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The response of first- and second-order systems

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THE RESPONSE OF FIRST- AND SECOND-ORDER SYSTEMS

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THE RESPONSE OF FIRST- AND SECOND-ORDER SYSTEMS

1. NOTATION

a_0, a_1	constants
$f(t)$	input to system
K, K_1	overshoot factors
k	damping index
k_1	constant in exponential-step input
s	parameter in Laplace transform
$-s_1, -s_2$	real roots of characteristic equation
T	time constant (time to $1/e$)
t_1, t_2	times associated with inputs (see Figure 1)
$w(t - \tau)$	response to a unit impulse applied at time τ
$X(s)$	Laplace transform of $x(t)$ $\left(= \int_0^{\infty} e^{-st} x(t) dt \right)$
$x(t)$	response variable
Δt	small increment in time
ε	phase angle
ζ	relative damping ratio
λ	constant in numerator of transfer function
ω	angular frequency of second-order system
τ	time of application of a specimen pulse or step (see Section 4)
ω	“undamped” angular frequency of second-order system
ω_0	angular frequency of sinusoidal input

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2. INTRODUCTION

This Item presents sets of normalised response curves for simple first- and second-order systems.

The data are restricted to certain basic forms of input, which are presented in Figure 1, and to systems that can be described by a linear differential equation with constant coefficients. The characteristic equation, and also the denominator of the corresponding transfer function, is therefore a first or second degree polynomial.

The response curves for first- and second-order systems are presented in Figures 2 (2a, 2b, 2c, 2d, 2e, 2f, 2g and 2h) and Figures 3 (3a, 3b, 3c, 3d, 3e, 3f, 3g and 3h) respectively, and maxima in response to some of the input functions are given in Figure 4 and Figures 5 (5a, 5b, 5c, 5d and 5e).

A step input, its time integral (ramp input) and its derivative (impulsive or “Delta function” input), are considered independently although the responses to any two of these inputs can be deduced from the response to the third. Similarly the responses to related inputs of ramp-step and triangular form are given, although these could be deduced by superposition of the responses to suitably chosen ramp inputs. Brief mention is made of the convolution integral (Duhamel integral) method for determining the response to an arbitrary input when the response to an impulse is known.

Data for the simple frequency response, *i.e.* the ultimate steady state of a stable system subjected to a sinusoidal input, are presented in Figures 6 (6a and 6b) and Figures 7 (7a and 7b). In this case the diagrams give the ratio of the amplitudes and the difference in phase of the response variable and the input.

Finally Appendix A considers the response of a system having a modified form of transfer function for one of the inputs used, the ramp-step input.

3. DIFFERENTIAL EQUATIONS AND TRANSFER FUNCTIONS

The first-order system is described by the differential equation

$$\frac{dx(t)}{dt} + kx(t) = f(t), \quad (3.1)$$

where $x(t)$ and $f(t)$ are functions of time t , $f(t)$ being the input and $x(t)$ the response. Such an equation occurs when considering a one-degree of freedom approximation to the rolling motion produced by an aileron deflection.

The characteristic equation is

$$s + k = 0, \quad (3.2)$$

s being the Laplace parameter.

The transfer function is $1/(s + k)$. There is only one natural mode. It has the form $\exp(-kt)$, k being the damping index. The time constant τ (time to $1/e$) is equal to $1/k$.

The second-order system is described by the equation

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega \frac{dx(t)}{dt} + \omega^2x(t) = f(t), \quad (3.3)$$

where ζ and ω are constants. One example where this type of differential equation occurs is the “short-period” response of an aircraft to an elevator deflection.

The characteristic equation is

$$s^2 + 2\zeta\omega s + \omega^2 = 0, \quad (3.4)$$

where the transfer function is $1/(s^2 + 2\zeta\omega s + \omega^2)$.

The stable systems considered here have an undamped angular frequency ω and a relative damping ratio ζ greater than zero. When $\zeta < 1$, the only natural mode is oscillatory and of the form $\exp(-\zeta\omega t) \sin(\nu t + \varepsilon)$, where $\zeta\omega$ is the damping index and $\nu = \omega\sqrt{1-\zeta^2}$ is the angular frequency. When $\zeta = 1$, the mode is critically damped and of the form $(a_0 + a_1t) \exp(-\omega t)$. When $\zeta > 1$, there are two separate exponential modes of the form $\exp(-s_1t)$ and $\exp(-s_2t)$ where $-s_1, -s_2$ are the (real) roots of the characteristic equation.

The frequency response of first- or second-order systems is frequently used in the construction of Bode diagrams. It is convenient to write the transfer function of a first-order system in the form $k/(s + k)$ rather than $1/(s + k)$ and that of a second-order system in the form $\omega^2/(s^2 + 2\zeta\omega s + \omega^2)$ rather than $1/(s^2 + 2\zeta\omega s + \omega^2)$.

Figures 6 (6a and 6b) and Figures 7 (7a and 7b) have been presented accordingly. The numerator (k or ω^2) is simply a factor, and hence data for the transfer functions $1/(s + k)$ or $1/(s^2 + 2\zeta\omega s + \omega^2)$ can be obtained by merely dividing the amplitude k or ω^2 as the case may be.

4. RESPONSE TO AN ARBITRARY INPUT

From a knowledge of the step or impulse response of a function the response to any input can be determined.

The input shown in Sketch 4.1 has been approximated by a series of rectangular pulses. The response to any individual pulse will be approximately that to an impulse function of area $f(\tau)\Delta\tau$, occurring at time $t = \tau$.

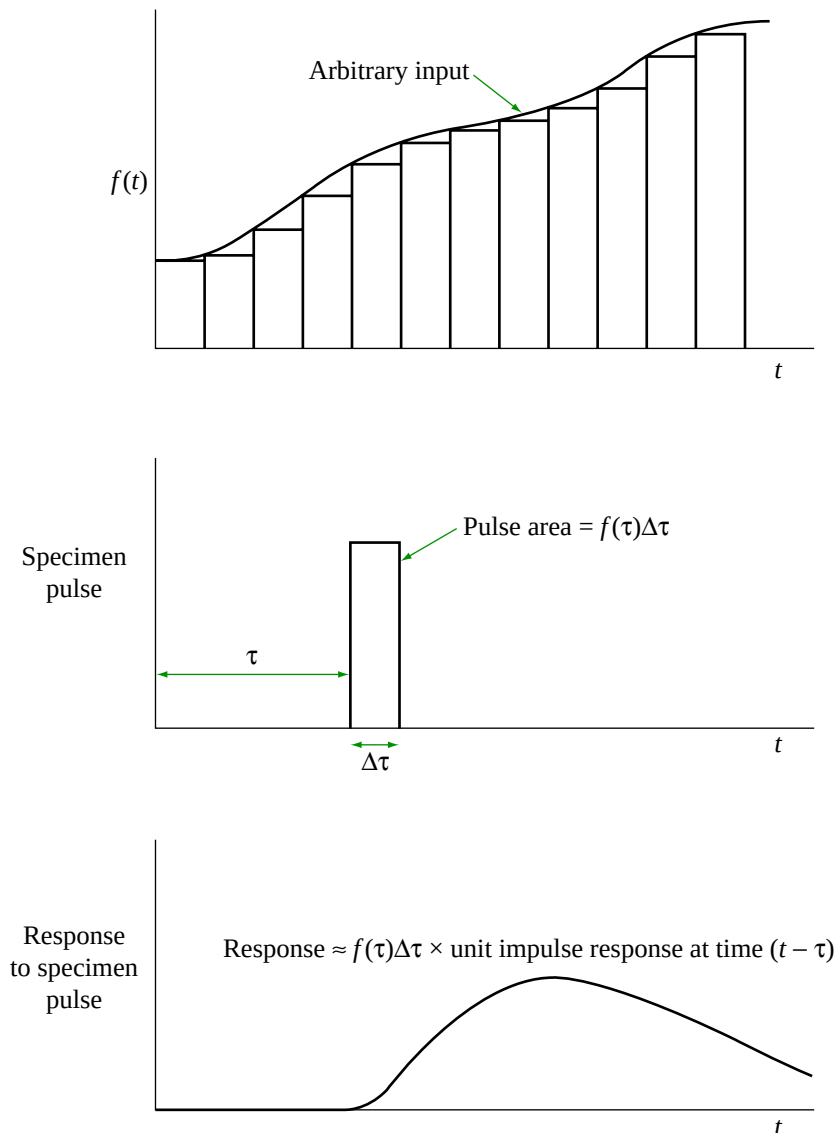
On the assumption that the system is initially at rest, the response to the full input is then given by the summation of the responses to the individual pulses. If the response to a unit impulse applied at τ is denoted by $w(t - \tau)$ then the full response will be given by

$$x(t) = \sum_{\tau=0}^{\tau=t} f(\tau) \Delta\tau w(t - \tau), \quad (4.1)$$

and in the limit, as $\Delta\tau$ tends to zero, this becomes

$$x(t) = \int_0^t f(\tau)w(t-\tau)d\tau. \quad (4.2)$$

The integral on the right-hand side of Equation (4.2) has the form known as the convolution integral. An arbitrary input may also be approximated in other ways by the use of step or triangular inputs.



Sketch 4.1

5. DERIVATION

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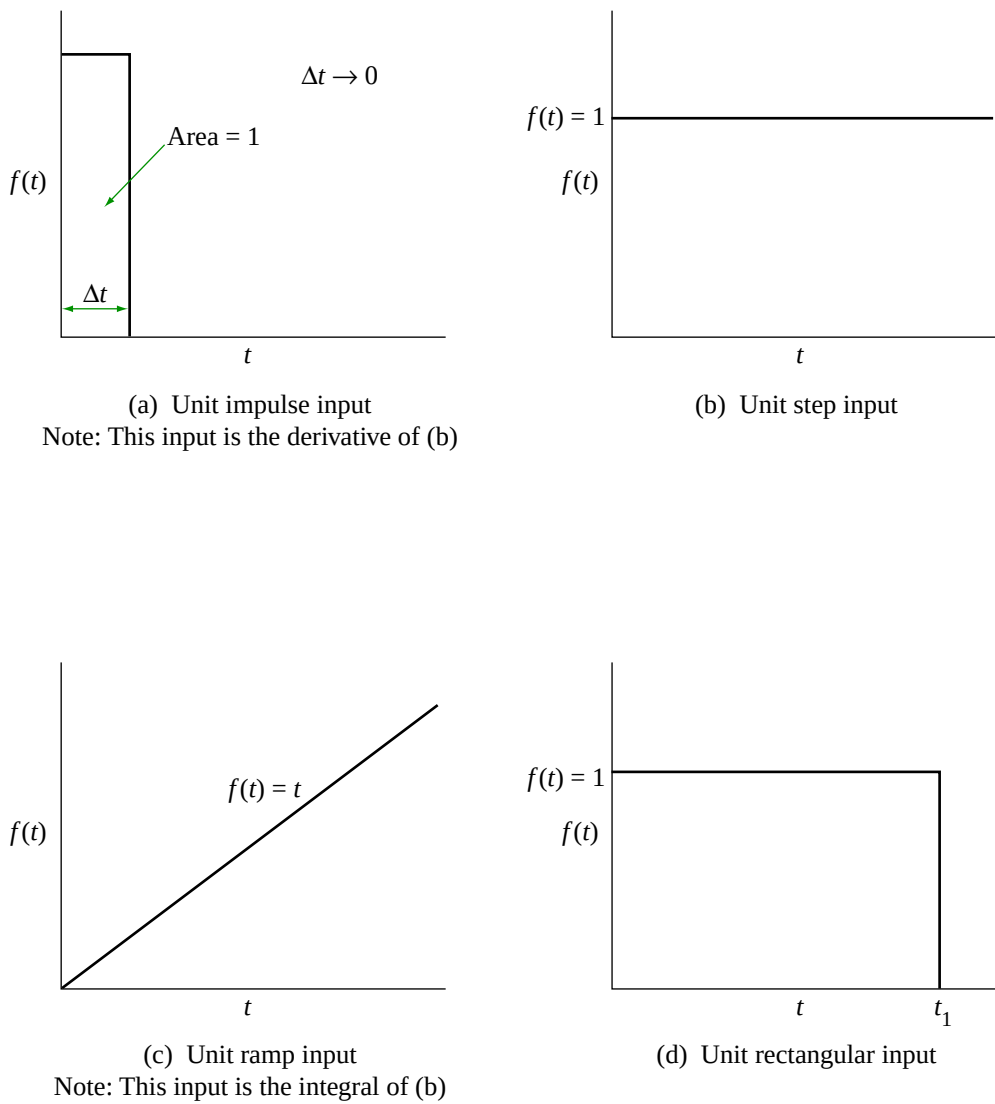
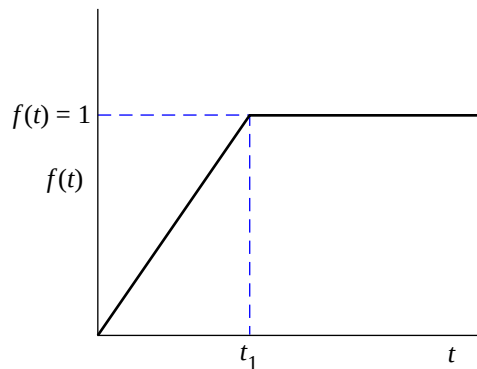
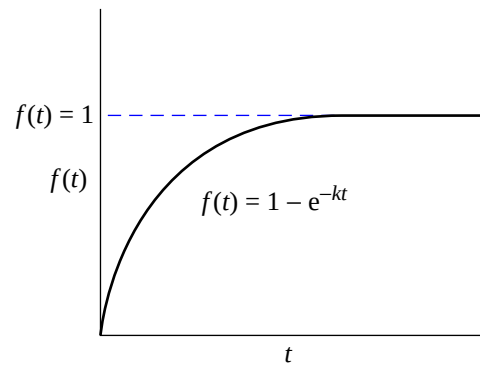


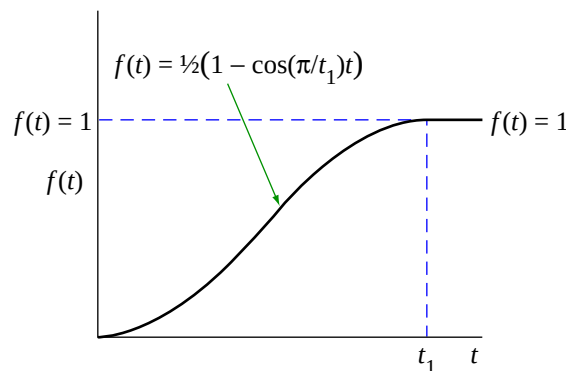
FIGURE 1 INPUT FUNCTIONS



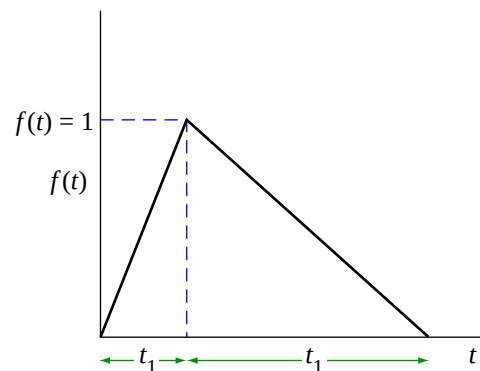
(e) Unit ramp-step input



(f) Unit exponential-step input (lagged-step)



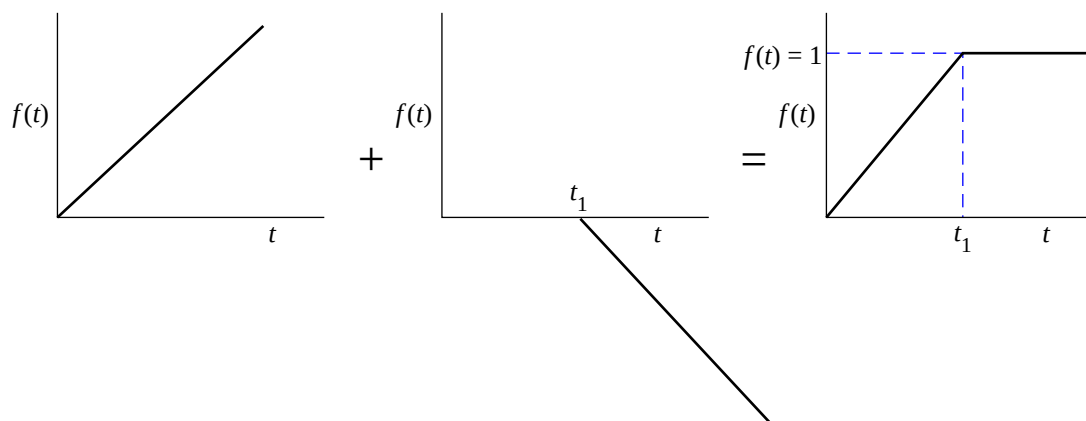
(g) Unit cosine-step input



(h) Unit triangular input

Note: The steady-state response of the system to a unit sinusoidal input is also considered.

Further notes: The unit ramp-step input (e) can be obtained from the superposition of two ramps of equal and opposite gradients, positive and negative, displaced by a time, t_1 .



Similarly, the unit triangular input (h) can be obtained from the superposition of three ramps.

FIGURE 1 (concluded)

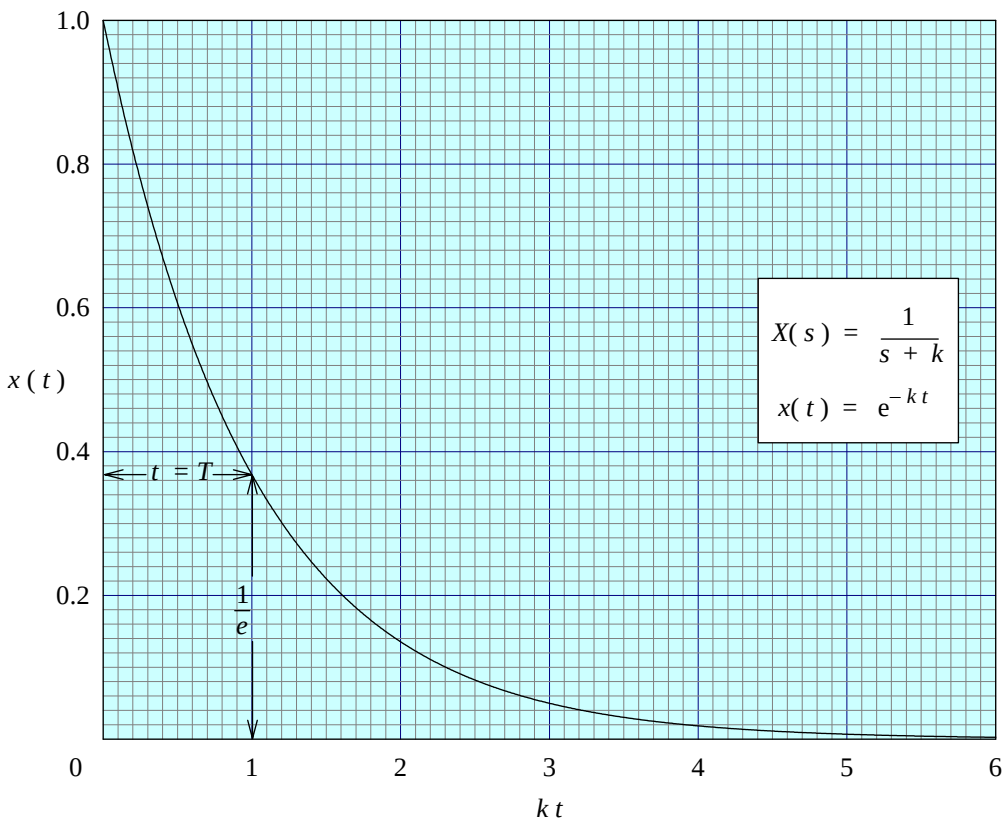


FIGURE 2a RESPONSE OF A FIRST-ORDER SYSTEM TO A UNIT IMPULSE INPUT

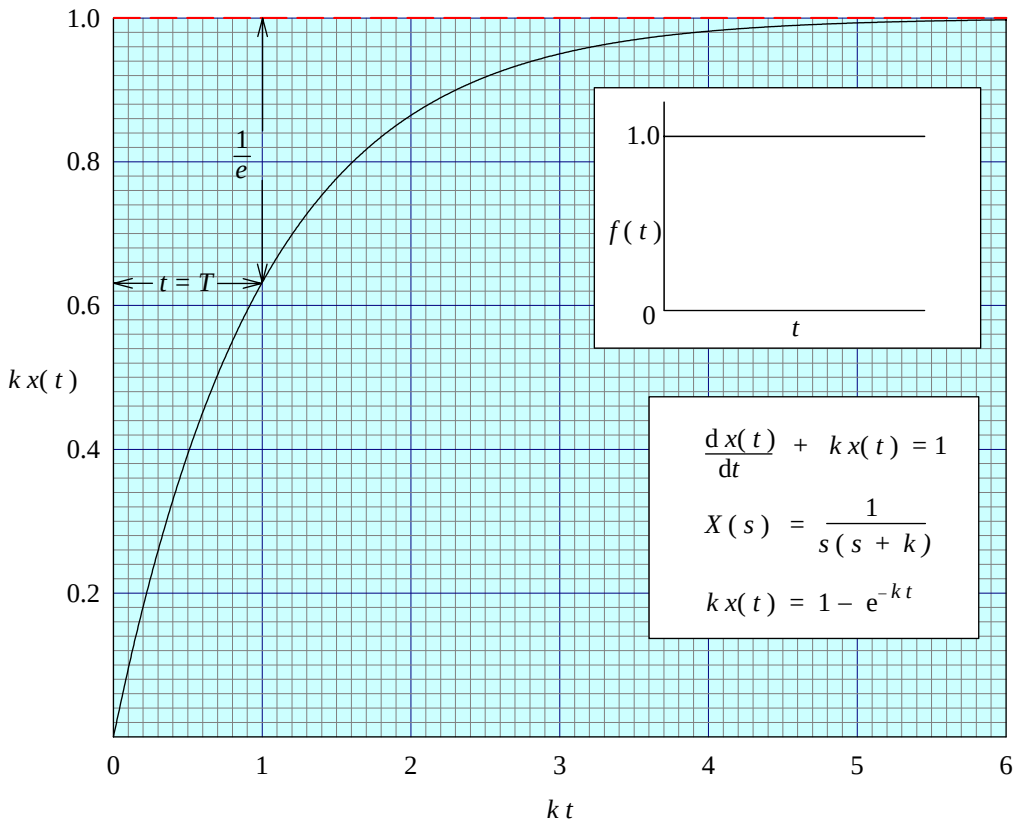


FIGURE 2b RESPONSE OF A FIRST-ORDER SYSTEM TO A UNIT STEP INPUT