

SOLUTION OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS BY THE LAPLACE TRANSFORM METHOD

1. INTRODUCTION

This Item presents a working scheme for solving ordinary linear differential equations with constant coefficients by means of the Laplace transform.

The merit of the method lies in the fact that the application of the transform reduces such an equation to a simple algebraic one from which the required solution is found by reference to tables of transforms. Thus with its use the only mathematical knowledge required to solve an ordinary linear differential equation with constant coefficients is the manipulation of algebraic equations. In addition, the method includes, as a matter of course, the initial conditions.

Sufficient information is given here to enable the method to be used but, if required, full details of the derivation of the method can be found in textbooks such as Reference 1. However, some elementary details of the mathematics involved will be found in Appendix B and an alternative transform and other notation which may be used are given in Appendix C.

2. METHOD OF SOLUTION

2.1 Outline of Method

Consider the differential equation

$$\frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + a_{n-2} \frac{d^{n-2} x}{dt^{n-2}} + \dots + a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = P(t), \quad (2.1)$$

where $a_{n-1}, a_{n-2}, \dots, a_2, a_1$ and a_0 are constant coefficients ($a_n = 1$) and $P(t)$ is a given function of time. The initial conditions at $t = 0$ are

$$x = x_0, \quad \frac{dx}{dt} = x_1, \quad \frac{d^2 x}{dt^2} = x_2, \quad \dots, \quad \frac{d^{n-1} x}{dt^{n-1}} = x_{n-1}. \quad (2.2)$$

The first step in the method is to write a subsidiary equation which takes the form

$$\begin{aligned} & (s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots + a_2s^2 + a_1s + a_0) \bar{x} \\ &= \bar{P}(s) + (a_1x_0 + a_2x_1 + a_3x_2 + \dots + a_{n-2}x_{n-3} + a_{n-1}x_{n-2} + x_{n-1}) \\ &+ s(a_2x_0 + a_3x_1 + a_4x_2 + \dots + a_{n-1}x_{n-3} + x_{n-2}) \\ &+ s^2(a_3x_0 + a_4x_1 + \dots + x_{n-3}) \\ &\dots \dots \dots \\ &\dots \dots \dots \\ &+ s^{n-2}(a_{n-1}x_0 + x_1) \\ &+ s^{n-1}x_0, \end{aligned} \quad (2.3)$$

where $\bar{P}(s)$ is Laplace transform of $P(t)$ obtained using Tables 5.1 and 5.2, and $\bar{x}$ is the Laplace transform of x .

The subsidiary equation is then rearranged to solve for $\bar{x}$ and by using the transform tables in the reverse direction on the resulting equation the solution $x(t)$ is obtained.

The details of the procedure are given in the following step-by-step statement illustrated by a simple example. Appendix A illustrates three physical systems which can be described by a second-order differential equation such as that used in this example.

2.2 Detailed Solution Method

Steps in method	Simple example
1. Divide differential equation throughout by coefficient of term of highest order of differential operator to obtain form of Equation (2.1)	Solve the equation $m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = Be^{-bt}.$ Let $\gamma = \frac{c}{m}, \omega^2 = \frac{k}{m}, \beta = \frac{B}{m}.$ Then the equation becomes $\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega^2 x = \beta e^{-bt}$ Consider the case of $\gamma < 2\omega$
2. Write down initial conditions at $t = 0$ in form of Equations (2.2)	$x = x_0, \frac{dx}{dt} = x_1$
3. Write down subsidiary equation in form of Equation (2.3). $\bar{P}(s)$ is obtained from $P(t)$ by using transform tables. Other terms are obtained by reference to Equations (2.1), (2.2) and (2.3).	$(s^2 + \gamma s + \omega^2) \bar{x} = \frac{\beta}{s+b} + (\gamma x_0 + x_1) + s x_0$ (because $a_2 = 1$, $a_1 = \gamma$ and $a_0 = \omega^2$ following Equation (2.1).
4. Rearrange to obtain $\bar{x}$. Solution will be a number of fractions with numerators and denominators which are, in general, polynomials in s .	$\bar{x} = \frac{\beta}{(s+b)(s^2 + \gamma s + \omega^2)} + \frac{\gamma x_0 + x_1}{s^2 + \gamma s + \omega^2} + \frac{s x_0}{s^2 + \gamma s + \omega^2}.$
5. Factorise denominators into linear factors when roots are real and into quadratic factors when roots are complex and conjugate, i.e. $(f \pm ig)$	Denominators are already in required form because $(s+b)$ is linear and, since $\gamma < 2\omega$, the roots of quadratic factor $(s^2 + \gamma s + \omega^2)$ are complex and conjugate.
6. Break down all fractions into simpler fractions whose denominators are the factors found in Step 5.	The first term must be broken down into partial fractions, thus $\bar{x} = \frac{\beta}{b^2 - b\gamma + \omega^2} \left[\frac{1}{s+b} - \frac{s+\gamma-b}{s^2 + \gamma s + \omega^2} \right] + \frac{\gamma x_0 + x_1}{s^2 + \gamma s + \omega^2} + \frac{s x_0}{s^2 + \gamma s + \omega^2}.$
7. Arrange denominators into forms similar to those appearing on right-hand sides of Tables	The complex roots of factor $(s^2 + \gamma s + \omega^2)$ are $(-\frac{1}{2}\gamma \pm iv)$ where $v^2 = \omega^2 - \frac{1}{4}\gamma^2$. With this substitution $s^2 + \gamma s + \omega^2 = (s + \frac{1}{2}\gamma)^2 + v^2$ which is of the form appearing in Table 5.1 and

Steps in method	Simple example
8. Cast terms on right-hand side into standard forms appearing on right-hand sides of Tables.	$\bar{x} = \frac{\beta}{b^2 - b\gamma + \omega^2} \left[\frac{1}{s+b} - \frac{s+\gamma-b}{(s+\frac{1}{2}\gamma)^2 + v^2} \right] + \frac{\gamma x_0 + x_1}{(s+\frac{1}{2}\gamma)^2 + v^2} + \frac{s x_0}{(s+\frac{1}{2}\gamma)^2 + v^2}$ The first term is already in the form of Transform 4. Table 5.1 and the other terms are easily cast into the form of Transforms 8 and 7, thus $\bar{x} = \frac{\beta}{b^2 - b\gamma + \omega^2} \left[\frac{1}{s+b} - \frac{s+\frac{1}{2}\gamma}{(s+\frac{1}{2}\gamma)^2 + v^2} - \frac{v}{(s+\frac{1}{2}\gamma)^2 + v^2} \cdot \frac{\frac{1}{2}\gamma - b}{v} \right]$ $+ \frac{v}{(s+\frac{1}{2}\gamma)^2 + v^2} \cdot \frac{\frac{1}{2}\gamma x_0 + x_1}{v} + \frac{s+\frac{1}{2}\gamma}{(s+\frac{1}{2}\gamma)^2 + v^2} x_0.$
9. Take inverse transform of each term in $\bar{x}$ using Tables in reverse direction (right to left) to obtain solution, $x(t)$.	Solution is $x(t) = \frac{\beta}{b^2 - b\gamma + \omega^2} \left[e^{-bt} - e^{-\frac{1}{2}\gamma t} \left(\cos vt + \frac{\frac{1}{2}\gamma - b}{v} \sin vt \right) \right]$ $+ \frac{\frac{1}{2}\gamma x_0 + x_1}{v} e^{-\frac{1}{2}\gamma t} \sin vt + x_0 e^{-\frac{1}{2}\gamma t} \cos vt.$
10. In all cases, check that solution satisfies initial conditions and original differential equation.	$\frac{dx}{dt} = \frac{\beta}{b^2 - b\gamma + \omega^2} \left[-b e^{-bt} + e^{-\frac{1}{2}\gamma t} \left(b \cos vt + \frac{\omega^2 - \frac{1}{2}\gamma b}{v} \sin vt \right) \right]$ $+ e^{-\frac{1}{2}\gamma t} \left(x_1 \cos vt - \frac{x_0 \omega^2 + x_1 \frac{1}{2}\gamma}{v} \sin vt \right)$ At $t = 0$, $x = x_0$ and $\frac{dx}{dt} = x_1$ so the initial conditions are satisfied. Also $\frac{d^2x}{dt^2} = \frac{\beta}{b^2 - b\gamma + \omega^2} \left[b^2 e^{-bt} + e^{-\frac{1}{2}\gamma t} \left\{ (\omega^2 - b\gamma) \cos vt - \frac{\frac{1}{2}\gamma \omega^2 - \frac{1}{4}\gamma^2 b + b v^2}{v} \sin vt \right\} \right]$ $+ e^{-\frac{1}{2}\gamma t} \left[-(x_0 \omega^2 + x_1 \gamma) \cos vt + \frac{x_0 \frac{1}{2}\gamma \omega^2 + x_1 (\frac{1}{4}\gamma^2 - v^2)}{v} \sin vt \right]$ and substitution into the original equation will demonstrate that the solution satisfies that equation.

Note (i) When roots can be extracted algebraically at Step 5, as in the example, it is possible to solve differential equations in general terms. In practice, however, it will often be found advisable or even necessary to introduce numerical values of the coefficients of the differential equations in the early states of the calculation. In such cases it is important to select the most suitable methods of extracting linear and/or quadratic factors of the denominator polynomials. A very useful collection of such methods can be found in Reference 2.

Note (ii) The labour of breaking down the fractions at Step 6 into simpler fractions might be eliminated by using fuller tables of Laplace transforms, such as those of Reference 3[†].

[†] To be compatible with the Laplace transform used here, each transform in Reference 3 should be divided by s (i.e. by D in the notation of that report) (see Appendix C).