

NATURAL FREQUENCIES OF THIN-WALLED ISOTROPIC, CIRCULAR-CYLINDRICAL SHELLS

1. NOTATION

		<i>Units</i>	
		<i>SI</i>	<i>British</i>
c	$(E/\rho)^{1/2}$ for shell material*	m/s	in/s
c_T	$(G/\rho)^{1/2}$ for shell material*	m/s	in/s
E	Young's modulus for shell material	N/m ²	lbf/in ²
f	natural frequency of initially unstressed shell	Hz	Hz
f_N	natural frequency of shell subjected to an initial stress	Hz	Hz
f_R	shell ring frequency [†] ($f_R = c/2\pi R$)	Hz	Hz
G	shear modulus for shell material	N/m ²	lbf/in ²
j	torsional mode number		
K	frequency coefficient		
K_I	inextensional mode frequency coefficient		
K_T	torsional mode frequency coefficient		
L	shell length	m	in
M	modal density		
m	axial mode number		
N_x	uniform initial axial force per unit length (positive tensile)	N/m	lbf/in
N_θ	uniform initial circumferential force per unit length (positive tensile)	N/m	lbf/in
n	circumferential mode number (number of circumferential waves)		
p	pressure (internal pressure positive)	N/m ²	lbf/in ²
R	shell mean radius	m	in
t	shell thickness	m	in

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ν Poisson's ratio of shell material

ρ shell material density kg/m^3 ‡

Shell end conditions

C denotes clamped end

F denotes free end

SD SD denotes shear diaphragm end

Additional notation is given in the Appendix, Section A1.

* For most common structural metallic materials the values of c and c_T are approximately 5080 m/s (200 000 in/s) and 3150 m/s (124 000 in/s) respectively.

† See footnote in Section 5.

‡ A density value expressed in British units as pounds per cubic inch has to be divided by 386.4 before being used to calculate c and c_T . (A force of 1 lbf acting on a mass of 1 lb produces an acceleration of 386.4 in/s².) For example, if the material density is 0.097 lb/in³, the value used to calculate c and c_T should be $2.51 \times 10^{-4} \text{ lbs}^2/\text{in}^{-4}$.

2. INTRODUCTION

This Item provides graphical and tabulated data for the estimation of the lower natural frequencies of thin-walled unstiffened shells. Natural frequencies of initially unstressed shells for both flexural and torsional vibration are considered in Section 3. Flexural vibrations are defined in terms of axial and circumferential mode numbers. For each combination of these mode numbers there are three natural frequencies in which the vibrating motion is either primarily radial, axial or circumferential. A computer program, referred to as ESDUpac A8040, that may be used to evaluate the flexural modes of vibration of thin-walled, unstiffened shells is described in Item No. 80040.*

In Section 4 natural frequencies of shells subjected to uniform static loads are considered and in Section 5 shell modal density data are given. The data in Section 5 are applicable to response calculations of shells under wide band excitation, when using statistical energy analysis methods.

Section 6 provides information on the limitations of data presented and in Section 7 some guidance on the calculation of stiffened shell natural frequencies is given. Worked examples which illustrate the application of the Item are provided in Section 9.

3. NOTES ON DATA PRESENTED FOR INITIALLY UNLOADED SHELLS

3.1 General

The figures and tables provide data for calculating the lower natural frequencies of vibration of thin-walled isotropic, circular-cylindrical shells for all combinations of shear diaphragm (SD)[†], clamped (C) and free (F) end conditions.

* Free vibration of thin-walled, orthogonally stiffened, circular-cylindrical shells.

† In other technical literature the shear diaphragm condition may be referred to as "simply-supported" or "freely-supported". However these terms are not strictly applicable to shells because they do not indicate that there is zero radial and tangential displacement of the shell surface at its ends.

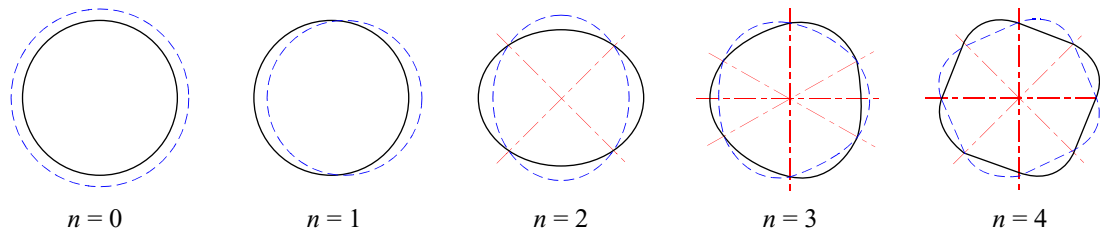
For the shear diaphragm end condition the shear diaphragm is assumed to be thin and stiff only within its own plane. Because the shear diaphragm is thin it has little stiffness in the shell axial direction and generates negligible bending moment about a tangent to the shell circumference at the shell-to-diaphragm junction; also negligible longitudinal membrane force is generated in the shell at the shell-to-diaphragm junction. The in-plane stiffness of the diaphragm ensures that there is no radial or tangential displacement of the shell surface at its ends.

The physical restraints which apply to the various end conditions considered are defined in the Appendix, Section A2.

3.2 Mode Numbers

3.2.1 Circumferential mode number (n)

The circumferential mode number is defined for all shell end conditions as the number of circumferential waves around the shell circumference ($n = 0, 1, 2, \text{etc.}$). The circumferential mode shapes for the lower values of n are shown in Sketch 3.1.



Sketch 3.1 Circumferential mode shapes

For all sets of end conditions there are $2n$ longitudinal nodal lines on which both radial and longitudinal displacements are zero. In general, shells do not have surface lines on which the radial, longitudinal and circumferential displacements are all zero.

3.2.2 Axial mode number (m)

The axial mode number, m , is defined as one plus the number of circumferential nodal circles ($m = 1, 2, 3, \text{etc.}$). On this definition the supported shell ends are not counted as nodes and the nodal circles are lines of zero radial displacement. For SD-SD shells both the radial and tangential displacements are zero on nodal lines.

In the case of the SD-SD shell the wave form is sinusoidal and the axial mode number is equal to the number of half waves in the length of the shell. For this case mode shapes may be calculated using the equations given in the Appendix, Section A3.

The axial mode shapes of the natural modes of vibration for which frequency data are presented are sketched in Table 10.1

3.2.3 Torsional mode number (j)

The torsional mode number j ($j = 1, 2, 3 \text{ etc.}$) defines the order in which torsional natural frequencies occur as frequency increases.

3.3 Natural Frequencies

3.3.1 Coupled modes

For each combination of m and n , for thin shells, there are three natural frequencies of vibration. The modes associated with each of these frequencies correspond to motion which is primarily radial, axial or circumferential. The lower natural frequencies are usually associated with a motion which is primarily radial.

Shell natural frequencies, except for inextensional and torsional modes, are obtained from the expression

$$f = K \frac{c}{R}$$

where values of K are presented in Figures 1 to 55 as detailed in Table 10.1.

Graphical data are provided for the following four combinations of end conditions:

shear diaphragm	– shear diaphragm	(SD – SD),
clamped	– clamped	(C – C),
clamped	– free	(C – F),
free	– free	(F – F).

Values of K for the SD-SD shell are presented in two forms. On Figure 1 all data ($2 \leq n < 21$) are plotted against nL/mR for a range of values of R/n^2t and on Figures 2 to 7 values of K , for fixed values of R/t , are plotted against L/mR for all $n < 21$. The presentation in Figure 1 allows easy interpolation for any R/t value but does not readily show the lowest frequency and order in which modes occur as frequency increases.

Values of K for the end condition clamped-shear diaphragm (C – SD) and free-shear diaphragm (F – SD) may be obtained from the longitudinally antisymmetric modes of the C – C and F – F cases respectively. For $m = 1$ and 2 for a C – SD shell of length L values of K are identical to those of the $m = 2$ and 4 modes respectively of the C – C shell of length $2L$. Similarly for the F – SD shell of length L values of K for modes $m = 1, 2$ and 3 are identical to those of the $m = 2, 4$ and 6 modes respectively of a F – F shell of length $2L$.

The values of K presented in this Item for the higher values of n are for the predominantly radial motion mode. An approximate “cut-off” line is shown on the figures to the right of which a natural frequency occurs in a mode, other than the predominantly radial motion mode, having a lower natural frequency than the intersected radial motion mode. This lower frequency mode will have a lower value of n than the intersected radial motion mode.

When it is necessary to interpolate for values of K at intermediate R/t ratios it is recommended that either graphical interpolation or the interpolation equation given in the Appendix, Section A4, should be used.

3.3.2 Inextensional modes for free-free shell and free-shear diaphragm shell

For inextensional modes the middle surface of the shell deforms without stretching. For this condition to apply to circular-cylindrical shells the shell generating line must remain straight during vibration.

For the case of a F – F shell there are two sets of modes of vibration for which approximate natural frequencies may be derived from inextensional theory. These modes are referred to as the Rayleigh modes ($m = 1$) and the Love modes ($m = 2$). The $m = 1$ modes of the F – SD shell may also be derived from inextensional theory. These latter modes are derived from the Love mode and values of the frequency

parameters are obtained as described in Section 3.3.1. The mode shapes for the three inextensional theory frequencies are shown in Table 10.1. The natural frequencies for these modes are given by

$$f = K_I \frac{c_t}{R^2}.$$

The values of K_I for the Rayleigh mode are given in the Table 10.2. The values of K_I for the Love mode are plotted against L/R for a range of values of n in Figure 56. It should be noted that the difference between the natural frequencies for the Rayleigh and Love modes is only significant for low values of L/R .

The $n = 0, m = 1$ and 2 modes of F – F shells and $n = 0, m = 1$ modes of F – SD shells are predominantly extensional and their natural frequencies are approximately given by

$$\begin{aligned} f &= \frac{c}{2\pi R \sqrt{1 - \nu^2}} \\ &= 0.167 \frac{c}{R} \text{ (for } \nu = 0.3 \text{)}. \end{aligned}$$

The $n = 1, m = 1$ and 2 modes of F – F shells and $n = 1, m = 1$ modes of F – SD shells are rigid body modes for which frequencies are zero.

3.3.3 Torsional mode of vibration

The natural frequencies of torsional vibration of circular-cylindrical shells are independent of the cross-sectional dimensions of the shell. The only geometric parameter to be considered in calculating torsional vibration frequencies is therefore the shell length, L . Natural frequencies of torsional vibration are given by

$$f = \frac{c_T}{L} K_T$$

where $K_T = \frac{j}{2}$ for F – F, SD – SD and C – C shells

and $K_T = \frac{2j-1}{4}$ for F – SD, and C – F shells ($j = 1, 2, 3 \text{ etc.}$).

3.4 Effect of Poisson's Ratio

In calculating frequency parameters presented in this Item a value of $\nu = 0.3$ was used.

For each value of m , for all flexural modes having n greater than the n corresponding to the lowest frequency, frequency is in most cases approximately inversely proportional to $\sqrt{1 - \nu^2}$. For flexural modes having n lower than the lowest frequency, the inverse proportionality relationship does not generally apply. However, for these latter cases, the error in the calculated frequency obtained using the value of K for $\nu = 0.3$ is, in general, less than 5 per cent for $n > 2$ in the range of ν from 0 to 0.5. The effect of variations in ν may be investigated using the computer program in ESDUpac A8040.

4. EFFECT OF INITIAL STRESS

Natural frequencies of flexural modes of vibration of shells subjected to static axial and circumferential

loads may be estimated using the relationships given here for modes in which n is not too small as discussed below. Alternatively, the effect of initial stress due to static axial or pressure loading may be investigated using the computer program in ESDUpac A8040.

For uniform axial load N_x (tension positive)

$$f_N^2 = f^2 + \frac{N_x m^2}{4\rho L^2 t},$$

for uniform circumferential load N_θ (tension positive)

$$f_N^2 = f^2 + \left(\frac{n}{2\pi R}\right)^2 \frac{N_\theta}{\rho t},$$

for combined uniform axial and circumferential loads

$$f_N^2 = f^2 + \frac{N_x m^2}{4\rho L^2 t} + \left(\frac{n}{2\pi R}\right)^2 \frac{N_\theta}{\rho t},$$

and for pressure load $p(N_x = N_\theta/2)$

$$f_N^2 = f^2 + \frac{pR}{4\rho t} \left\{ \frac{m^2}{2L^2} + \left(\frac{n}{\pi R}\right)^2 \right\}.$$

Uniform initial circumferential stress can arise from internal or external pressure or through constant velocity rotation about the axis of the shell.

The above results are approximate due in part to the modifying effects of the end conditions on the uniformity of the initial stress field in finite length shells. However for thin shells the frequency characteristics are not greatly affected by the local edge stress variations so the relationships may be used for finite length shells. The relationships for stressed shell natural frequencies should not be used for shells which are initially loaded such that a buckling condition is approached.

The case of torsional prestress is more complicated than the case of uniform axial or circumferential initial stress so that reliable simple frequency relationships cannot be given for shells under initial static torque. A suitable theory for estimating the effects of torsional prestress on natural frequencies of shells may be found in Reference 7.

Correlation of measured natural frequencies, for initially stressed shells, with those calculated using the equations given shows reasonable agreement, irrespective of shell boundary conditions, for modes with low m and n greater than that of the lowest frequency mode.

Limited measured data on initially stressed shell natural frequencies suggest that the equations given become less accurate as m increases so for $m > 2$ predicted values should be treated with caution since for these cases the calculated frequency obtained under tensile load is likely to be too high. Insufficient data are available to give guidance on the correlation between measured and calculated frequencies for compressive initial stress but it is expected that calculated frequencies would become increasingly too low as m is increased.

5. MODAL DENSITIES

The calculation of the response of shells to wide band random excitation using classical methods becomes impractical at high frequencies. The reason for this is that the natural frequencies are close together and therefore many modes participate in the overall motion. In such a situation the vibration levels may be estimated using statistical energy methods as described in Reference 9. In using these methods it is necessary to have a knowledge of the number of natural frequencies occurring in a particular frequency interval. This quantity is conveniently expressed in terms of the modal density of the structure. Modal density is defined as the number of natural frequencies per unit frequency and is frequency dependent.

In Figure 57 the parameter Mt/L is plotted against the ratio of frequency to ring frequency, f_R . In the parameter Mt/L , M is the non-dimensionalised modal density. To estimate the number of modes occurring within a frequency band of width Δf multiply the value of Mt/L , obtained from Figure 57 at the geometric band-centre frequency, by $(L/t) \times (\Delta f/f_R)$. Values of Mt/L were computed using the modified Bolotin method in Derivation 3. The ring frequency*, as used in this Item, is the reciprocal of the time for a wave to travel round the shell circumference so that $f_R = c/2\pi R$.

The theory used to derive the data presented in Figure 57 was based on SD – SD shell frequency relationships and is concerned with modes which have predominant displacements in the radial direction. The theory determines the asymptotic distributions of eigenvalues for high mode numbers and it has been shown theoretically that boundary conditions do not affect such distributions so that Figure 57 may be used for all shell boundary conditions. As frequency is increased above the ring frequency, the modal density of a circular-cylindrical shell approaches that of a flat plate having the same area as the shell.

Correlation of modal density data, obtained by counting modes in 1/3-octave frequency bands, with values predicted at the band centre frequency shows greatest scatter at the lower frequencies. This is to be expected since in practice the effect of boundary conditions is important at the lower frequencies.

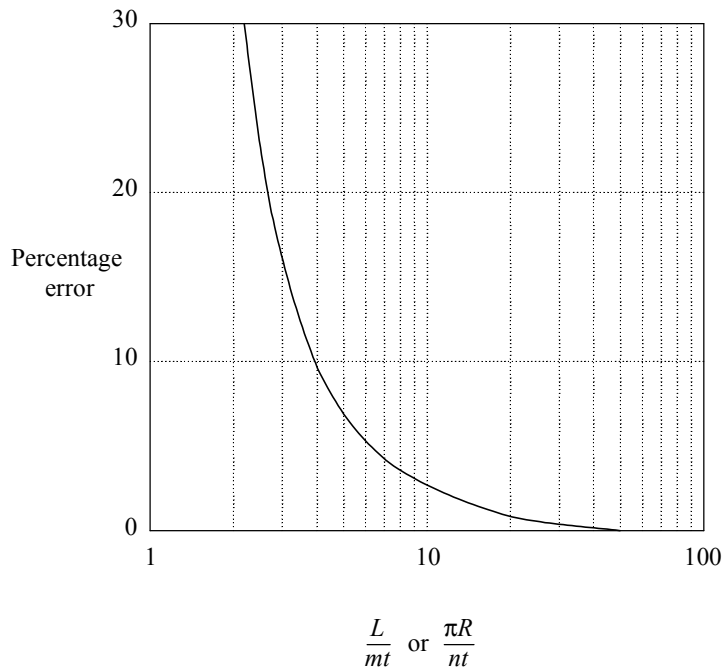
6. GENERAL LIMITATIONS ON DATA PRESENTED

Values of the natural frequency coefficients for the F – F, SD – SD and C – C shells were obtained from exact solutions of the differential equations of motion. For the C – F shell the frequency coefficients were obtained using a Rayleigh-Ritz analysis method which utilises beam mode shapes so that for this case the frequency coefficients are upper bounds.

Data provided are limited to thin shells. For most practical purposes this condition is satisfied for shells having $R/t > 20$. Also in calculating the natural frequency parameters it was assumed that the shells are initially unstressed and their vibration amplitudes are small.

In calculating the frequency parameters three-dimensional elastic theory was used for SD – SD shells and thin shell theory for other end conditions. In thin shell theory calculations the effects of shear deformation and rotary inertia were neglected. The omission is valid when the half wavelength of the mode is much larger than the shell thickness, a condition which is satisfied when $L/mt > 10$ and $\pi R/nt > 10$. Percentage errors in natural frequencies due to neglecting effects of shear deformation and rotary inertia are shown plotted against L/mt and $\pi R/nt$ in Sketch 6.1. Data shown on this Sketch are obtained by comparing SD – SD shell frequencies calculated using the three-dimensional elastic theory and thin shell theory.

* Ring frequency is defined as $f_R = c/\{2\pi R(1-\nu^2)^{1/2}\}$ in some technical papers. See Reference 10 for a discussion of the two definitions. The ring frequency should be evaluated without the $(1-\nu^2)^{1/2}$ parameter when using Figure 57.



Sketch 6.1 Errors due to neglecting effects of shear deformation and rotary inertia

Data presented on the figures which fall outside the quoted range of thin shell theory are shown as broken lines. These lines are included to allow interpolation for intermediate values of R/t .

7. NATURAL FREQUENCIES OF STIFFENED SHELLS

For the case of shells with many close-pitched, uniform ring, and/or stringer, stiffeners the lower natural frequencies may be estimated using an analysis in which it is assumed that the stiffener properties are averaged over the shell surface and account is taken of stiffener eccentricity. This “smeared” stiffener theory provides estimates of natural frequencies, for uniform closely spaced stiffeners, within close agreement with those given by the theory in which stiffeners are treated as discrete elements. In Item No. 80040 two computer programs are provided for the calculation of natural frequencies of shells having uniform closely spaced stiffeners. The first of those programs, in ESDUpac A8040, considers various boundary conditions, uses a “smeared” stiffener analysis and account is taken of stiffener eccentricity. The second program, in ESDUpac B8040, considers SD – SD shells only but uses both a “smeared” stiffener and “discrete” stiffener analysis. The latter program should be used to provide a guide to the validity of the “smeared” stiffener analysis for particular shell and stiffener combinations.

For shells having few stiffeners at arbitrary spacing it is necessary to estimate natural frequencies by a method in which the stiffeners are treated as discrete elements. An analysis of this type is described in Reference 8.

The “smeared” stiffener type of analysis breaks down for stringer-stiffened shells at high frequencies since the stiffeners tend to act as supports and the shell panels vibrate independently as cylindrical plates. Natural frequencies of rectangular single-curved plates may be estimated using Item No. 87002*. Also the “smeared” stiffener analysis is unsuitable for ring-stiffened shells if the rings are sufficiently stiff to allow the shell sections between rings to act independently as C – C shells.

* Natural frequencies of rectangular singly-curved plates.

Although adding stiffeners to a shell generally increases natural frequencies there are cases when the addition of stiffeners may result in a reduction of frequency. The situation arises when the increase in mass, due to the added stiffeners, is more significant than the increase in shell stiffness. An example where adding stiffeners to a shell may reduce natural frequencies is on a C – F shell vibrating in its cantilever mode when a single ring stiffener is attached at the free end.

8. DERIVATION AND REFERENCES

8.1 Derivation

The derivation lists selected sources that have assisted in the preparation of this Item.

1. ARNOLD, R.N.
WARBURTON, G.B. The flexural vibrations of thin cylinders. *Proc. Instn mech. Engrs*, A, Vol. 167, pp.62-80, 1953.
2. WARBURTON, G.B. Vibration of thin cylindrical shells. *J. mech. Engng Sci.*, Vol. 7, No. 4, pp. 399-407, December 1965.
3. MILLER, D.K.
HART, F.D. Modal density of thin circular cylinders. NASA CR-897, December 1967.
4. SHARMA, C.B.
JOHNS, D.J. Vibration characteristics of a clamped-ring-stiffened circular cylindrical shell. *J. Sound Vib.*, Vol. 14, No. 4, pp. 459-474, February 1971.
5. LEISSA, A.W. Vibration of shells. NASA SP-288, 1973.
6. SHARMA, C.B. Calculation of natural frequencies of fixed-free circular cylindrical shells. *J. Sound Vib.*, Vol. 35, No. 1, pp. 55-76, July 1974.

8.2 References

The references given are recommended sources of information supplementary to that in this Item.

7. KOVAL, L.R.
CRANCH, E.T. On the free vibrations of thin cylindrical shells subjected to an initial static torque. *Proceedings of the fourth U.S. National Congress of Applied Mechanics*, Vol. 1, pp.107-117, 1962.
8. EGLE, D.M.
SEWALL, J.L. An analysis of free vibration of orthogonally stiffened cylindrical shells with stiffeners treated as discrete elements. *AIAA J.*, Vol. 6, No. 3, pp. 518-526, March 1968.
9. LYON, R.H. *Statistical energy analysis of dynamical systems: theory and applications*. Massachusetts Institute of Technology Press, 1975.
10. LANGLEY, R.S. The modal density and mode count of thin cylinders and curved panels. *J. Sound Vib.*, Vol. 169, No. 1, pp.43-53, 1994.

9. EXAMPLES

- 9.1 It is required to estimate the lowest natural frequency for an initially unstressed circular cylindrical shell with shear diaphragms at either end. The shell has the following dimensions and physical properties:

$$\begin{aligned}
 R &= 240 \text{ mm}, & L &= 1.05 \text{ m}, & t &= 1.5 \text{ mm}, \\
 E &= 70\,000 \text{ MN/m}^2, & \rho &= 2710 \text{ kg/m}^3 & \nu &= 0.3.
 \end{aligned}$$

Firstly $\frac{R}{t} = \frac{240}{1.5} = 160.$

$$c = \sqrt{\frac{70\,000 \times 10^6}{2710}} = 5082 \text{ m/s.}$$

Since the lowest natural frequency occurs in a mode having an axial mode number $m = 1$,

$$\frac{L}{mR} = \frac{1.05 \times 10^3}{1 \times 240} = 4.38.$$

The end conditions for this shell are SD – SD and from Table 10.1 the frequency parameter K is found from Figures 3, 4 and 5 using the interpolation relationship given in the Appendix, Section A4.

From Figures 3, 4 and 5 the lowest natural frequency for R/t ratios 50, 100 and 500 occurs in modes having n of 3, 4 and 5 respectively. It is therefore necessary to consider each of these modes to find the lowest natural frequency at the intermediate R/t value. Values of K are tabulated as follows.

R/t	K		
	$n = 3$	$n = 4$	$n = 5$
50	0.0113	0.0153	0.0233
100	0.00908	0.00871	0.0120
500	0.00817	0.00506	0.00393
160	0.00848	0.00662	0.00792

Hence the lowest natural frequency occurs in the mode where $m = 1$ and $n = 4$ and has a value

$$f = 0.00662 \times \frac{5082}{240 \times 10^{-3}} = 140 \text{ Hz.}$$

- 9.2 If the shell of Section 9.1 has a shear diaphragm at one end and the other end clamped estimate the lowest natural frequency.

For this case the natural frequency is obtained using the lowest value of K for the antisymmetric axial mode of a C – C shell of twice the true length.

The required value of $\frac{L}{R} = \frac{1.05 \times 2}{240 \times 10^{-3}} = 8.75.$

The lowest antisymmetric axial mode frequency of a C – C shell has $m = 2$, and as shown in Table 10.1 the required value of K is found by interpolation from Figures 13, 14 and 15. From these figures, modes having $n = 3$ to 7 must be considered in order to locate the lowest natural frequency. Values of K are tabulated with the interpolated value for $R/t = 160$.

R/t	K				
	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
50	0.0138	0.0162	0.0236	0.0340	0.0463
100	0.0122	0.0102	0.0126	0.0172	0.0233
1000	0.0115	0.00703	0.00486	0.00377	0.00342
160	0.0118	0.00844	0.00876	0.0110	0.0147

Hence the lowest natural frequency occurs in the mode where $m = 1$ and $n = 4$ and has a value

$$f = 0.00844 \times \frac{5082}{240 \times 10^{-3}} = 179 \text{ Hz.}$$

- 9.3** For the shell in Section 9.1 calculate the modal density at the band centre frequency and estimate the number of modes within the 1/3-octave band having lower and upper frequencies of 1120 Hz and 1410 Hz respectively.

Firstly
$$f_R = \frac{c}{2\pi R} = \frac{5082}{2\pi \times 240 \times 10^{-3}} = 3370 \text{ Hz.}$$

The geometric band centre frequency f_c is given by

$$f_c = \sqrt{1120 \times 1410} = 1257 \text{ Hz.}$$

(When using standard tables of 1/3-octave bands the nominal centre frequency for the preferred band having the limiting frequencies in this example is 1250 Hz.)

Now
$$\frac{f_c}{f_R} = \frac{1257}{3370} = 0.373,$$

and
$$\frac{\Delta f}{f_R} = \frac{1410 - 1120}{3370} = 0.0861.$$

From Figure 57 at $\frac{f_c}{f_R} = 0.373$, $\frac{Mt}{L} = 0.484.$

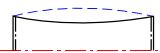
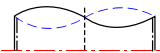
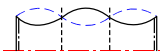
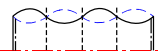
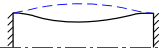
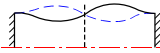
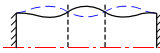
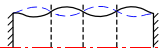
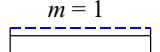
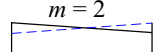
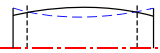
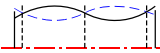
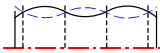
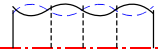
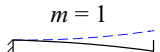
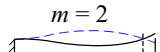
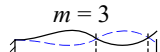
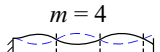
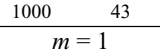
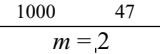
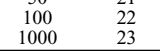
Hence
$$M = 0.484 \times \frac{1.05}{1.5 \times 10^{-3}} = 339$$

and the number of modes within the 1/3-octave band is

$$M \frac{\Delta f}{f_R} = 339 \times 0.0861 \approx 29.$$

10. TABLES

TABLE 10.1 Radial Displacements and Figure Locations for Frequency Coefficients of Initially Unstressed Shells

Shell boundary condition	Mode shapes and figure numbers																																											
SD SD	$m = 1$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td>All</td><td>1</td></tr><tr><td>20</td><td>2</td></tr><tr><td>50</td><td>3</td></tr><tr><td>100</td><td>4</td></tr><tr><td>500</td><td>5</td></tr><tr><td>1000</td><td>6</td></tr><tr><td>5000</td><td>7</td></tr></table>	R/t	Figure No.	All	1	20	2	50	3	100	4	500	5	1000	6	5000	7	$m = 2$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td colspan="2">as $m = 1$</td></tr></table>	R/t	Figure No.	as $m = 1$		$m = 3$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td colspan="2">as $m = 1$</td></tr></table>	R/t	Figure No.	as $m = 1$		$m = 4$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td colspan="2">as $m = 1$</td></tr></table>	R/t	Figure No.	as $m = 1$													
	R/t	Figure No.																																										
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C C	$m = 1$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td>20</td><td>8</td></tr><tr><td>50</td><td>9</td></tr><tr><td>100</td><td>10</td></tr><tr><td>1000</td><td>11</td></tr></table>	R/t	Figure No.	20	8	50	9	100	10	1000	11	$m = 2$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td>20</td><td>12</td></tr><tr><td>50</td><td>13</td></tr><tr><td>100</td><td>14</td></tr><tr><td>1000</td><td>15</td></tr></table>	R/t	Figure No.	20	12	50	13	100	14	1000	15	$m = 3$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td>20</td><td>16</td></tr><tr><td>50</td><td>17</td></tr><tr><td>100</td><td>18</td></tr><tr><td>1000</td><td>19</td></tr></table>	R/t	Figure No.	20	16	50	17	100	18	1000	19	$m = 4$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td>20</td><td>20</td></tr><tr><td>50</td><td>21</td></tr><tr><td>100</td><td>22</td></tr><tr><td>1000</td><td>23</td></tr></table>	R/t	Figure No.	20	20	50	21	100	22	1000	23
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F F	$m = 1$ See Table II for values of K_f For $n = 0$, $K = 0.1668$	$m = 2$ All values of R/t K_f given in Figure 56 For $n = 0$, $K = 0.1668$	Inextensional theory modes $m = 1$, Rayleigh mode $m = 2$, Love mode																																									
	$m = 3$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td>20</td><td>24</td></tr><tr><td>50</td><td>25</td></tr><tr><td>100</td><td>26</td></tr><tr><td>1000</td><td>27</td></tr></table>	R/t	Figure No.	20	24	50	25	100	26	1000	27	$m = 4$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td>20</td><td>28</td></tr><tr><td>50</td><td>29</td></tr><tr><td>100</td><td>30</td></tr><tr><td>1000</td><td>31</td></tr></table>	R/t	Figure No.	20	28	50	29	100	30	1000	31	$m = 5$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td>20</td><td>32</td></tr><tr><td>50</td><td>33</td></tr><tr><td>100</td><td>34</td></tr><tr><td>1000</td><td>35</td></tr></table>	R/t	Figure No.	20	32	50	33	100	34	1000	35	$m = 6$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td>20</td><td>36</td></tr><tr><td>50</td><td>37</td></tr><tr><td>100</td><td>38</td></tr><tr><td>1000</td><td>39</td></tr></table>	R/t	Figure No.	20	36	50	37	100	38	1000	39
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C F	$m = 1$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td>20</td><td>40</td></tr><tr><td>50</td><td>41</td></tr><tr><td>100</td><td>42</td></tr><tr><td>1000</td><td>43</td></tr></table>	R/t	Figure No.	20	40	50	41	100	42	1000	43	$m = 2$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td>20</td><td>44</td></tr><tr><td>50</td><td>45</td></tr><tr><td>100</td><td>46</td></tr><tr><td>1000</td><td>47</td></tr></table>	R/t	Figure No.	20	44	50	45	100	46	1000	47	$m = 3$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td>20</td><td>48</td></tr><tr><td>50</td><td>49</td></tr><tr><td>100</td><td>50</td></tr><tr><td>1000</td><td>51</td></tr></table>	R/t	Figure No.	20	48	50	49	100	50	1000	51	$m = 4$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td>20</td><td>52</td></tr><tr><td>50</td><td>53</td></tr><tr><td>100</td><td>54</td></tr><tr><td>1000</td><td>55</td></tr></table>	R/t	Figure No.	20	52	50	53	100	54	1000	55
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SD F	$m = 1$ For all values of R/t for $n \geq 2$, K_f given on Figure 56 For $n = 0$, $K = 0.1668$	$m = 2$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td>20</td><td>28</td></tr><tr><td>50</td><td>29</td></tr><tr><td>100</td><td>30</td></tr><tr><td>1000</td><td>31</td></tr></table>	R/t	Figure No.	20	28	50	29	100	30	1000	31	$m = 3$ <table><tr><th>R/t</th><th>Figure No.</th></tr><tr><td>20</td><td>36</td></tr><tr><td>50</td><td>37</td></tr><tr><td>100</td><td>38</td></tr><tr><td>1000</td><td>39</td></tr></table>	R/t	Figure No.	20	36	50	37	100	38	1000	39																					
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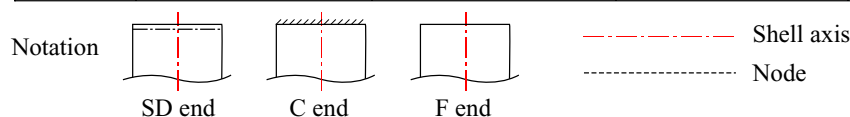


TABLE 10.2 Rayleigh Inextensional Mode Frequency Coefficients

n	K_I	n	K_I	n	K_I
		7	2.289	14	9.368
1	0	8	3.011	15	10.76
2	0.1292	9	3.829	16	12.26
3	0.3655	10	4.744	17	13.85
4	0.7009	11	5.756	18	15.53
5	1.133	12	6.863	19	17.32
6	1.663	13	8.068	20	19.19

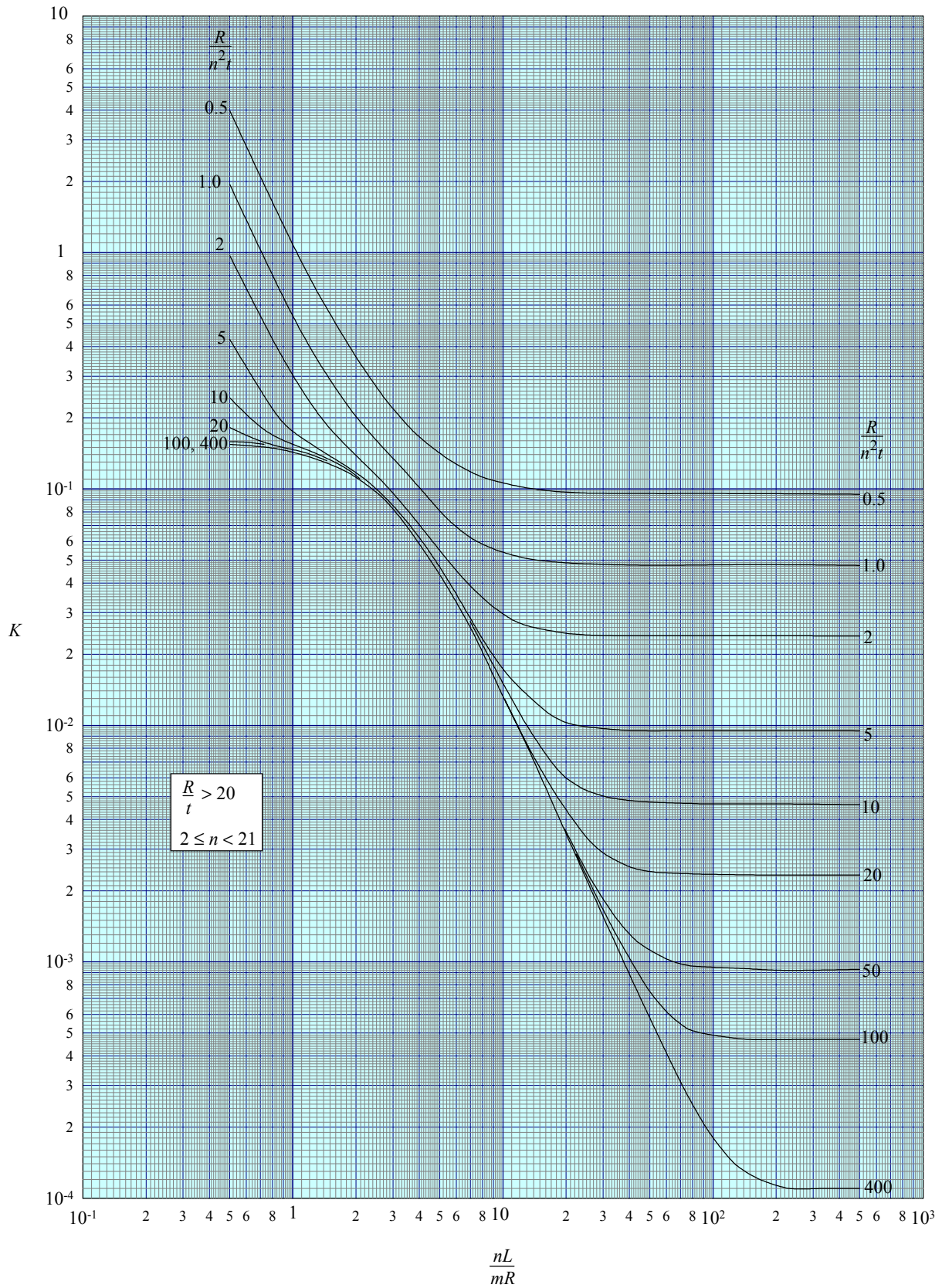


FIGURE 1 SD – SD

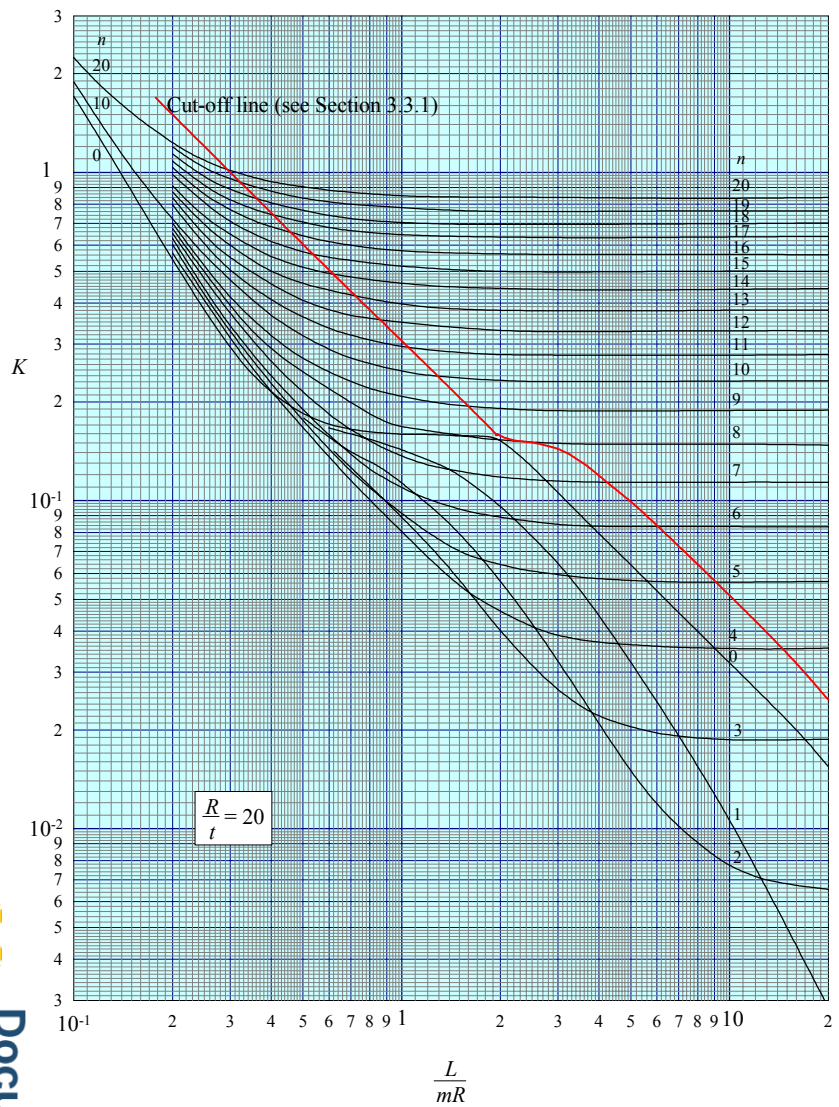


FIGURE 2 SD – SD

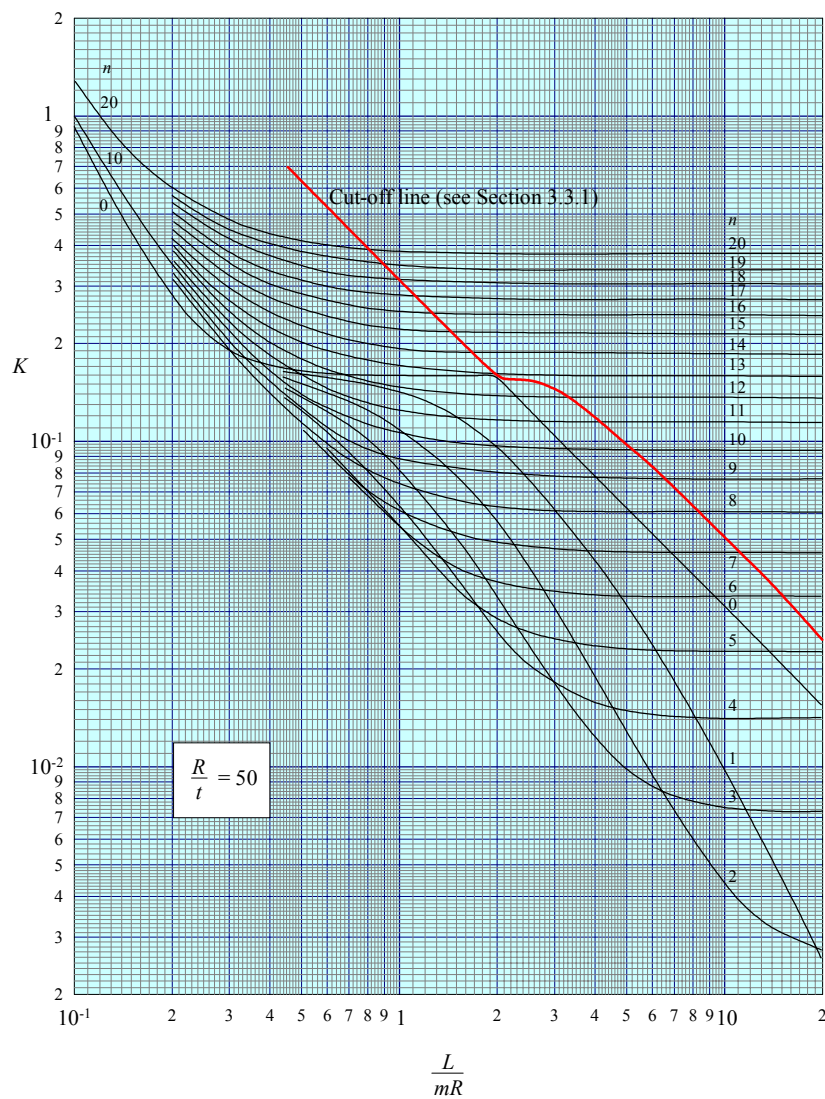


FIGURE 3 SD – SD