

BUCKLING OF RECTANGULAR SPECIALLY ORTHOTROPIC PLATES

1. NOTATION

a, b	plate length in x - and y -directions, respectively	m	in
A	in-plane stiffness sub-matrix for plate (see Section 2)	N/m	lbf/in
B	coupled in-plane and flexural stiffness sub-matrix for plate (see Section 2)	N	lbf
C	coefficient dependent upon plate edge conditions (see Section 3)		
D	flexural stiffness sub-matrix for plate (see Section 2)	N m	lbf in
	$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ & D_{22} & 0 \\ \text{symmetric} & & D_{33} \end{bmatrix}$		
D_o	$D_{12} + 2D_{33}$	N m	lbf in
H_{11}, H_{22}	transverse (through-the-thickness) shear stiffnesses with respect to xz - and yz -planes, respectively	N m	lbf/in
E	modulus of elasticity of isotropic plate material	N/m ²	lbf/in ²
K	$\begin{bmatrix} -\partial^2 w / \partial x^2 \\ -\partial^2 w / \partial y^2 \\ -2\partial^2 w / \partial x \partial y \end{bmatrix}$ column sub-matrix of curvatures of plate	m ⁻¹	in ⁻¹
K_o	elastic buckling coefficient for plates under uniaxial or biaxial loading defined by		
	$N_{xb} = \frac{K_o (D_{11} D_{22})^{1/2}}{b^2} + \frac{C \pi^2 D_o}{b^2}$		
M	$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix}$ column sub-matrix of bending moment intensities acting in xz - and yz -planes and twisting moment intensity acting about x - and y -axes, respectively	N m/m	lbf in/in
M	elastic restraint stiffness per unit length against rotation of edges $y = 0$ and $y = b$	(N m/m)/rad	(lbf in/in)/rad

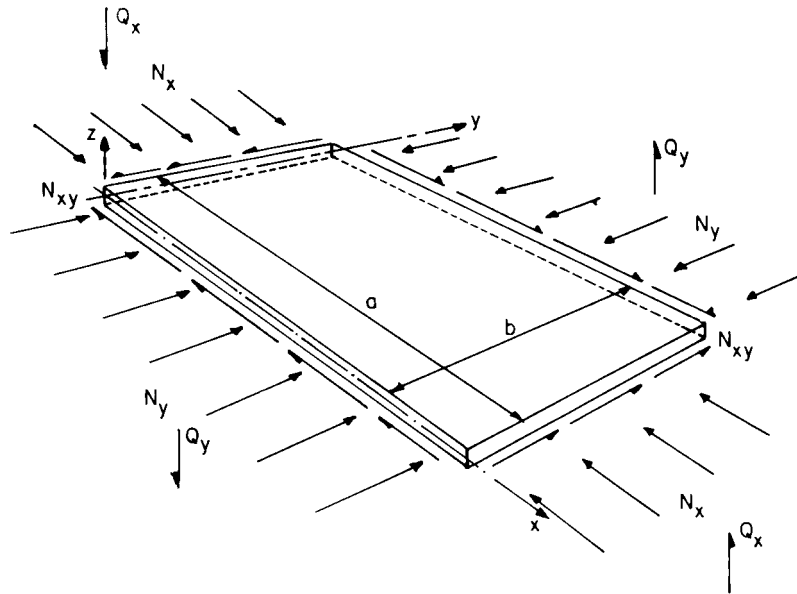
m, n	number of half-waves into which plate buckles in x - and y -directions, respectively		
$\mathbf{N}$	$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix}$ column sub-matrix of direct load intensities and shear load intensity acting in neutral plane of plate, referred to (x, y) coordinates (see Section 2)	N/m	lbf/in
N_{xb}, N_{yb}	values of N_x and N_y , respectively, at which plate buckles	N/m	lbf/in
N_{xyb}	value of N_{xy} at which plate buckles	N/m	lbf/in
Q_x, Q_y	transverse (through-the-thickness) shear forces per unit length acting in xz - and yz -planes, respectively	N/m	lbf/in
t	thickness of plate	m	in
x, y, z	plate and material coordinate system (see Sketch 1.1 and Section 2)		
γ_{xz}, γ_{yz}	through-the-thickness shear strains in xz - and yz -directions, respectively (see Section 2)		
ϵ	$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{bmatrix}$ column sub-matrix of direct strains and shear strain acting in neutral plane of plate referred to (x, y) coordinates		
λ	half-wavelength of buckle measured in x -direction	m	in
μ	coefficient of rotational restraint acting along sides $y = 0, y = b$ defined by $\mu = Mb/D_{22}$	rad ⁻¹	rad ⁻¹
ν	Poisson's ratio for isotropic plate material		

Both SI and British Units are quoted but any coherent system of units may be used.

1.1 Sign Convention

For convenience of use with the graphs the following sign convention* is used. Compressive loads and strains are positive, tensile loads and strains are negative. Shear loads and strains are assumed positive.

* Other Data Items that deal with buckling and that have an associated computer program use the following, different, sign conventions; tensile loads and strains positive, compressive loads and strains negative and shear load and strains positive or negative as required.



Sketch 1.1

2. DATA PROVIDED

This Item gives the elastic buckling loads of thin flat rectangular homogeneous specially orthotropic plates in which the through-the-thickness shear deformations have negligible effect on the buckling loads. The curves are based upon an elastic small deflections theory in which the plate is assumed to be made of a homogeneous orthotropic material whose principal axes of orthotropy are aligned with the edges of the plate (axes, x, y). Table 3.1 lists the combinations of loadings and edge support conditions dealt with and identifies the related figures. Section 4 provides a criterion for assessing whether or not the effect of through-the-thickness shear stiffnesses are negligible.

2.1 Plate Stiffness Relationships

For a homogeneous orthotropic plate the following in-plane and flexural stiffness matrix applies.

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon} \\ \mathbf{K} \end{bmatrix} \quad (2.1)$$

For the case of special orthotropy where the principal axes of orthotropy are aligned with the plate edges Equation (2.1) takes the following form

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ & A_{22} & 0 & 0 & 0 & 0 \\ & & A_{33} & 0 & 0 & 0 \\ \text{---} & \text{---} & \text{---} & D_{11} & D_{12} & 0 \\ & & & D_{22} & 0 & \\ \text{Symmetric} & & & & D_{33} & \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \\ -\partial^2 w / \partial x^2 \\ -\partial^2 w / \partial y^2 \\ -2\partial^2 w / \partial x \partial y \end{bmatrix}. \quad (2.2)$$

Thus, specially orthotropic plates satisfy the following three conditions.

- (i) The in-plane stiffness sub-matrix **A** is orthotropic with direct and shear stiffnesses uncoupled so that A_{13} and A_{23} are zero.
- (ii) The flexural stiffness sub-matrix **D** is orthotropic with the bending and twisting terms uncoupled so that D_{13} and D_{23} are zero.
- (iii) The in-plane and flexural stiffness sub-matrices are uncoupled, that is **B** = 0.

In addition to the conditions of special orthotropy the following through-the-thickness shear stiffness matrix is taken to apply.

$$\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix} \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}. \quad (2.3)$$

Thus, the shear stiffnesses in the xz - and yz -planes are assumed to be uncoupled, that is, $H_{12} = H_{21} = 0$.

3. PRESENTATION OF DATA

Table 3.1 lists all the load and edge support conditions for which curves are provided in this Item. The data for plates loaded either uniaxially or biaxially are presented in terms of the buckling coefficient K_o and a coefficient C , the latter being only dependent upon plate edge conditions. To obtain the buckling load the following equation must be evaluated.

$$N_{xb} = \frac{K_o (D_{11} D_{22})^{1/2}}{b^2} + \frac{C \pi^2 D_o}{b^2}. \quad (3.1)$$

TABLE 3.1

Plate loading	Conditions at plate edges	Figure number(s)
Uniaxial	All sides simply-supported All sides clamped One pair of opposite sides simply-supported, the other pair clamped against rotation	1 1 1
Biaxial	All sides simply-supported One pair of opposite sides simply-supported, the other pair clamped Long plates with the long sides simply-supported	2 3 4a and 4b
Shear	All sides simply-supported Long plates with the long sides normally supported and elastically restrained against rotation	5 6a and 6b
Shear plus uniaxial	All sides simply-supported	7 to 12

The exact value of C for plates with all edges simply-supported is 2.0. For plates with other edge conditions the value of C is approximate; however, errors in N_{xb} resulting from errors in C do not exceed 5 per cent and are, in general, unlikely to exceed 2 per cent. Figures 4a and 4b present the value of K_o for long plates with simply-supported sides under biaxial loading. They therefore provide the asymptotes to the curves of Figures 2 and 3 and the asymptotes to the lower two curves of Figure 1.

In all the loading cases that include shear the expression on the ordinate contains the required buckling load. In order to provide for the full range of values of D for long plates loaded in shear Figures 6a and 6b are presented.

For isotropic plates

$$D_{11} = D_{22} = Et^3/(12(1-\nu^2)), D_{12} = \nu D_{11} \text{ and } D_{33} = (1-\nu)D_{11}/2.$$

Thus, for isotropic plates the parameters used with the figures reduce to

$$D_{22}/D_{11} = D_o/(D_{11}D_{22})^{1/2} = 1 \text{ and } (D_{11}D_{22})^{1/2} = (D_{11}D_{22}^3)^{1/4} = D_{11}.$$

Other data for isotropic plates may be located by means of the ESDU Index.

Data Item No. 81047* provides a computer program that is capable of calculating the buckling load for the majority of loading conditions and edge conditions covered by the curves of this Item and for many more not covered by this Item. It also takes account of the through-the-thickness shear stiffnesses (see Section 4). By selecting high values of H_{11} and H_{22} the program may be used to calculate values that will reproduce the curves of this Item.

* Data Item No. 81047 "Buckling of flat rectangular plates (isotropic, orthotropic and laminated plates and sandwich panels)".

4. INFLUENCE OF THROUGH-THE-THICKNESS SHEAR STIFFNESS ON BUCKLING

Expression (4.1) gives an indication of the validity of the assumption that the through-the-thickness shear stiffnesses are so large that shear deformations do not significantly affect the buckling loads. The buckling loads in this expression are determined using the curves of this Item. Where a combination of loads causes buckling those loads should be used in Expression (4.1) (see Example 2, Section 6.2).

$$\max \{ N_{xb}/H_{11}, N_{yb}/H_{22}, N_{xyb}/\sqrt{H_{11}H_{22}} \} < 0.1. \quad (4.1)$$

If any of the above ratios approaches 0.1 then the through-the-thickness shear stiffnesses are such that the buckling loads may be significantly below the values given by this Item. In such cases the reduction in buckling load can be ascertained by comparing buckling loads given by the curves of this Item with those computed by the program of Data Item No. 81047. The example in Section 6.2 of this Item carries out such a comparison.

For simply-supported plates with N_y fixed the equations of Appendix A may also be used to calculate N_{xb} when H_{11} and H_{22} are such that their influence on N_{xb} is significant.

5. DERIVATION

This section lists selected sources that have assisted in the preparation of this Item.

1. SMITH, R.C.T. The buckling of flat plywood plates in compression. Australian Council for Aeronautics, Report ACA-12, December 1944.
2. WITTRICK, W.H. Correlation between some stability problems for orthotropic and isotropic plates under bi-axial and uni-axial direct stress. *Aeronaut. Q.*, Vol. IV, pp. 83-92, August 1952.
3. BRUCE, J.
ROBINSON, J.R. The buckling and bending of orthotropic sandwich panels in compression and shear. Unpublished work at Handley Page Ltd, 1956.
4. PETERSON, J.P.
CARD, M.F. Investigation of the buckling strength of corrugated webs in shear. NASA tech. Note D-424, June 1960.

6. EXAMPLES

6.1 Example 1

It is required to determine the buckling load of a simply-supported plate subjected to a shear load, N_{xy} , and a compressive uniaxial load, N_x , in ratio 1.0 to 0.65. The plate dimensions are $a = 225$ mm and $b = 173$ mm. The flexural stiffness sub-matrix is

$$\mathbf{D} = \begin{bmatrix} 7.64 & 1.29 & 0 \\ 1.29 & 0.645 & 0 \\ 0 & 0 & 1.42 \end{bmatrix} \text{ N m}.$$

From Table 3.1 it can be seen that the relevant figures are 7 to 12. For use of these figures the following non-dimensional parameters are required. Note that the calculation is carried out in newtons and metres throughout.

$$\frac{a}{b} \left\{ \frac{D_{22}}{D_{11}} \right\}^{1/4} = \frac{0.225}{0.173} \left\{ \frac{0.645}{7.64} \right\}^{1/4} = 0.701,$$

$$\frac{D_o}{(D_{11}D_{22})^{1/2}} = \frac{1.29 + 2 \times 1.42}{(7.64 \times 0.645)^{1/2}} = 1.86$$

and
$$\frac{a}{b} \frac{N_{xyb}}{N_{xb}} = \frac{0.225}{0.173} \times \frac{1.0}{0.65} = 2.0.$$

Thus, using Figure 8, and interpolating for the above value of $D_o/(D_{11}D_{22})^{1/2}$, gives

$$\frac{N_{xyb}ab}{(D_{11}D_{22})^{1/2}} = 81.5,$$

and
$$N_{xyb} = \frac{81.5(7.64 \times 0.645)^{1/2}}{(225 \times 10^{-3} \times 173 \times 10^{-3})} = 4650 \text{ N/m}$$

$$N_{xb} = 4.65 \times 10^3 \times 0.65 = 3020 \text{ N/m}.$$

6.2 Example 2

It is required to examine the influence of the through-the-thickness shear stiffnesses on the buckling of the plate of Example 1, see Section 6.1. The plate as initially designed has stiffnesses $H_{11} = 8000 \text{ N/m}$ and $H_{22} = 2000 \text{ N/m}$.

From the given values of H_{11} and H_{22} and the buckling loads calculated in Example 1, Equation (4.1) can be evaluated as follows.

$$\frac{N_{xb}}{H_{11}} = \frac{3020}{8000} = 0.378 > 0.1$$

and
$$\frac{N_{xyb}}{\sqrt{H_{11}H_{22}}} = \frac{4650}{\sqrt{8000 \times 2000}} = 1.16 > 0.1.$$

Both of the above values exceed 0.1. The through-the-thickness shear stiffnesses will therefore reduce the buckling loads to values significantly below those calculated in Example 1.

To account for the effect of through-the-thickness shear stiffnesses it is necessary to use the program of Data Item No. 81047. Working in units of newtons and metres, and noting that the sign convention of that Data Item requires the compressive loadings to be negative, the program was run within ESDUview. Section 6.2.1 presents the results. This shows that the $N_{xyb} = 2400 \text{ N/m}$ and $N_{xb} = 1560 \text{ N/m}$ (compressive).

As predicted the above computed values are much lower than the values calculated in Example 1. The effect of varying the through-the-thickness shear stiffnesses may be examined by running the program of Data Item No. 81047 for a range of values of H_{11} and H_{22} . Sketch 6.1 presents the results of several runs and shows the reduction in N_{xyb} with reducing values of H_{11} when H_{22} is taken to be $H_{11}/4$. Note that the right

hand end of the curve is asymptotic to the value of 4650 N/m calculated in Example 1. Also shown on Sketch 6.1 is the value of H_{11} corresponding to $N_{xyb}/\sqrt{H_{11}}H_{22} = 0.1$. Since $H_{22} = H_{11}/4$ this is given by

$$\frac{4650}{\sqrt{H_{11} \times H_{11}/4}} = 0.1$$

or $H_{11} = 93000 \text{ N/m}$.

6.2.1 Results File Obtained Using ESDUpac A8147 to Illustrate Effect of Through-the-thickness Shear

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ESDU International Plc.
Program A8147V40 (Previously ESDS01 and 1031B)
ESDUpac Number: A8147 (Previously ESDS01 and 1031B)
ESDUpac Title: BUCKLING OF FLAT RECTANGULAR ORTHOTROPIC PLATES
                (ISOTROPIC, ORTHOTROPIC AND LAMINATED COMPOSITE
                PLATES AND SANDWICH PANELS)STIFFNESSES

Data Item Number: 81047
Data Item Title: BUCKLING OF FLAT RECTANGULAR ORTHOTROPIC PLATES
                (ISOTROPIC, ORTHOTROPIC AND LAMINATED COMPOSITE
                PLATES AND SANDWICH PANELS)

ESDUpac Version: 4.0 DECEMBER 1994
(See Data Item for full input/output specification and interpretation.)
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Data Item 80023
Example 6.2
Investigation of the influence of through-the-thickness shear stiffnesses
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Declaration of units used for all dimensional input.
Units Length metres
      Force newtons
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ORTHOTROPIC PLATE PROPERTIES (Plate code 2).
D FLEXURAL STIFFNESS MATRIX
      .764E+01      .129E+01      .000E+00
      .129E+01      .645E+00      .000E+00
      .000E+00      .000E+00      .142E+01
Input stiffnesses are for a specially orthotropic plate.
Plate through-the-thickness shear stiffnesses
H11 = .800E+04  H22 = .200E+04
Non-dimensionalising parameter, see Section 2.4 of Data Item.
Pe = .851E+03
PLATE GEOMETRY AND BOUNDARY CONDITIONS
Plate dimension in x-direction      = .225E+00
Plate dimension in y-direction      = .173E+00
Plate edge supports: condition input code = 1
      Edges parallel to x simply-supported.
      Edges parallel to y simply-supported.
SETTINGS FOR ROOT SOLVING TECHNIQUE (ALL FOLLOWING LOAD CASES)
Error in root is reciprocal of 50000 = .200E-04
Limit set on number of iterations = 200
-----
BUCKLING LOADS
-----
UNITS DECLARED AT INPUT:
Units Length metres
      Force newtons
```