

INTRODUCTION TO THE USE OF LINEAR ELASTIC FRACTURE MECHANICS IN ESTIMATING FATIGUE CRACK GROWTH RATES AND RESIDUAL STRENGTH OF COMPONENTS

1. INTRODUCTION

This Item introduces the concepts of linear elastic fracture mechanics (LEFM) and shows how these concepts may be used for analysing the behaviour of cracked structures under static or fatigue loadings. The concepts provide an analytical method based upon the stress intensity factor, which characterises the stress distribution in the vicinity of the crack tip, and are valid in design applications provided that gross yielding does not occur. Linear elastic fracture mechanics can be used to describe ultimate static failure of low toughness high strength materials used in aerospace and other specialised applications. It should not be used to describe static failure of high toughness low strength materials (for example, materials used in the construction of pressure vessels). Under fatigue loading, crack growth rates can be correlated by the stress intensity factor for a wide range of materials, because even in the more ductile materials the amount of plastic flow that can occur under fatigue loading is restricted. It should not be used in cases where the fatigue cycles involve extensive plastic deformation, often referred to as “low cycle” fatigue.

This Item describes the basic equations of the linear elastic fracture mechanics analysis and the limitations of usage. Many of the data required in the application of these methods are provided in ESDU Data Items. They may be located via the current ESDU Index.

2. NOTATION

a	crack half-length for a crack free to extend at both ends, or crack length for a crack free to extend at one end only (see Section 3.1)	m	in
a_c	critical crack length at failure	m	in
a_0	initial crack length	m	in
a_1	effective crack length	m	in
$2b$	characteristic dimension of component width	m	in
C	coefficient in simple crack growth rate law defined by $da/dN = C(\Delta K)^n$	$m(m^{3/2}/N)^n$	$m(m^{3/2}/lbf)^n$
E	modulus of elasticity	N/m^2	lbf/in^2
G	release rate of potential energy per unit area	N/m	lbf/in
H	effective modulus of elasticity (see Section 5.1.2)	N/m^2	lbf/in^2
K	stress intensity factor (Mode I)	$N/m^{3/2}$	$lbf/in^{3/2}$
K_c	critical value of stress intensity factor (also plane stress fracture toughness)	$N/m^{3/2}$	$lbf/in^{3/2}$

K_{cl}	stress intensity factor value at crack closure	$\text{N/m}^{3/2}$	$\text{lbf/in}^{3/2}$
K_p	stress intensity factor corrected for plasticity	$\text{N/m}^{3/2}$	$\text{lbf/in}^{3/2}$
K_R	crack growth resistance	$\text{N/m}^{3/2}$	$\text{lbf/in}^{3/2}$
K_0	reference value of K (see Section 3)	$\text{N/m}^{3/2}$	$\text{lbf/in}^{3/2}$
K_{Ic}	plane strain fracture toughness in Mode I	$\text{N/m}^{3/2}$	$\text{lbf/in}^{3/2}$
ΔK	stress intensity factor range	$\text{N/m}^{3/2}$	$\text{lbf/in}^{3/2}$
ΔK_{eff}	effective stress intensity factor range defined as $K_{max} - K_{cl}$	$\text{N/m}^{3/2}$	$\text{lbf/in}^{3/2}$
ΔK_p	stress intensity factor range corrected for plasticity	$\text{N/m}^{3/2}$	$\text{lbf/in}^{3/2}$
ΔK_{TH}	threshold value of ΔK	$\text{N/m}^{3/2}$	$\text{lbf/in}^{3/2}$
M	bending moment per unit thickness	$(\text{N m})/\text{m}$	$(\text{lbf in})/\text{in}$
N	number of load cycles		
n	characteristic of slope of crack growth rate curve		
P	point load per unit thickness	N/m	lbf/in
R	stress ratio, S_{min}/S_{max}		
r	polar co-ordinate of stress in component	m	in
r'	hole radius in infinite plate	m	in
r_p	plasticity correction term	m	in
S	nominal gross section stress in crack-free part of component	N/m^2	lbf/in^2
S_c	critical stress at failure	N/m^2	lbf/in^2
t	thickness of component in region of crack tip	m	in
a	geometrical factor in stress intensity factor		
a_1	geometrical factor in stress intensity factor corrected for plasticity		
σ_y	yield (or 0.2 per cent proof) stress of material	N/m^2	lbf/in^2
σ_r	stress along axis of polar co-ordinate r	N/m^2	lbf/in^2
$\sigma_{r\theta}$	shear stress in $r\theta$ plane	N/m^2	lbf/in^2

σ_θ	stress perpendicular to polar co-ordinate r	N/m ²	lbf/in ²
θ	angle of polar co-ordinate r from x-axis	rad	rad
ν	Poisson's ratio		

Suffixes

I, II and III	indicate crack opening modes (see Sketch 3.1)
<i>max</i> , <i>min</i>	indicate maximum and minimum values of stress or stress intensity factor in loading cycle
<i>f</i> , <i>i</i>	indicate final and initial crack length (see Equation (4.10))

SI and British units are shown but any coherent system of units may be used.

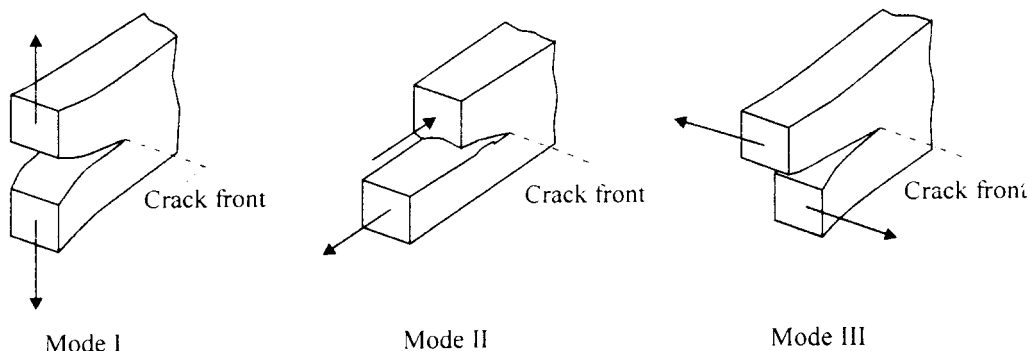
3. ANALYSIS OF CRACKED BODIES USING LINEAR ELASTIC FRACTURE MECHANICS

In analysing the stresses close to a crack tip, the concept of using elastic stress concentration factors breaks down. This is because the crack tip radius is assumed to approach zero and hence the stress concentration factor tends to approach infinity. The linear elastic fracture mechanics approach avoids this difficulty by identifying certain characteristics of the stress field that surrounds the crack tip and analysing them, rather than attempting analysis in terms of the 'infinite' stress at the crack tip. Allowances for a limited amount of plastic flow at the crack tip may be made (see Section 3.1.1).

Cracks grow in a variety of ways and the stress fields that develop around the crack tips may be divided into three basic modes. The modes are listed here and illustrated in Sketch 3.1. The Sketch shows small sections, in the vicinity of the crack, of much larger components.

- Mode I Tension, normal to the faces of the crack (opening mode)
- Mode II Shear, normal to the crack front in the plane of the crack (edge sliding mode).
- Mode III Shear, parallel to the crack front (tearing mode).

Mode I is the mode of crack opening that occurs most commonly and this Item is concerned with the analysis of the Mode I opening only. Data obtained under one particular mode cannot be applied to any other mode.



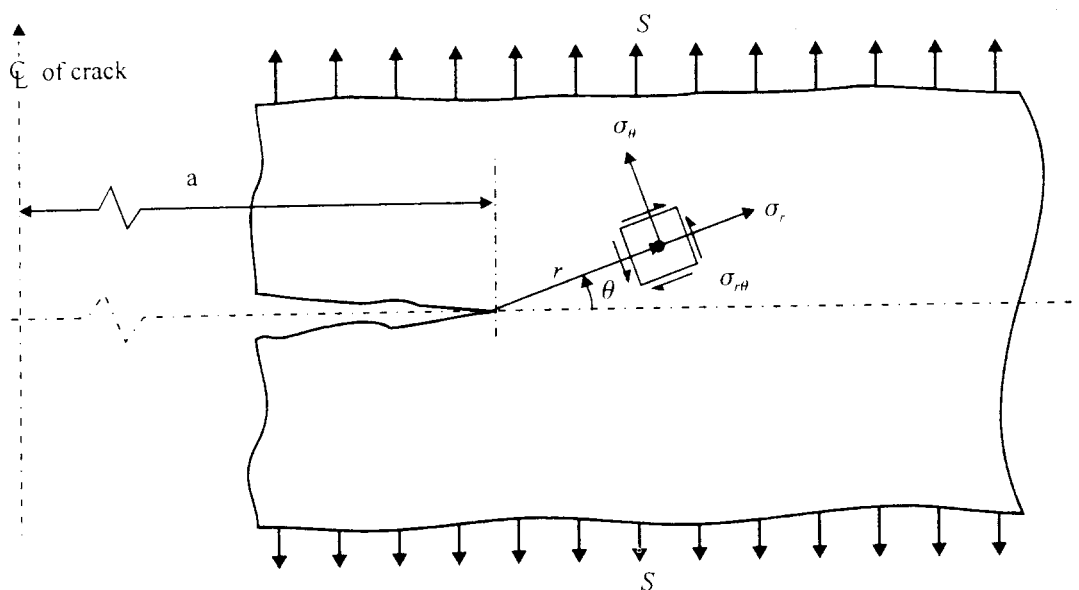
Sketch 3.1 Illustrating the three basic modes of crack growth

A two-dimensional loading model is used to describe the stress field but this model may be used to approximate to many three-dimensional loading cases that are met in practice. Assuming the material behaves in a linear elastic manner, the stresses σ_r , σ_θ , and $\sigma_{r\theta}$, developed in an element in the immediate neighbourhood ($r \ll a$) of the tip of a crack in any loaded body, can be expressed in terms of the polar co-ordinates r , θ (see Sketch 3.2), in the form (see References 3 and 10)

$$\sigma_r, \sigma_\theta, \sigma_{r\theta} = \frac{K}{(2\pi r)^{1/2}} f(\theta) \text{ for } r \ll a, \quad (3.1)$$

where $f(\theta)$ is a different function for each of the three stresses. The terms $(2\pi r)^{1/2}$ and $f(\theta)$ map the stress field in the vicinity of the crack tip and they are the same for all cracks under any external loading system that causes crack opening of the same mode. They are linked by a scale factor, the so-called crack-tip stress-field-intensity factor, denoted K ; the similarity of the mapping terms and the use of the scale factor are the essence of linear elastic fracture mechanics. The scale factor, K , is a function of the nature and magnitude of the applied stress levels, the crack size and other geometrical features, and is commonly referred to as the stress intensity factor.*

The form of Equation (3.1) is applied to all the modes of crack opening.



Sketch 3.2 Illustrating co-ordinate system and notation used in crack tip analysis

The geometry of a structure and the nature of applied loads determine the intensity of the stress field in the vicinity of the crack tip and hence the value of K , but do not affect the form of Equation (3.1). **The stress intensity factor, K , is a combination of various factors as described below and, where K is the same for different cracks even though the applied stress levels and geometries may be different, the stress fields very close to the crack tips will be identical.** Experimental data have shown that crack growth and residual strength data can be correlated better with K than other parameters.

* In some solutions found in the literature the stress intensity factor is denoted by k where $k = K/(\pi)^{1/2}$ (for example see References 14 and 18).

The value of K depends upon the nominal applied stress, which is usually taken as the nominal gross section stress, S , if the component were crack free, the crack length, a , and a term α that accounts for the effect of different configurations of the crack and its surrounding boundaries. It is usually expressed by*

$$K = S(\pi a)^{1/2} \alpha. \quad (3.2)$$

The term α is non-dimensional and is usually expressed as a ratio of crack length to any convenient local characteristic dimension in the plane of the component. Methods for finding solutions for α are described in Section 3.1. For the simple case of a crack in an infinite sheet or plate subjected to a uniform stress, S , remote from the crack, $\alpha = 1.0$ and K becomes $S(\pi a)^{1/2}$.

In some cases K may have an alternative form to that of Equation (3.2) and need not necessarily be expressed in terms of gross stress. For instance in cases where opposing loads P are applied at points close to the plane of the crack K can be expressed as

$$K = \left[\frac{P}{(\pi a)^{1/2}} \right] \alpha, \quad (3.3)$$

where P is the load per unit thickness. Note that for this loading configuration the value of α will be different from that in Equation (3.2) even for the same geometry.

Equations (3.2) and (3.3) may be expressed in terms of normalised stress intensity factors (see Reference 23) thus

$$K = K_0 \alpha, \quad (3.4)$$

where K_0 is the reference value of stress intensity appropriate to the type of loading considered. In simple cases, such as the remotely loaded plate in tension, it is convenient to use

$$K_0 = S(\pi a)^{1/2}, \quad (3.5)$$

so that Equations (3.4) and (3.2) are identical and hence, for a given ratio of crack length to plate width, the value of α is the same for both formulations. In more complex cases, such as the in-plane bending of a plate of finite width $2b$ with central crack of length $2a$, it may be convenient to express K_0 as (Reference 24)

$$K_0 = \frac{3Ma}{4b^3} (\pi a)^{1/2}, \quad (3.6)$$

where M is bending moment per unit thickness. By comparison with Equation (3.2) this implies that $S = 3Ma/4b^3$ which is not the conventional definition, but is the stress in the uncracked body at a distance $\pm a/2$ from the central axis. Note that the value of α depends on the value of S that is chosen, for example, $S = 3M/2b^2$ for the outer fibre stress might have been chosen, instead of the value $S = 3Ma/4b^3$ used in Equation (3.6).

The final value, K , must be independent of the form of presentation used. The use of Equations (3.2), (3.3) and (3.4) will depend on the form of solutions available for K_0 . Care must be taken that the formula used

* In some presentations the term $(\pi)^{1/2}$ is incorporated into the geometrical factors so that Equation (3.2) may be expressed as $K = S(a)^{1/2} \alpha'$, where $\alpha' = (\pi)^{1/2} \alpha$ (see Reference 26).

and the definition of nominal stress are consistent with those used in the derivation of the selected value of α .

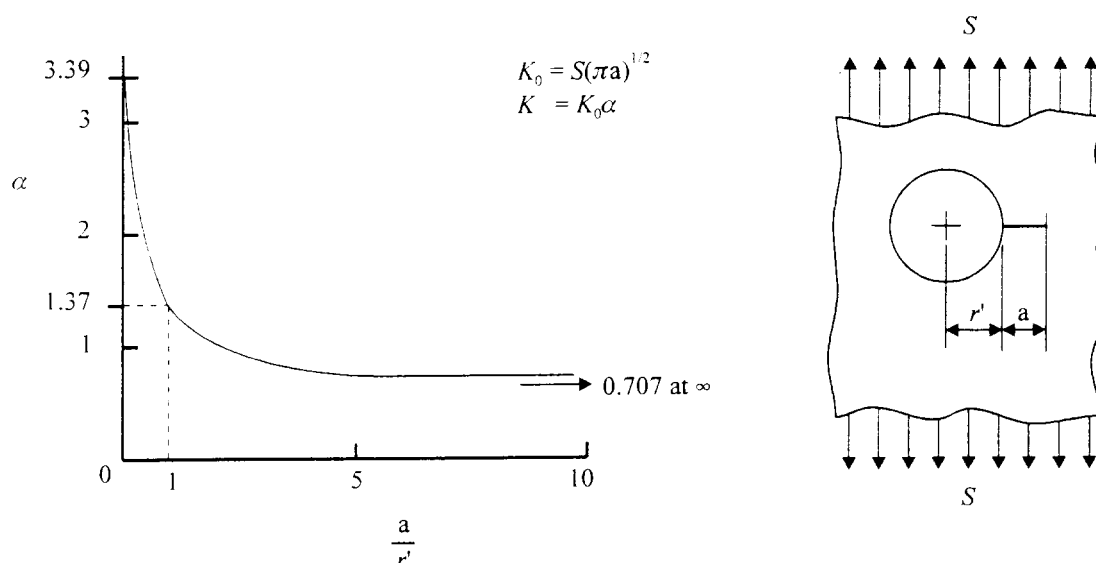
For any one material and environmental condition, test data relating crack propagation rate and residual strength to K may be generated from relatively simple specimens and applied to cracks in other bodies for which the K values are the same but result from different combinations of load and crack length. It may be necessary to consider the extent of plastic flow in the vicinity of the crack tip and this is discussed in Section 3.1.1.

3.1 Solutions for K and α

There are several methods of obtaining K (see References 15, 17, 18, 23 and 24). The two most commonly used methods are as follows.

- (i) Standard solutions for a wide variety of geometries and loading conditions are available from a number of sources (see References 3, 5, 9, 14, 17, 18, 25 and 30). Where a component has more than one type of loading, linear superposition may be used to obtain solutions for K . In cases where a crack is influenced by a combination of geometries, and a standard solution is not available, the compounding method may be used (see Reference 23 and Data Item No. 78036*).
- (ii) Finite element analysis methods using special procedures are also possible where the discontinuity associated with a crack is incorporated.

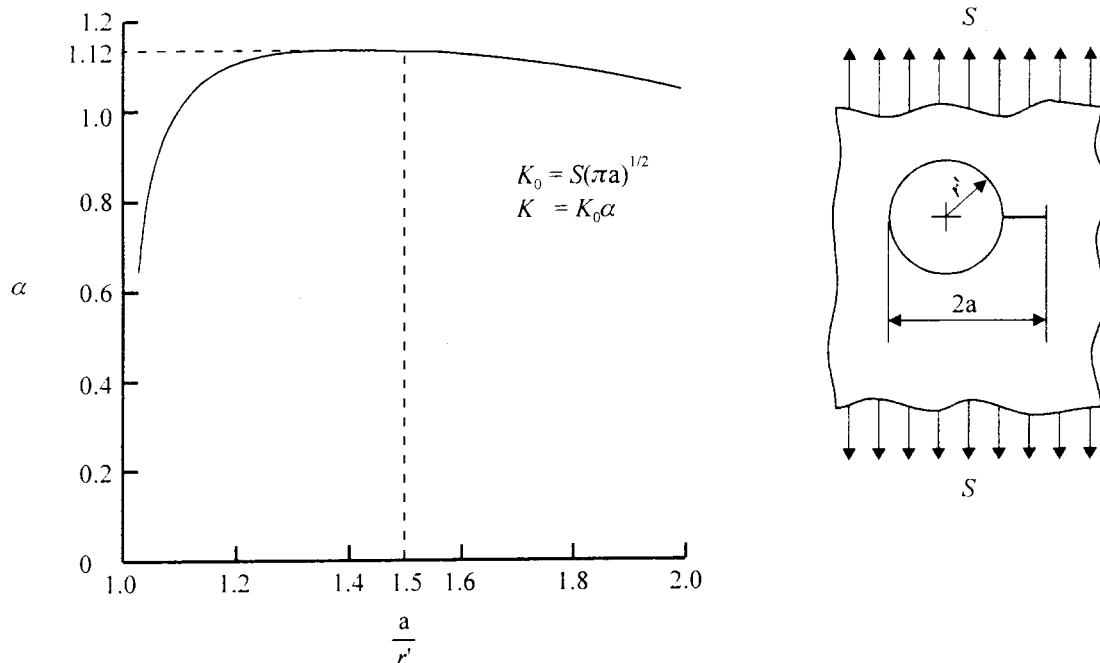
The particular definition of crack length, a , used in solutions for K and α is a matter of convenience, but care must be taken to ensure that the correct value of a is substituted into the appropriate expression. For example, Sketches 3.3 and 3.4 give alternative presentations of the solution for a single crack starting from a circular hole in an infinite sheet. Sketch 3.3 shows how α varies with the ratio of crack length to hole radius (a/r') where a is defined as the crack length from the edge of the hole and r' is the "convenient local characteristic dimension in the plane of the component" described in Section 3, earlier.



Sketch 3.3 Illustrating a definition of crack length that does not include the local geometry

* Data Item No. 78036 "The compounding method of estimating stress intensity factors for cracks in complex configurations using solutions from simple configurations".

An alternative presentation of the same case, but where the hole is considered as part of the crack length, $2a$, is shown in Sketch 3.4. Note that for the two presentations K_0 is different according to the definition of a .



Sketch 3.4 Illustrating a definition of crack length that includes the local geometry

To show that the two presentations give identical values for K , the following example is given.

(a) *Solution for K from Sketch 3.3*

Assuming that the crack length is $a = r'$, then from Sketch 3.3 for $a/r' = 1.0$, $\alpha = 1.37$. Substituting into Equation (3.2) the values of α and a , the solution for K is given by

$$K = 1.37S(\pi r')^{1/2}. \quad (3.7)$$

(b) *Solution for K from Sketch 3.4*

In this case the crack length is $a = 1.5r'$, and from Sketch 3.4 for $a/r' = 1.5$, $\alpha = 1.12$. Substituting into Equation (3.2) the values of α and a , the solution for K is given by

$$K = 1.12S(\pi 1.5r')^{1/2} = 1.37S(\pi r')^{1/2}. \quad (3.8)$$

Hence, both methods give the same value of K .

A third solution for α would be obtained for this crack configuration if the crack length, a , were measured from the crack tip to the centre of the hole (see Reference 25).

3.1.1 Plasticity correction under steadily increasing loading

As noted in Section 3, a modification may be necessary to account for the effect of plastic flow that occurs in the vicinity of the crack tip. In such cases the size of the plastic zone is estimated and added to the real

crack length to form an effective crack length, a_1 , which is used in place of a .

The size of the plastic zone is described in terms of a plasticity correction*, r_p . For a crack in a component the effective crack length is therefore estimated as

$$a_1 = a + r_p. \quad (3.9)$$

Since α is a function of the crack length for a given geometry and loading, this factor should also be corrected for plasticity to become α_1 corresponding to a_1 . The stress intensity factor corrected for plasticity is denoted by K_p and Equation (3.2) may then be re-written thus

$$K_p = S(\pi a_1)^{1/2} \alpha_1. \quad (3.10)$$

When calculating the value of r_p it is necessary to consider the influence of component thickness on the stress state which is triaxial very close to the crack tip. In linear elastic fracture mechanics it is accepted that this triaxiality is less pronounced if the plastic zone surrounding the crack tip is large in relation to the thickness (usually in thin components). In practical applications the stress state at the crack tip can be described by one of the following three conditions,

- (i) plane strain – a triaxial stress state where the plastic zone is very small compared with component thickness,
- (ii) plane stress – a biaxial stress state where the plastic zone is large compared with component thickness and
- (iii) somewhere between (i) and (ii) above.

These stress states are more fully described in Section 5.1.1.

An estimate of the extent of the plastic zone, r_p , for conditions of plane stress may be obtained for a particular value of K (see Reference 4) from

$$r_p = \frac{1}{2\pi} \left(\frac{K}{\sigma_y} \right)^2 \text{ or } r_p = \frac{a}{2} \left(\frac{S}{\sigma_y} \right)^2 \alpha^2 \quad (3.11)$$

and it has been estimated empirically that, for plane stress conditions to apply when using linear elastic fracture mechanics, $r_p > (t/2)$ so that, from Equation (3.11),

$$t < \frac{1}{\pi} \left(\frac{K}{\sigma_y} \right)^2 \approx 0.32 \left(\frac{K}{\sigma_y} \right)^2. \quad (3.12)$$

In the condition of plane strain the shape of the plastic zone is more complex but it is small and is taken to be about one third of the plane stress plastic zone size (see Reference 4) so that

$$r_p = \frac{1}{6\pi} \left(\frac{K}{\sigma_y} \right)^2. \quad (3.13)$$

* The quantity r_p is commonly known as the radius of the plastic zone from analysis of the Mode III crack opening case in which the plastic zone for small scale yielding is circular (see Reference 4). For Mode I crack opening the plastic zone is not circular but r_p is retained as a dimensional correction.

For plane strain conditions to apply when using linear elastic fracture mechanics, it has been estimated empirically that $r_p < (t/50)$ so that, from Equation (3.13),

$$t > \frac{50}{6\pi} \left(\frac{K}{\sigma_y} \right)^2 \approx 2.5 \left(\frac{K}{\sigma_y} \right)^2. \quad (3.14)$$

When the stress situation is between plane stress and plane strain the plasticity correction employed should be chosen so as to ensure that the more conservative answer is obtained. Although r_p is a function of K , and hence of K_p , it is not implied that the correction for a_1 in Equation (3.10) should be re-estimated by iteration.

4. USE OF FRACTURE MECHANICS TO ESTIMATE FATIGUE CRACK PROPAGATION RATES

In fatigue loading situations the process that eventually leads to failure of components can be conveniently divided into three successive stages, although the transition from one stage to the next is a continuous process and boundaries cannot be precisely defined. The three stages are as follows.

- (i) Crystallographic slip and nucleation of cracks occurring on the atomic scale, of the order 10^{-9} m (3.9×10^{-8} in) in length or less.
- (ii) Micro-growth of cracks ranging in length from 10^{-9} m (3.9×10^{-8} in) to about 10^{-4} m (3.9×10^{-3} in). The final stages of this phase may be just visible.
- (iii) Macro-growth commencing in cracks of about 10^{-4} m (3.9×10^{-3} in) in length and continuing up to failure. By the time this stage is reached the cracks will be sufficiently large to grow as though the material were a homogeneous continuum.

References 7 and 22 give detailed descriptions of the above stages. Where the crack size is of the order of stages (i) and (ii) (often referred to as crack initiation), and where those stages take up a major part of the life, then estimates of the fatigue life of components can be obtained using material or component $S-N$ data provided in the Data Items. However, if a component has a pre-existing crack of the order of the stage (iii) size then the early stages may be by-passed and the life will be much shorter. In this condition the use of linear elastic fracture mechanics is appropriate and the theory is applied primarily in two areas.

- (a) Estimating the life of structures or components that contain cracks or crack-like defects.
- (b) Establishing inspection intervals or permissible stress levels for cracked components. This will generally necessitate prediction of the growth of a crack under the load history appropriate to one inspection interval (or more) from an initial postulated crack. The length of the initial crack is generally taken to be the maximum that might be missed by the inspection method to be used for the component concerned. Crack growth prediction over one inspection interval, or more, is made to check that the crack will not grow within the inspection interval to such a length that the residual strength becomes inadequate. If the performance of the component is shown to be inadequate, it may be necessary to reduce the stress level in the component or to consider shorter inspection intervals in service.

The Data Items on crack propagation (referred to in Section 4.2) relate to the growth of visible macro-cracks, Stage (iii) of the fatigue process. However, for short cracks (that is, less than 1 – 2 mm (0.039 – 0.079 in) in many metallic materials) special data should be used, see Section 4.2.2.