $(F_x L^2 / EI) \operatorname{cosec} \alpha$.

3.5.4 Detailed Design of Strips with Regard to Fatigue

Where repeated angular deflections of the flexure-pivots are required, the fatigue properties of the strips are important. This becomes critical where the flexure-pivot is to operate in a corrosive environment. The avoidance of fretting at the strip attachments is particularly important. (See Derivation 4 and Item No. 90031.)

In manufacturing strips sudden changes of cross section should be avoided in order to minimise the adverse effects of stress concentration on the fatigue life. Further guidance is given in other Data Items (see the current Engineering Sciences Data Index).

4. DERIVATION

1. WITTRICK, W.H. The theory of symmetrical crossed flexure pivots. *Aust. J. scient. Res., Series A, Physical Sciences*, Vol. 1, No. 2, pp. 121-134, June 1948.
2. HARINGX, J.A. The cross-spring pivot as a constructional element. *Appl. scient. Res., Section A*, Vol. 1, No. 4, pp. 313-332, 1949.
3. WITTRICK, W.H. The properties of crossed flexure pivots and the influence of the point at which the strips cross. *Aeronaut. Q.*, Vol. II, Part IV, pp. 272-292, February 1951.
4. NPL Notes on Applied Science No. 15. Application of spring strips to instrument design. National Physical Laboratory. HMSO, London, 1960.

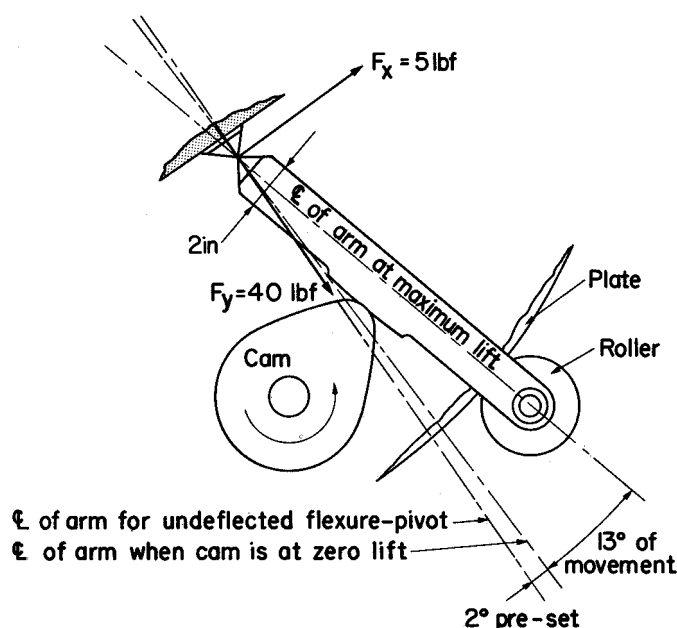
5. EXAMPLES

5.1 Example 1

A 4 in diameter roller is required to ink a plate as shown in Sketch 5.1. The roller is spring loaded to ensure contact between it and the plate. For control of the roller load the maximum deviation of the roller from a circular path must not be more than 0.025 in. A single strip flexure-pivot would give too great a spring deflection so that suitable two strips crossed flexure-pivots are required.

The arm moves through 13° and the flexure-pivots are pre-set by 2° to ensure contact between the cam and arm. An angular spring rate of at least 25 lbf in/rad is required to overcome friction in the system.

The forces on the flexure-pivot in the deflected position are $F_x = 5$ lbf and $F_y = 40$ lbf, as shown on the sketch.



Sketch 5.1

The "Flow Diagram of Design Procedure", Figure 1, is followed. No space limitations have been specified. It is therefore assumed that the size should be as small as possible.

The angular movement ϕ at the maximum deflection position is $13 + 2 = 15^\circ$ so that

$$\theta = \pi \times 15/180 = 0.262 \text{ rad.}$$

The loads F_x and F_y have been specified, and calculations indicate that, although the loads will not be constant through the stroke of the roller (due to varying cam reaction), the loads at the maximum deflection position will give the most critical loading condition.

From Figure 2 it is now possible to choose a suitable combination of material and bL . For economy, the use of an ordinary spring steel is preferred. A design stress must be chosen to give a long life and the primary consideration will therefore be failure through fatigue. Assuming at this stage that the fatigue cycle ranges from zero (at the 2° deflection position) to f_p at the 15° deflection position (*i.e.* mean stress = $f_p/2$), an unfactored design stress of 67 000 lbf/in² is chosen corresponding to the fatigue limit of a spring steel tempered to a tensile strength of 200 000 lbf/in² or greater.

However, a factor of safety will be required to allow for uncertainties in loading, unavoidable stress concentration and surface imperfections and approximations in calculation. It is judged reasonable to take a relatively small factor of 1.5, since it is likely that the maximum stress will depend mainly on the maximum angle of deflection, which is known accurately, rather than on the less certain loading. The chosen design stress of 67 000 lbf/in² is therefore reduced by this factor to 45 000 lbf/in², giving, taking $E = 30 \times 10^6$ lbf/in²,

$$f_p/E = 45\,000/(30 \times 10^6) = 0.0015.$$

Referring now to Figure 2, from curve 10 for $\phi = 15^\circ$, the maximum permissible value of $(|F_y| + F_x)/bL$ is 9.6 lbf/in². Hence the minimum value of bL is given by

$$bL = \frac{|F_y| + F_x}{9.6} = \frac{40 + 5}{9.6} = 4.7 \text{ in}^2.$$

This is not too large. Continuing, there are no large moments at right angles to the axis of rotation, so the next stage is to calculate P_1 and P_2 from Equations (3.1) and (3.2). With $\alpha = 45^\circ$,

$$P_1 = \frac{1}{2}(F_y \sec \alpha + F_x \operatorname{cosec} \alpha) = \frac{1}{2}[(40 \times 1.414) + (5 \times 1.414)] = 31.8 \text{ lbf}$$

$$\text{and } P_2 = \frac{1}{2}(F_y \sec \alpha - F_x \operatorname{cosec} \alpha) = \frac{1}{2}[(40 \times 1.414) - (5 \times 1.414)] = 24.7 \text{ lbf.}$$

Before continuing to determine suitable values for bL^2 , bL and t/L it is necessary to consider the angular spring rate required. A minimum value of K of 25 lbf in/rad has been specified, and it is assumed that values greatly in excess of this will lead to excessive cam wear. In case of accidental overload it is undesirable that the pivot should approach the zero or negative stiffness condition, and this must be borne in mind in making the selection of dimensions.

Referring then to Figure 7 (for $f_p/E = 0.0015$), the design must lie on or below the curve for $\phi = 15^\circ$. For minimum bL it is clear that a point near the peak of this curve must be chosen, but this conflicts with the requirement to avoid zero or negative stiffness. It is clear therefore that it is necessary to work away from the peak to the right. There is no single 'correct' solution, and a point is tentatively selected on the above

considerations at

$$\frac{10^8 P}{EbL} = 16, \quad \frac{t}{L} = 0.0086 \quad \text{and} \quad \frac{K}{EbL^2} = 5 \times 10^{-8}.$$

Then for $P = P_1 = 31.8$ lbf (only the higher value is of interest for stressing),

$$bL = \frac{10^8 P}{16E} = \frac{10^8 \times 31.8}{16 \times 30 \times 10^6} = 6.625 \text{ in}^2,$$

and for $K = 25$ lbf in/rad (ignoring at this stage the effect of non-zero side load F_x),

$$bL^2 = \frac{K}{E \times 5 \times 10^{-8}} = \frac{25}{30 \times 10^6 \times 5 \times 10^{-8}} = 16.7 \text{ in}^3.$$

Hence these values correspond to

$$L = \frac{bL^2}{bL} = \frac{16.7}{6.625} = 2.52 \text{ in}, \quad b = \frac{bL}{L} = \frac{6.625}{2.52} = 2.63 \text{ in}$$

and $t = (t/L) \times L = 0.0086 \times 2.52 = 0.0217 \text{ in},$

which appear to be reasonable. However, it is desirable for economic reasons to use a standard strip thickness and the nearest higher gauge, $t = 0.022$ in (24 S.W.G.) is therefore chosen, and b and L rounded off to 2.60 in and 2.50 in, respectively.

It remains now to check and finalise this tentative design. The value of EI/L^2 will be used frequently and is therefore calculated at this stage.

$$\frac{EI}{L^2} = \frac{Ebt^3}{12L^2} = \frac{30 \times 10^6 \times 2.60 \times 0.022^3}{12 \times 2.50^2} = 11.1 \text{ lbf}.$$

Then

$$\frac{F_y L^2}{EI} \sec \alpha = \frac{40 \times 1.414}{11.1} = 5.10$$

and $\frac{F_x L^2}{EI} \operatorname{cosec} \alpha = \frac{5 \times 1.414}{11.1} = 0.637.$

From Figure 11,

$$\begin{aligned} \frac{KL}{EI} &= 1.15 \text{ so } K = 1.15 \times EI/L^2 \times L \\ &= 1.15 \times 11.1 \times 2.50 \\ &= 31.9 \text{ lbf in/rad.} \end{aligned}$$

The discrepancy from the original value of $K = 25$ lbf in/rad, attributable to inaccuracy in reading Figure 7