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The stability and response of linear systems. Part II: method of displaying stability characteristics

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THE PREPARATION OF THIS DATA ITEM

The work on this particular Item was monitored and guided by the following Working Party:

Chairman

Mr H.H.B.M. Thomas — Royal Aircraft Establishment

Members

Mr G.C. Howell — Royal Aircraft Establishment

Mr D.K. Mendela — Hawker Siddeley Aviation Ltd, Hatfield

Mr R.M. Taunton — British Aircraft Corporation Ltd, Stevenage

on behalf of the Dynamics Committee which first met in 1962 and now has the following membership:

Chairman

Mr H.H.B.M. Thomas — Royal Aircraft Establishment

Vice-Chairman

Mr R. Melling — Hawker Siddeley Aviation Ltd, Brough

Members

Mr J. Arnall — Hawker Siddeley Aviation Ltd

Mr D. Booker — British Aircraft Corporation Ltd, Preston

Prof. G.J. Hancock — Queen Mary College, London

Mr D.K. Mendela — Hawker Siddeley Aviation Ltd, Hatfield

Mr R.L. Scotland — British Aircraft Corporation Ltd, Filton

Mr R.M. Taunton — British Aircraft Corporation Ltd, Stevenage.

The work on this Item was carried out in the Aircraft Motion Group of ESDU. The Group Head is Mr P.D. Chappell. The member of staff who undertook the technical work involved in the construction and development of the Item was Mr J.A. Shaw.

THE STABILITY AND RESPONSE OF LINEAR SYSTEMS PART II: METHODS OF DISPLAYING STABILITY CHARACTERISTICS

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THE STABILITY AND RESPONSE OF LINEAR SYSTEMS PART II: METHODS OF DISPLAYING STABILITY CHARACTERISTICS

1. INTRODUCTION

This Item is the second in a group dealing with methods of assessing the stability and response of linear systems. Part I, Item No. 74019^[2], introduces the mathematical description of linear systems, both with and without feedback, and introduces the concept of stability.

This Item considers methods of graphical presentation that provide the means of displaying the effects on the characteristics of the system of varying some of its parameters. The methods discussed are the root locus and the one-parameter stability diagram, which are alternative ways of showing how variations in one parameter affect the damping and frequency of the modes of motion, and the two-parameter stability diagram which displays the effect on the system stability of varying two parameters and also shows the damping and frequency of the mode whose stability is of concern.

Part III, Item No. 74021^[3], treats the methods of system assessment that are based on frequency response. Part IV, Item No. 80002^[4], considers the specification and measures of performance that may be used in conjunction with these methods of assessment. Part V, Item No. 80018^[5], deals with the function of the different system elements and with design techniques used to construct a system with the required characteristics. Part VI, Item No. 81008^[6], considers the factors that, in particular circumstances, affect the selection of a specific method of system assessment, and Part VII, Item No. 81040^[7], provides examples that illustrate use of the methods of Parts I to VI.

2. GENERAL

The techniques considered in Item Nos 74019^[2] and 74021^[3] determine whether a given system is stable or not and allow (in the case of graphical methods) the degree of stability to be assessed. However, more information may be required; for example, the damping and frequency of particular modes is of interest and diagrams that show how they are affected by the variation of system parameters are useful. The real and imaginary parts of the roots of the characteristic equation, or the related damping and frequency of the modes, are plotted as functions of the parameters that enter into the design of the system. Some features of the response of the system are accordingly revealed.

The simplest diagram shows the effect of variation in one parameter only and two commonly used forms are considered in Section 3. When the effect of varying two parameters is studied the damping and frequency loci that are drawn only apply to the mode whose stability is of particular interest, as discussed in Section 4.

When the design of a system involves more than two parameters it is usual to treat the variations of one or two of them at a time.

3. ONE-PARAMETER STABILITY DIAGRAMS

The characteristic equation can in general be written in the form

$$R(s) + Q(K, s) = 0 ,$$

where K is the parameter the influence of whose variation is to be considered. The equation can be solved for various values of K and the variation of real and imaginary parts of the roots can be displayed either in the complex plane when appropriate values of K are marked along the locus or by plotting them, each on a separate figure, against K . The former diagram is called a root locus diagram while the latter is a root parameter diagram.

3.1 The Root Locus

Relatively cheap and therefore readily available computing facilities have eased the difficulties involved in solving high-order equations (see References 3 to 5 of Item No.74019^[2] for suitable algorithms) so that the need for approximate procedures is now correspondingly less important. However, the methods that have been developed for sketching the root locus diagram without the necessity for calculating numerical values for the roots are sufficiently simple for them to retain their value for establishing trends before any detailed calculations are required. Such a sketch may also be useful in indicating how roots calculated for selected values of the variable parameter K are joined by the branches of the locus. For many practical systems the characteristic equation can be written in the form

$$R(s) + K U(s) = 0 , \quad (3.1)$$

$$\text{i.e.} \quad 1 + K \frac{U(s)}{R(s)} = 0 . \quad (3.2)$$

Thus, all values of the variable s satisfying Equation (3.1), i.e. those that are roots of the equation, must satisfy the modulus and argument conditions

$$|K| = \left| \frac{R(s)}{U(s)} \right| \quad (3.2a)$$

$$\text{and} \quad \arg U(s) - \arg R(s) = \begin{cases} (2n + 1)\pi, & K > 0 \\ 2n\pi, & K < 0 \end{cases} \text{ where } n = 0, 1, 2, \dots \quad (3.2b)$$

A method has been devised for sketching the loci of the roots with variations in K . The method is illustrated in the Examples and is as follows.

- (i) The number of branches of the locus is equal to the order of the characteristic equation, Equation (3.1).
- (ii) Provided the coefficients in Equation (3.1) are real, the locus will be symmetrical about the real axis.
- (iii) The roots of $R(s) = 0$, i.e. the poles of $U(s)/R(s)$, correspond to $K = 0$.
- (iv) The roots of $U(s) = 0$, i.e. the zeros of $U(s)/R(s)$, correspond to $K = \pm \infty$.

- (v) There may exist r asymptotes which the locus approaches as $K \rightarrow \pm \infty$, where r is given by the number of poles of $U(s)/R(s)$ minus the number of zeros. These asymptotes will make angles with the real axis given by

$$\frac{(2n+1)\pi}{r} \quad \text{as} \quad K \rightarrow \infty \quad (3.3a)$$

$$\text{and} \quad \frac{2n\pi}{r} \quad \text{as} \quad K \rightarrow -\infty, \quad (3.3b)$$

where n is an integer taking values $0, 1, 2, \dots, (r-1)$.

- (vi) The asymptotes intersect each other on the real axis at a point with coordinates

$$\left(\frac{\Sigma p_j - \Sigma z_j}{r}, 0 \right), \quad (3.4)$$

where Σp_j and Σz_j are the sums of the poles and of the zeros of $U(s)/R(s)$ respectively.

- (vii) The existence of a real pole or a real zero of $U(s)/R(s)$ implies that all the real axis will be made up of branches of the locus.
- (viii) A segment of the real axis will correspond to $K < 0$ if the total number of real poles and real zeros of $U(s)/R(s)$ to the right of that segment is even (in this context zero is an even integer) and to $K > 0$ if the total number of real poles and zeros to the right is odd.
- (ix) Any points of intersection of the branches of the locus (frequently termed breakaway points) occur at values of s given by $dK/ds = 0$. The order of the equation $dK/ds = 0$ may be even higher than that of the characteristic equation so that finding from it exact values of s at breakaway points may not be easy without some computing facilities. However, it will usually be possible, from the portions of the locus already determined, to decide whether breakaway points are likely to exist and calculation of K at a few points along the branch where the breakaway point is expected to occur will enable its approximate position to be located quickly.

Such a procedure will usually be adequate since exact solutions are rarely required. However, where the equation $dK/ds = 0$ is used, it should be noted that the corresponding values of K should be calculated for the solutions obtained because they may prove to be complex and, in that case, will not be significant.

- (x) The intersections of the locus with the imaginary axis are found by inserting $s = iv$ in Equation (3.1) and equating real and imaginary parts to zero. Either one equation will be independent of K and will yield the value of v or both may be solved as simultaneous equations for v and K .
- (xi) The slope of the locus at a complex pole or zero of $U(s)/R(s)$ can be determined from the angles made with the real axis by the lines drawn from all the other poles and zeros to that complex pole or zero. Denoting the angle of departure from a pole by x and the angle of arrival at a zero by y , x and y can be found from the equations

$$\begin{aligned} \Sigma \theta_p - \Sigma \phi_p - x &= \begin{cases} (2n+1)\pi, & \text{when } K > 0 \\ 2n\pi, & \text{when } K < 0, \end{cases} \\ \Sigma \theta_p - \Sigma \phi_p + y &= \end{aligned} \quad (3.5)$$

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where $\Sigma\theta_p$ is the sum of the angles made with the real axis by the lines drawn from the zeros, $\Sigma\phi_p$ is the sum of the angles made by the lines drawn from poles and n is a suitable integer ($n = 0, \pm 1, \pm 2, \dots$).

- (xii) The values of K on the locus are given, from Equation (3.2), by $-R(s)/U(s)$ and may be readily evaluated if required. For any value of s , $R(s)/U(s)$ is given by the product of the constant factor (a_n/b_m of Equation (3.5) of Item No. 74019 where Equation (3.5) is the 'open loop' transfer function) and all the distances of the poles of $U(s)/R(s)$ from the point in question divided by the product of the distances of the zeros of $U(s)/R(s)$ from the point. The sign of K is given by paragraph (v).

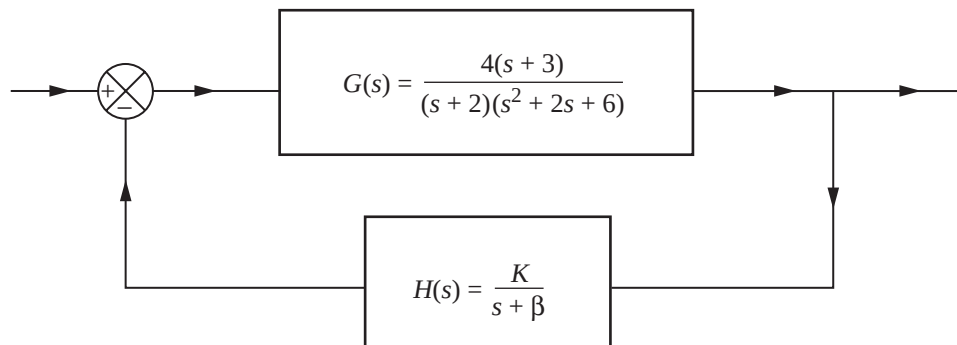
It is worth noting that for an oscillatory mode, from Item No. 67038^[1], circles centred on the origin are contours of constant undamped frequency; also straight lines through the origin are contours of constant damping and the angle such lines make with the imaginary axis is the damping angle, $\sin^{-1}\zeta$ where ζ is the relative damping.

The system will be stable for values of K for which the branches of the locus are in the left-hand half of the complex plane.

3.2 Examples of Root Locus Diagrams

3.2.1 Example 1

Construct the root locus diagram for the feedback system shown in Sketch 3.1 for K varying between 0 and $+\infty$ and $\beta = 5$.



Sketch 3.1 Feedback system of Example 1

For this system, the characteristic equation, Equation (3.2), becomes

$$\begin{aligned}
 1 + K \frac{U(s)}{R(s)} &= 1 + G(s) H(s) \\
 &= 1 + K \frac{4(s+3)}{(s+5)(s+2)(s^2+2s+6)} = 0.
 \end{aligned}
 \tag{3.6}$$

The root locus can now be constructed and is developed in three stages. Sketch 3.2a corresponds to steps (i) to (iv), Sketch 3.2b includes the additional results from steps (v) to (viii), and Sketch 3.2c shows the complete diagram. The step numbers quoted correspond to those in Section 3.1.

- (i) There are four branches of the locus since Equation (3.6) is of order four.
- (ii) The locus is symmetrical about the real axis because the coefficients of Equation (3.6) are real.
- (iii) The poles of $G(s)H(s)$ are

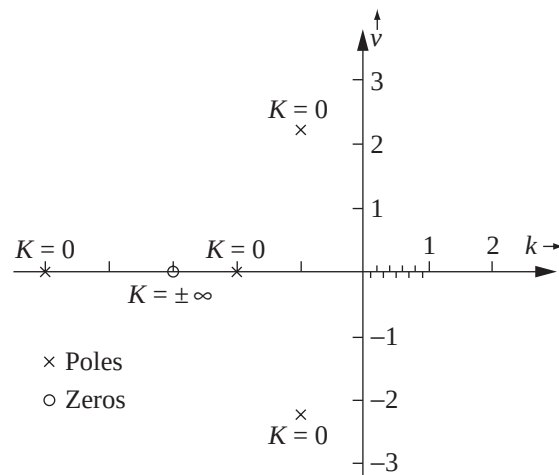
$$s = -5,$$

$$s = -2$$

$$\text{and } s = \frac{-2 \pm (4 - 4 \times 6)^{1/2}}{2} = -1 \pm i\sqrt{5}$$

and, at a pole, $K = 0$. The poles are denoted in the sketches by crosses.

- (iv) The zero of $G(s)H(s)$ is $s = -3$ and, at a zero, $K = \pm \infty$. The zero is denoted in the sketches by a circle.



Sketch 3.2a The root locus sketch corresponding to steps (i) to (iv)

- (v) The value of r , the number of poles minus the number of zeros, is 3. Hence, from Equations (3.3a) and (3.3b) with $n = 0, 1$ and 2 , the angles the asymptotes make with the real axis are

$$\frac{\pi}{3}, \quad \pi \quad \text{and} \quad \frac{5\pi}{3}, \quad \text{as } K \rightarrow \infty.$$

- (vi) From Equation (3.4) the asymptotes intersect at the point

$$\left[\frac{-5 - 2 - (1 + i\sqrt{5}) - (1 - i\sqrt{5}) + 3}{3}, 0 \right] = (-2, 0).$$

- (vii) The real poles and zero at $-5, -3$, and -2 divide the real axis into four segments of which those from $-\infty$ to -5 and -3 to -2 correspond to $K > 0$.
- (viii)