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The stability and response of linear systems. Part I: introductory concepts of stability

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THE STABILITY AND RESPONSE OF LINEAR SYSTEMS PART I: INTRODUCTORY CONCEPTS OF STABILITY

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THE STABILITY AND RESPONSE OF LINEAR SYSTEMS PART I: INTRODUCTORY CONCEPTS OF STABILITY

1. INTRODUCTION

This Item is the first in a group dealing with methods of assessing the stability and response of linear systems. It introduces the mathematical description of linear systems and the formulation of the characteristic equation. The assessment of stability from a knowledge of the roots of the characteristic equation, or from its coefficients by means of the Routh-Hurwitz Criteria, is discussed and the calculation of the modes of motion is explained. The concept of a system transfer function is introduced together with its application to systems with feedback.

Part II, Item No. 74020^[4], considers methods of graphical presentation that provide the means of displaying the effects on the characteristics of the system of varying some of its parameters. Part III, Item No. 74021^[5], treats the methods of system assessment that are based on frequency response. Part IV, Item No. 80002^[6], considers the specification and measures of performance that may be used in conjunction with these methods of assessment. Part V, Item No. 80018^[7], deals with the function of the different system elements and with design techniques used to construct a system with the required characteristics, and Part VI, Item No. 81008^[8], considers the factors that, in particular circumstances, affect the selection of a specific method of system assessment. Part VII, Item No. 81040^[9] provides examples that illustrate use of the methods of Parts I to VI.

2. THE CHARACTERISTIC EQUATION AND ITS USES

2.1 Characteristic Equation of Linear Systems

A system relates a number of output variables (or state variables or independent variables) $X_i(t)$, $i = 1, 2, \dots, N$, to a number of input variables $F_i(t)$, $i = 1, 2, \dots, M$, where $M \leq N$. In a linear system all the relationships between the state variables are linear, *i.e.* no terms involving products, squares or higher powers of the state variables appear.

It is further assumed in these Items that the system characteristics do not vary with time so that the particular type of linear system considered can be expressed by a set of differential equations with constant coefficients having, for the case of $M = N$, the form

$$\left. \begin{aligned} h_1 X_1(t) + h_2 X_2(t) + \dots + h_N X_N(t) &= F_1(t), \\ k_1 X_1(t) + k_2 X_2(t) + \dots + k_N X_N(t) &= F_2(t), \\ \dots &\dots \\ q_1 X_1(t) + q_2 X_2(t) + \dots + q_N X_N(t) &= F_N(t), \end{aligned} \right\} \quad (2.1)$$

where h_1, k_1 , etc. are differential operators of the form

$$\left. \begin{aligned} h_i &= h_{i,m} D^m + h_{i,m-1} D^{m-1} + \dots + h_{i,1} D + h_{i,0}, \\ k_i &= k_{i,m} D^m + k_{i,m-1} D^{m-1} + \dots + k_{i,1} D + k_{i,0}, \end{aligned} \right\} \quad (2.2)$$

where $D = d/dt$ and $h_{i,m}, h_{i,m-1}, \dots, k_{i,m}, k_{i,m-1}, \dots$ are real constants. One or more of the input variables $F_i(t)$, $i = 1, 2, \dots, N$, start at $t = 0$ and all the initial values of all the state variables are specified at $t = 0$. In most practical systems the differential operators are usually of second order, *i.e.* $m = 2$, and the initial conditions to be specified are then x_i and dx_i/dt at $t = 0$.

Any set of differential equations similar to Equations (2.1) can be recast in canonical form as a larger set of mN first-order differential equations in which the original state variables $X_i(t)$, $i = 1, 2, \dots, N$, are replaced by new state variables x_k , $k = 1, 2, \dots, mN$. They then take the form

$$\left. \begin{aligned} a_0 Dx_1 + a_1 x_1 + a_2 x_2 + \dots + a_{mN} x_{mN} &= f_1(t), \\ b_0 Dx_2 + b_1 x_1 + b_2 x_2 + \dots + b_{mN} x_{mN} &= f_2(t), \\ \dots & \\ r_0 Dx_{mN} + r_1 x_1 + r_2 x_2 + \dots + r_{mN} x_{mN} &= f_{mN}(t), \end{aligned} \right\} \quad (2.3)$$

where the input functions, $f_1(t)$, $f_2(t)$, *etc.*, are each functions of all the original input functions, $F_1(t)$, $F_2(t)$, *etc.*, or are zero. This is illustrated in the example in Appendix A.

A system that is described by Equations (2.3) is stable when, with the input functions, $f_i(t)$, equal to zero for $t > 0$, all non-zero solutions for $x_i(t)$, $i = 1, 2, \dots, mN$, decay to zero with increasing time, t . Equating all the input functions $f_i(t)$, $i = 1, 2, \dots, mN$, in Equations (2.3) to zero, and writing the general solution of the resulting equations in the form

$$x_i(t) = \alpha_i e^{\lambda t}, \quad (2.4)$$

where the coefficients α_i , $i = 1, 2, \dots, mN$, are independent of t , non-zero solutions exist if, and only if,

$$\begin{vmatrix} a_0 \lambda + a_1 & a_2 & a_3 \dots a_{mN} \\ b_1 & b_0 \lambda + b_2 & b_3 \dots b_{mN} \\ \dots & \dots & \dots \\ r_1 & r_2 & r_3 \dots r_0 \lambda + r_{mN} \end{vmatrix} = 0. \quad (2.5)$$

The determinantal equation, Equation (2.5), can be expanded into the form

$$A_{mN} \lambda^{mN} + A_{mN-1} \lambda^{mN-1} + \dots + A_1 \lambda + A_0 = 0, \quad (2.6)$$

where A_i , $i = 0, 1, 2, \dots, mN$, are real constants that depend on a_i , b_i , *etc.* Equation (2.6) is known as the characteristic equation. It has mN roots which may be real or complex and in general can be written as

$$k_1, k_2, \dots, k_p, [k_{p+1} \pm i v_{p+1}], \dots, [k_{1/2(mN+p)} \pm i v_{1/2(mN+p)}],$$

where k_j and v_j are real.

The state variables from Equation (2.4) are

$$x_i(t) = \sum_{r=1}^p \beta_{ir} e^{k_r t} + \sum_{r=p+1}^{\frac{mN+p}{2}} \beta_{ir} e^{k_r t} \sin(v_r t + \varepsilon_{ir}), \quad (2.7)$$

where β_{ir} and ε_{ir} are real constants. This solution, $x_i(t)$, decays with time when *all* k_r are negative; it follows from the definition already given that when this condition is satisfied the system is stable.

The quantities β_{ir} (and ε_{ir}) in Equation (2.7) are not independent. Consider the r^{th} root of the characteristic equation, λ_r , where λ_r may be real or imaginary. Then that part of the response of a state variable x_i due to that root will be

$$x_i = \beta_{ir} e^{\lambda_r t}. \quad (2.8)$$

If the contributions of that root to each of the mN state variables in the form of Equation (2.8) are substituted in Equations (2.3) with the right-hand sides set equal to zero, a set of simultaneous equations in the coefficients β_{ir} is obtained. Since the determinant of the coefficients of that set of equations is zero, only the ratios

$$\beta_{1r} : \beta_{2r} : \beta_{3r} : \dots : \beta_{mN,r} \quad (2.9)$$

can be obtained. The ratios denoted by Expression (2.9) define the shape of the r^{th} mode corresponding to the r^{th} root.

Although Expression (2.9) defines the mode, in the case when λ_r is complex it is necessary to combine the conjugate complex roots $\lambda_r = k_r + i v_r$ and $\lambda_{r+1} = k_r - i v_r$ to obtain a single oscillatory mode expressed in the form of the second term of Equation (2.7). In a physical problem the coefficients in Expression (2.9) β_{ir} and $\beta_{i,r+1}$ must be conjugate complex pairs, say $A_r \pm i B_r$, in order that the solution for x_i shall be real. Then the contribution of the two roots to x_i will be

$$\begin{aligned} x_i &= (A_r + i B_r) e^{(k_r + i v_r)t} + (A_r - i B_r) e^{(k_r - i v_r)t} \\ &= C_r e^{k_r t} \sin(v_r t + \varepsilon_r), \end{aligned}$$

where $\varepsilon_r = \tan^{-1}(A_r / (-B_r))$

and $C_r = 2A_r / \sin \varepsilon_r = -2B_r / \cos \varepsilon_r$.

There are many algorithms available for the numerical evaluation of the characteristic roots and modes (see, for example, References 10, 12 and 14).

2.1.1 Example illustrating formation of the characteristic equation and determination of roots and modes

Obtain the characteristic equation for the system defined by

$$\begin{aligned} D^2x + 3Dx + 4x - 3Dy &= \sin t \\ D^2y + 3Dy + 4y - 3Dx &= \sin 3t \end{aligned}$$

(this is the system used as an example in Item No. 69025^[3] with $\gamma = 3$ and $\omega = 2$).

To write the equations in canonical form the following substitutions are made

$$\begin{aligned} Dx &= x_1, \\ x &= x_2, \\ Dy &= x_3, \\ y &= x_4. \end{aligned}$$

The characteristic equation then becomes

$$\left. \begin{aligned} (D+3)x_1 + 4x_2 - 3x_3 &= \sin t \\ -3x_1 + (D+3)x_3 + 4x_4 &= \sin 3t \\ -x_1 + Dx_2 &= 0 \\ -x_3 + Dx_4 &= 0 \end{aligned} \right\} \quad (2.10)$$

and the characteristic equation is given by

$$\begin{vmatrix} \lambda + 3 & 4 & -3 & 0 \\ -3 & 0 & \lambda + 3 & 4 \\ -1 & \lambda & 0 & 0 \\ 0 & 0 & -1 & \lambda \end{vmatrix} = 0, \quad (2.11)$$

or $\lambda^4 + 6\lambda^3 + 8\lambda^2 + 24\lambda + 16 = 0.$

To obtain the modes, the roots of this equation have first to be calculated. They are

$$\lambda = -0.764, -5.236 \text{ and } \pm 2i.$$

Now consider a mode of motion represented by the root $\lambda = \lambda_r$. The state variables would then be

$$x_1 = \beta_{1r} e^{\lambda_r t}, \quad x_2 = \beta_{2r} e^{\lambda_r t}, \quad x_3 = \beta_{3r} e^{\lambda_r t} \quad \text{and} \quad x_4 = \beta_{4r} e^{\lambda_r t}.$$

Substituting these values in Equations (2.10), setting the right-hand sides equal to zero and dividing throughout by β_{4r} yields a set of four equations in the three unknowns β_{1r}/β_{4r} , β_{2r}/β_{4r} and β_{3r}/β_{4r} . As a consequence of the condition that the values of λ_r must satisfy the characteristic equation any three of the four equations can be solved to obtain the three unknowns. Taking the first three of Equations (2.10) which become

$$\left. \begin{aligned} (\lambda_r + 3) \frac{\beta_{1r}}{\beta_{4r}} + 4 \frac{\beta_{2r}}{\beta_{4r}} - 3 \frac{\beta_{3r}}{\beta_{4r}} &= 0 \\ -3 \frac{\beta_{1r}}{\beta_{4r}} + (\lambda_r + 3) \frac{\beta_{3r}}{\beta_{4r}} + 4 &= 0 \\ -\frac{\beta_{1r}}{\beta_{4r}} + \lambda_r \frac{\beta_{2r}}{\beta_{4r}} &= 0 \end{aligned} \right\}$$

and solving yields

$$\begin{aligned} \beta_{1r} : \beta_{2r} : \beta_{3r} : \beta_{4r} &= \begin{vmatrix} 0 & 4 & -3 \\ -4 & 0 & \lambda_r + 3 \\ 0 & \lambda_r & 0 \end{vmatrix} : \begin{vmatrix} \lambda_r + 3 & 0 & -3 \\ -3 & -4 & \lambda_r + 3 \\ -1 & 0 & 0 \end{vmatrix} : \begin{vmatrix} \lambda_r + 3 & 4 & 0 \\ -3 & 0 & -4 \\ -1 & \lambda_r & 0 \end{vmatrix} : \begin{vmatrix} \lambda_r + 3 & 4 & -3 \\ -3 & 0 & \lambda_r + 3 \\ -1 & \lambda_r & 0 \end{vmatrix} \\ &= 12\lambda_r : 12 : 4\lambda_r(\lambda_r + 3) + 16 : -(\lambda_r^3 + 6\lambda_r^2 + 4\lambda_r + 12) . \end{aligned}$$

Substituting the real roots gives for the first mode with $\lambda_1 = -0.764$,

$$\beta_{11} : \beta_{21} : \beta_{31} : \beta_{41} = -0.764 : 1 : 0.764 : -1$$

so that $x_2 = -x_4$ or $x = -y = \beta_{21}e^{-0.764t}$,

and for the second mode with $\lambda_2 = -5.236$,

$$\beta_{12} : \beta_{22} : \beta_{32} : \beta_{42} = -5.236 : 1 : 5.236 : -1$$

so that $x_2 = -x_4$ or $x = -y = \beta_{22}e^{-5.236t}$.

The two pure imaginary roots must be combined to yield the third mode as follows.

For $\lambda_3 = +2i$, $\beta_{13} : \beta_{23} : \beta_{33} : \beta_{43} = 2i : 1 : 2i : 1$ so that $x_2 = x_4$ or $x = y = \beta_{23}e^{2it}$ while,

for $\lambda_4 = -2i$, $\beta_{14} : \beta_{24} : \beta_{34} : \beta_{44} = -2i : 1 : -2i : 1$ so that $x_2 = x_4$ or $x = y = \beta_{24}e^{-2it}$.

Hence the mode is given by

$$\begin{aligned} x = y &= \beta_{23}(e^{2it} + e^{-2it}) \\ &= 2\beta_{23} \cos 2t . \end{aligned}$$

In this case, since there is no damping, the phase angle, ε , is zero.

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