

SECOND-ORDER METHOD FOR ESTIMATING THE SUBCRITICAL PRESSURE DISTRIBUTION ON A TWO-DIMENSIONAL AEROFOIL IN COMPRESSIBLE INVISCID FLOW

SUMMARY

A method is presented, accurate to second-order in thickness, camber and incidence, for estimating the subcritical pressure distribution on a two-dimensional aerofoil in compressible inviscid flow.

A few examples are given of comparisons with a nominally-exact method and it is shown how the method differs from a first-order method widely used as the basis for the calculation of pressure distributions on finite swept wings and on two-dimensional aerofoils in compressible viscous flow.

A critical evaluation of the second-order method is reported in Reference 7, wherein it is shown that over a wide range of aerofoil shapes, incidence and Mach number, pressure distributions calculated by the second-order method differ appreciably from nominally-exact calculations only for local regions of aerofoils where high local surface curvature occurs and over the first 20 per cent, or so, of chord for aerofoils having the bluffer leading-edge profiles.

Two suggested techniques for enumerating the method are outlined and reference is made to a computer program which has been used to obtain results from the method.

1. INTRODUCTION

In Reference 1 Weber described a simple, essentially first-order, method for calculating the pressure distribution on a two-dimensional aerofoil in incompressible potential flow. Subsequently Küchemann and Weber² developed a compressibility factor which enabled this method to be used for compressible flow, whilst an improvement upon this factor, derived empirically by Wilby³, is currently used with the Weber method as the standard first-order[†] approximate method.

The subsequent development of exact methods (*e.g.* Reference 4) highlighted the necessity for the inclusion of second-order, particularly camber, terms in approximate methods. A requirement for approximate methods still exists because at present it is not feasible to adapt existing exact methods for inviscid two-dimensional flow to the three-dimensional problem nor to take into account viscous effects, whilst the approximate methods can be so applied. Furthermore, the approximate methods are the more readily adapted to the design (inverse) problem and, compared with the exact methods, have an advantage in their considerably smaller requirement for computer time and size.

Van Dyke⁵ and others have suggested approximate solutions to the full second-order equation for the velocity potential while modifications to these solutions by Lock *et al.*⁶ have led to a second-order approximate method with some allowance for third-order effects.

The accuracy of this last method for calculating the compressible inviscid flow on two-dimensional aerofoils has now been assessed in some detail, in Reference 7, by comparison with, principally, the nominally-exact method of Sells. The basic formula is given in Section 3, its development is described in Section 4 and procedures for its enumeration are described in Section 6.

[†] See footnote to Section 4.1.

2. NOTATION

B	Wilby compressibility factor (Equation (4.10))
$\bar{B}_w$	Küchemann-Weber compressibility factor (Equation (4.5))
B_w	modified Küchemann-Weber compressibility factor (Equation (4.8))
C_p	pressure coefficient
C_L	lift coefficient
$\mathcal{F}$	$= C_p/[1 - q^2]$
K	coefficient in Equation (4.13)
M	Mach number
q	non-dimensional total velocity
$s_{\mu\nu}^{(1)} \dots s_{\mu\nu}^{(5)}$	coefficients in series approximations to $S^{(n)}(x)$ functions (Reference 9)
$S^{(1)} \dots S^{(5)}$	functions in standard first-order equation for velocity on aerofoil (Reference 9)
$S^{(40)} \dots S^{(44)}$	additional functions in second-order equation for velocity on aerofoil (Reference 13)
t	non-dimensional aerofoil maximum thickness
u	non-dimensional streamwise perturbation velocity
w	non-dimensional transverse perturbation velocity
x	non-dimensional coordinate in chordwise direction
z	non-dimensional coordinate measured from and normal to chord
$Z(x)$	$z = Z(x)$, equation of aerofoil surface
α	incidence (measured in degrees)
β	Prandtl-Glauert compressibility factor (Equation (4.7))
γ	ratio of specific heats of air (taken to be 1.4)
ρ	non-dimensional leading-edge radius
ϕ	perturbation velocity potential function

Subscripts

i	denotes incompressible flow value
s, t, α	taken singly denote first-order terms due to camber, thickness and incidence respectively and in combination denote second-order terms
x, z	denote differentials with respect to x, z
L	denotes local value
∞	denotes free-stream value

Superscripts

$(1), (2)$, when used with q, u and w , denote first- and second-order contributions to perturbation velocity.

Other notation, principally that particular to Sections 5 and 6, is defined when first introduced.

3. THE BASIC FORMULA

The standard expression (accurate to second-order) for the velocity on the surface of a two-dimensional aerofoil in compressible inviscid flow is:

$$\begin{aligned}
 q = & \left[\frac{1}{1 + \left\{ \left[\frac{1}{B} (S^{(2)}(x) \pm S^{(5)}(x)) \right]^2 \right\}^{1/2}} \right] \\
 & \times \left\{ \cos \alpha \left[1 + \frac{1}{B} (S^{(1)}(x) \pm S^{(4)}(x)) + \frac{1}{B^2} (S^{(40)}(x) \pm S^{(41)}(x)) + \right. \right. \\
 & \left. \left. + \frac{1}{B^2} (z_s \pm z_t) (S^{(42)}(x) \pm S^{(43)}(x)) + \frac{1}{B^2} (S^{(2)}(x) \pm S^{(5)}(x))^2 \right] + \right. \\
 & \left. + \sin \alpha \left[\frac{1}{B^2} (S^{(44)}(x)) \pm \frac{1}{B} \sqrt{\frac{1-x}{x}} \left(1 + \frac{S^{(3)}(x)}{B} \right) \right] \right\}. \quad (3.1)
 \end{aligned}$$

The symbol $\pm$ refers to the upper and lower surfaces, the positive sign being taken for the upper surface and the negative sign for the lower surface.

The definitions of, and methods for evaluating, the various functions are given in Sections 5 and 6.

4. DEVELOPMENT OF THE METHOD

The full second-order equation for the velocity potential is

$$(1 - M_\infty^2)\phi_{xx} + \phi_{zz} = M_\infty^2[2 + (\gamma - 1)M_\infty^2]\phi_x^{(1)}\phi_{xx}^{(1)} + 2M_\infty^2\phi_z^{(1)}\phi_{xz}^{(1)}, \quad (4.1)$$

where $\phi^{(1)}$ is the solution to the first-order equation (with first-order boundary conditions).

4.1 The Method of Küchemann and Weber

Küchemann and Weber² considered a simplified form[†] of the second-order equation for the velocity potential

$$(1 - M_L^2)\phi_{xx} + \phi_{zz} = 0, \quad (4.2)$$

where, to a first approximation,

$$M_L^2 = M_\infty^2 \left\{ 1 + \phi_x [2 + (\gamma - 1)M_\infty^2] \right\}. \quad (4.3)$$

They neglected the term $\frac{(\gamma - 1)M_\infty^2}{2}$, since it is small compared with 2, and made the further approximation that $2\phi_x \approx -\overline{C_{pi}}$, where $\overline{C_{pi}}$ is a constant mean value of C_{pi} . This led to the result

$$u = \frac{u_i}{\bar{B}_w}, \quad \boxed{\text{KÜCHEMANN-WEBER VELOCITY RULE}} \quad (4.4)$$

where $\bar{B}_w = [1 - M_\infty^2(1 - \overline{C_{pi}})]^{1/2}$. $\boxed{\text{KÜCHEMANN-WEBER FACTOR}} \quad (4.5)$

In using the Weber method^{1, 8} for predicting the velocity distribution on aerofoils and wings it became customary apply this compressibility factor in the modified form of Equation (4.8) (below) which uses the *local* value of C_{pi} , in which form unfortunately it overestimates the rate of increase with Mach number of the velocity at the crest of an aerofoil at zero incidence (see Figure 1). However, some limited comparison with *experiment*[‡] suggested that for practical use on a lifting aerofoil this could be compensated for in the Weber method by applying the factor to the thickness terms whilst employing the Prandtl-Glauert, β , factor on the lifting terms, giving the equation

$$q = \frac{\cos \alpha \left[1 + \frac{S^{(1)}(x)}{B_w} \pm \frac{S^{(4)}(x)}{\beta} \right] \pm \frac{\sin \alpha}{\beta} \sqrt{\frac{1-x}{x}} \left[1 + \frac{S^{(3)}(x)}{B_w} \right]}{\left\{ 1 + \left[\frac{1}{B_w} (S^{(2)}(x) \pm S^{(5)}(x)) \right]^2 \right\}^{1/2}}, \quad (4.6)$$

[†] It has become customary⁷, following the developments described in Sections 4.2 and 4.3, to refer to the Küchemann-Weber method as being of first-order because of the large number of second-order terms that this simplification removes.

[‡] These experiments were probably inadequately corrected for blockage effects.

where $\beta = [1 - M_\infty^2]^{1/2}$, PRANDTL-GLAUERT FACTOR (4.7)

$$B_w = [1 - M_\infty^2(1 - C_{pi})]^{1/2}$$
MODIFIED KÜCHEMANN-WEBER FACTOR (4.8)

with C_{pi} defined as the local incompressible pressure coefficient due to thickness only

$$C_{pi} = 1 - \frac{[1 + S^{(1)}(x)]^2}{[1 + (S^{(2)}(x))^2]}$$
 (4.9)

but with the proviso that $B_w = \beta = [1 - M_\infty^2]^{1/2}$ should be used when $C_{pi} > 0$.

More recently Wilby³ obtained empirically[†] an improvement upon the modified compressibility factor of Küchemann-Weber (Equation (4.8)), namely

$$B = [1 - M_\infty^2(1 - M_\infty C_{pi})]^{1/2}$$
WILBY FACTOR (4.10)

and Figure 1 shows the improvement he achieved for velocities due to thickness.

Initially, this factor replaced the modified Küchemann-Weber factor in Equation (4.6) with the Prandtl-Glauert relationship being retained for the lifting terms. This proved⁷ to be a retrograde step.

The Küchemann-Weber and Wilby compressibility factors considered above defined C_{pi} as the incompressible pressure coefficient for the thickness distribution at zero incidence. Comparisons with exact solutions later suggested using the Wilby factor on both the thickness and lifting terms of the incompressible Weber formula, with C_{pi} taking its local total value (due to thickness and lift) on the aerofoil at incidence in incompressible flow. This then leads to the standard first-order equation

$$q = \frac{\cos \alpha \left[1 + \frac{S^{(1)}(x)}{B} \pm \frac{S^{(4)}(x)}{B} \right] \pm \frac{\sin \alpha}{B} \sqrt{\frac{1-x}{x}} \left[1 + \frac{S^{(3)}(x)}{B} \right]}{\left\{ 1 + \left[\frac{1}{B} (S^{(2)}(x) \pm S^{(5)}(x)) \right]^2 \right\}^{1/2}},$$
 (4.11)

STANDARD FIRST-ORDER

where B is given by Equation (4.10) in which $C_{pi} = [1 - q_i^2]$ and where q_i is obtained from Equation (4.11) by putting $B = 1.0$. Equation (4.11) is the two-dimensional form of a general expression⁹ for calculating the velocity at any point on the surface of a swept wing.

4.2 The Method of Van Dyke and Others

Hayes¹⁰, Van Dyke⁵ and Gretler¹¹ showed that the solution of Equation (4.1), with boundary conditions satisfied to second order at the aerofoil surface, could be expressed simply in terms of the corresponding second-order solution in incompressible flow.

[†] Initially by recourse to experiments on symmetrical aerofoils at zero incidence in a wind tunnel in which virtually blockage-free conditions had been achieved; later it was possible to check satisfactorily his relation by comparison with the exact methods already referred to.

If the expression for the non-dimensional total velocity q_i at a point on the surface in incompressible flow is expanded in the form

$$q_i = 1 + q_i^{(1)} + q_i^{(2)}, \quad (4.12)$$

where $q_i^{(1)} (= u_i^{(1)})$ represents the first-order perturbation velocity contribution to q_i

and $q_i^{(2)} (= u_i^{(2)}(x, z) + \frac{1}{2}w_i^{(1)2})$ the second-order contribution,

then Van Dyke's result is simply

$$q = 1 + \frac{u_i^{(1)}}{\beta} + \frac{1}{2}(K - 1)u_i^{(1)2} + Kq_i^{(2)}, \quad (4.13)$$

VAN DYKE
VELOCITY
RULE

where $\beta = [1 - M_\infty^2]^{\frac{1}{2}}$ (4.14)

and $K = \frac{(\gamma + 1)M_\infty^4 + 4\beta^2}{4\beta^4}$. (4.15)

The use of Equation (4.13) at the crest of an ellipse at zero incidence (for which $q_i^{(2)}$ is zero) is shown in Figure 1.

Now, in general, the term $q_i^{(2)}(x, z)$ appearing in Equation (4.13) takes the form

$$q_i^{(2)} = u_i^{(2)}(x, 0) + Z \frac{d^2 Z}{dx^2} + \frac{1}{2} \left(\frac{dZ}{dx} \right)^2 \quad (4.16)$$

where Z is $Z(x)$, the equation of the aerofoil surface. The last two terms in Equation (4.16) approach infinity as $x \rightarrow 0$ in the manner of x^{-1} . To overcome this difficulty, and hence to derive an expression uniformly valid up to the leading edge, Lock *et al*⁶ suggested writing $q_i^{(2)}$ in the form

$$\begin{aligned} q_i^{(2)} &= u_i^{(2)}(x, 0) + \frac{d}{dx} \left(Z \frac{dZ}{dx} \right) - \frac{1}{2} \left(\frac{dZ}{dx} \right)^2, \\ &= u_i^{(2)*} - \frac{1}{2} \left(\frac{dZ}{dx} \right)^2, \end{aligned} \quad (4.17)$$

where $u_i^{(2)*} = u_i^{(2)}(x, 0) + \frac{d}{dx} \left(Z \frac{dZ}{dx} \right)$, (4.18)

leading to $q = 1 + \frac{u_i^{(1)}}{\beta} + \frac{1}{2}(K - 1)u_i^{(1)2} + Ku_i^{(2)*} - \frac{1}{2}K \left(\frac{dZ}{dx} \right)^2$. (4.19)

This suggests the alternative expression

$$q = \frac{1 + \frac{u_i^{(1)}}{\beta} + \frac{1}{2}(K-1)u_i^{(1)2} + Ku_i^{(2)*}}{\left[1 + K\left(\frac{dZ}{dx}\right)^2\right]^{1/2}}, \quad (4.20)$$

in which the denominator $\left[1 + K\left(\frac{dZ}{dx}\right)^2\right]^{1/2}$ is a natural analogue of the Riegels factor for incompressible flow.

4.3 The Method of Lock and Wilby

Equation (4.20) gives satisfactory results provided the local Mach number does not exceed about 0.8. In order to extend its range of validity, and to bring it into line with the empirical expression due to Wilby (see Equations (4.10) and (4.11), use may be made of a third-order solution due to Hantzsche¹² for an ellipse at zero incidence (for which the term $u_i^{(2)*}$ in Equation (4.20) vanishes). This solution at the maximum thickness position takes the form

$$q = 1 + \frac{u_i^{(1)}}{\beta} + \frac{1}{2}(K-1)u_i^{(1)2} + f(M_\infty)u_i^{(1)3}, \quad (4.21)$$

where $f(M_\infty)$ is given in Reference 12.

Equation (4.17) is of the form

$$q = 1 + g(u_i^{(1)}, M_\infty). \quad (4.22)$$

Wilby³ (and also Lock *et al*⁶) suggested replacing the expression $g(u_i^{(1)}, M_\infty)$ in Equation (4.22) by the empirical relation $u_i^{(1)}/B$ (see Equation (4.10)) thus leading to the expression for an arbitrary aerofoil (for which $u_i^{(2)*}$ does not vanish)

$$q = \frac{1 + \frac{u_i^{(1)}}{B} + Ku_i^{(2)*}}{\left[1 + K\left(\frac{dZ}{dx}\right)^2\right]^{1/2}}. \quad (4.23)$$

Also, Wilby³ showed that in many practical examples the value of K is very similar to $1/B^2$ for the normal values of crest C_{pi} . He found that greater accuracy, by reducing the leading edge velocities, is obtained if K is replaced by $1/B^2$ in the denominator. A similar replacement can conveniently be made in the numerator, because this makes very little difference to the results. The final equation is then

$$q = \frac{1 + \frac{u_i^{(1)}}{B} + \frac{u_i^{(2)*}}{B^2}}{\left\{1 + \left[\frac{1}{B}\frac{dZ}{dx}\right]^2\right\}^{1/2}}. \quad (4.24)$$

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