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Undamped natural vibration of rectangular sway frame buildings and frame structures

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(Continued on inside back cover)

UNDAMPED NATURAL VIBRATION OF RECTANGULAR SWAY FRAME BUILDINGS AND FRAME STRUCTURES

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UNDAMPED NATURAL VIBRATION OF RECTANGULAR SWAY FRAME BUILDINGS AND FRAME STRUCTURES

1. NOTATION AND UNITS

Only the SI system of units is quoted, but any consistent system of units may be used.

		SI
A	cross-sectional area	m^2
D	beam-to-column stiffness ratio taper factor	
E	Young's modulus of elasticity	N/m^2
EI	uniform flexural rigidity term, see Equation (3.5)	N m^2
$(EI)_r$	flexural rigidity of beam or column in r th storey	N m^2
f_i	natural frequency for mode j	Hz
G	shear modulus	N/m^2
h	radius of gyration	m
I	second moment of area	m^4
K_j	normal-mode generalised stiffness for mode j	N/m
k_r	stiffness term, equal to $(EI)_r/l_r$ for r th storey	N m
k^b	overall stiffness term for substitute beam, see Equation (3.9)	N m
k^c	overall stiffness term for substitute column, see Equation (3.7)	N m
k'	member section shape factor	
l	uniform storey spacing, see Equation (3.6)	m
l_r	length of beam or column in r th storey	m
M_j	normal-mode generalised mass for mode j	kg
m	uniform lumped floor mass term, see Equation (3.10), in substitute frame method or lumped nodal mass in exact stiffness method	kg
m_r	lumped floor mass of r th storey	kg
N	number of structural nodes in frame structure	
n	number of storeys in sway frame building	

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P	static axial load, positive for compression, negative for tension	N
r	storey number (for ground storey $r = 1$ and for top storey $r = n$)	
u	column number	
v	number of columns in a given storey	
w	beam number	
x	number of beams parallel to sway plane in a given storey	
x'	member projection on global x' axis, see Sketch 4.2	m
y'	member projection on global y' axis, see Sketch 4.2	m
$\delta(\theta_j)$	correction factor for non-dimensional generalised mass ratio for mode j	
$\delta(v_j)$	correction factor for natural frequency ratio for mode j	
$\delta(\rho_1)$	correction factor for non-dimensional generalised mass for mode 1	
$\delta(\phi_1)$	correction factor for natural frequency parameter for mode 1	
θ_j	non-dimensional generalised mass ratio for mode j , M_j/M_1	
λf	foundation rotational spring stiffness	N m
μ	mass per unit length	kg/m
$\mu_j(\xi_r)$	non-dimensional lateral deflection of r th floor for mode j , normalised with respect to lateral deflection of the top storey, so $\mu_j(1) = 1$	
v_j	natural frequency ratio for mode j , f_j/f_1	
ξ_r	ratio r/n (for ground level $\xi_0 = 0$ and for top storey $\xi_n = 1$)	
ρ_1	non-dimensional generalised mass for mode 1	
ϕ_1	natural frequency parameter for mode 1	
ω	angular frequency	rad/sec
<i>Subscripts</i>		
j	relates to normal mode number	
r	relates to storey number (for ground level $r = 0$ and for top storey $r = n$)	

Superscripts

<i>b</i>	refers to beams
<i>c</i>	refers to columns

2. INTRODUCTION

With the tendency towards the construction of slender buildings of relatively high flexibility and low mass the dynamic loads on the structure have become increasingly important. The sway frame building is a popular structural option where such buildings are required up to 15 or 20 storeys. In a sway frame building, the rigidity of the connections between the columns and the beams or slabs usually suffices to provide lateral resistance. As the 20 storey height is approached, the rigidity of sway frame buildings can be insufficient and the sway due to wind may be unacceptable to the occupants of the building. In this situation, the introduction of shear walls interacting with the frame may increase the total rigidity, or the designer may choose another structural system, with frame-tube type buildings, shear buildings and core buildings being the most common structures.

For the calculation, by the modal superposition technique, of the response of a building to a fluctuating load induced by wind, earthquake or mechanical vibration, a knowledge of the free vibration characteristics is required. This Item provides a means of estimating the free vibration characteristics of rectangular sway frame buildings for use in these response calculations. In addition, a method of analysis is presented enabling the free vibration characteristics of more complex frame structures such as lattice towers, cranes and bridges to be computed efficiently. The way in which these data are used in the response of a building or structure under the action of a fluctuating wind load is described in ESDU 87035 and 88019. This Item is intended to complement ESDU 79005, for the undamped natural vibration of shear buildings, and ESDU 81036, for the undamped natural vibration of core buildings.

2.1 Vibration of Rectangular Sway Frame Buildings

In Section 3 of this Item, a method of analysis is presented for estimating the sway natural frequencies, mode shapes and normal-mode generalised masses and stiffnesses for rectangular sway frame buildings of up to 20 storeys. The method of analysis is based on substitute frame theory and reduces the sway frame building structure to an equivalent plane frame structure in the sway plane with one equivalent substitute beam and one equivalent substitute column at each storey. This approach enables the solution to be independent of the number of bays in a particular storey. A computer program which uses the substitute frame method of analysis is described in Appendix A, and an outline of the method of analysis is presented in Appendix B. For certain simple rectangular sway frame building structures, detailed in Section 3.5, this method of analysis allows a graphical presentation of the solution.

2.2 Vibration of Frame Structures

A method for calculating the natural frequencies, mode shapes, generalised masses and generalised stiffnesses of complex, rigidly jointed plane frame structures is described in Section 4 of this Item. The method of analysis uses classical beam theory to present an exact stiffness analysis in which the effects of axial load, shear deformation and rotatory inertia in the members may be accounted for if required. The method also provides the option of applying various elastic restraints, and lumped masses of finite mass and radius of gyration, at the structural nodes. A computer program using this method of analysis, enabling the natural vibration data corresponding to any chosen mode to be computed, is described in Appendix C, and an outline of the method of analysis is described in Appendix D.

2.3 Alternative Methods of Frame Vibration Analysis

In Appendix E of this Item, a short discussion is given of alternative methods for the estimation of the natural vibration parameters of frame structures. This is intended to assist the engineer should he decide that the methods of analysis presented in this Item are not suitable for a particular problem. This situation could arise when, for example, it is felt that the structure may not be idealised to a plane frame, perhaps where there is no plane of structural symmetry, or because the torsional modes of vibration are of interest.

3. UNDAMPED NATURAL VIBRATION OF RECTANGULAR SWAY FRAME BUILDINGS USING A SUBSTITUTE FRAME METHOD

3.1 Notes

This Section provides data for the estimation of the natural frequency characteristics for the sway modes of multistorey, multibay rectangular sway frame buildings. The data provided are applicable to those buildings which may be modelled as rectangular sway frame buildings. A sway frame building is one which has a frame structure consisting of continuous and uniform beams and columns with no cross-bracing members, rigid joints at each node of the frame and floor slabs attached to the beams of the frame at the discrete floor levels. Attaching the floor masses to the beams at each level allows the assumption of lumped masses at each floor level to be made. The increase in the effective stiffness of the beams due to the composite action effects between the floor slabs and the beams at a given level will generally have a small effect on the natural frequencies. In this Section it is assumed that the members of the frame are flexible in bending and not in shear, and the reductions in the stiffnesses of the columns due to the axial loads applied by the supported structure are negligible. Relaxation of either of these stiffness restraints would cause reductions in the frequencies and may modify the mode shapes, but such changes are generally small. A typical frame building structure is shown in Sketch 3.1. It is assumed that any soil/substructure interaction may be modelled by an equivalent rotational spring stiffness at ground level.

An idealised model of the frame building is shown in Sketch 3.2. This model idealises the multistorey, multibay sway frame as an equivalent substitute multistorey sway frame consisting of one substitute column and one substitute beam at each storey. A computer program is idealised as described in Section 3.3. Further idealisation of the model, as described in Section 3.5, allows a graphical presentation of natural vibration data relating to the first four sway modes of vibration.*

3.2 Data Provided

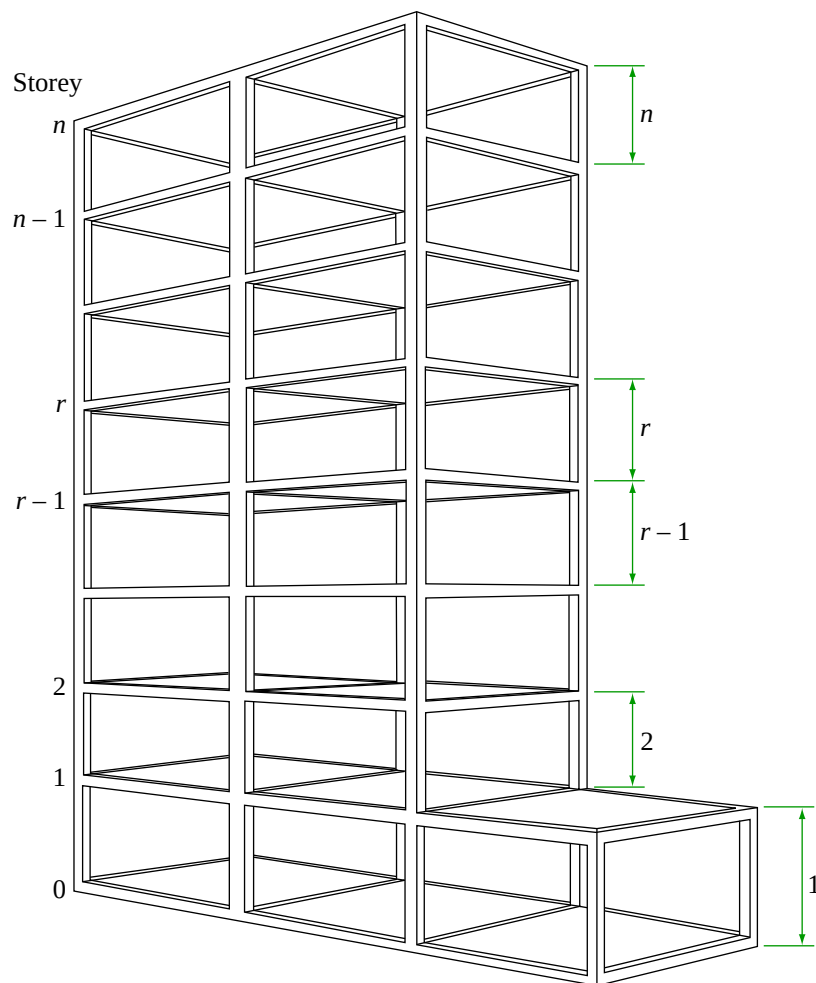
The natural frequencies, mode shapes and normal-mode generalised masses for the first four sway modes of vibration of idealised frame buildings from 1* to 20 storeys are provided in graphical form. These data are provided for buildings with a uniform or a uniformly varying ratio of the beam stiffness to column stiffness at each storey. Frame buildings that cannot be represented by these idealisations should be analysed using the computer program described in Appendix A.

* The four sway solutions are given only for a frame building of four or more storeys. For the vibration parameters referring to the second, third and fourth modes, the graphical data are presented for a minimum of two, three and four storeys, respectively.

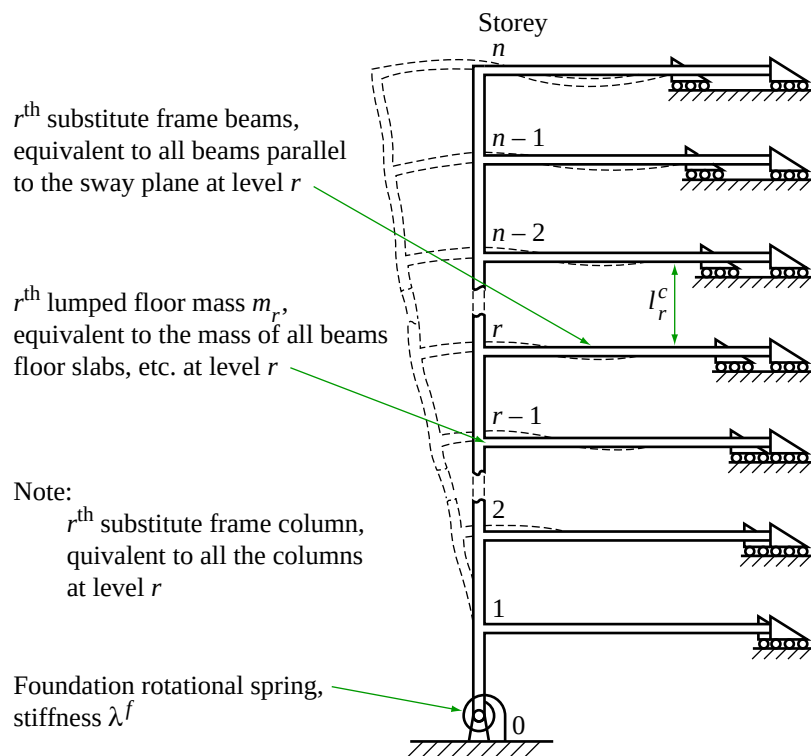
3.3 Sway Frame Building Model

A sway frame building may be modelled by the substitute sway frame shown in Sketch 3.2. In this model of the sway frame building, the following assumptions are made:

- (i) each joint is rigid,
 - (ii) the rotations at each joint at a given storey in the sway frame building are equal,
 - (iii) the foundations may be modelled by an equivalent rotational spring stiffness at ground level and are considered infinitely rigid in the translational plane,
 - (iv) each member of the frame is continuous and uniform, with finite mass and flexural rigidity,
- and
- (v) the masses of the floors are lumped at the floor levels.



Sketch 3.1 Typical frame building



Sketch 3.2 n -story substitute frame model

The lumped floor masses, m_r , must include the whole of the fixed mass and a proportion of the moveable mass associated with the floor loadings (see References 6, 7 and 8). The fixed mass includes the floor slabs and all the beams at the floor level, together with cladding, fittings, partitions and fixed equipment.

The building is considered to 'sway' in one plane only, with no torsional rotation of the floor masses. This may not apply if there is no coincident plane of structural and mass symmetry in which the sway motion would take place. The deformation of the building is defined by $2n + 1$ degrees of freedom, n lateral and $n + 1$ rotational, where n is the number of storeys in the frame building.

3.4 Method of Solution

A substitute sway frame method of analysis (Derivation 5) has been used to calculate the vibrational properties of sway frame buildings. The theory for this method of analysis is outlined in Appendix B. The computer program in Appendix A requires the values of the storey height, column flexural rigidities and masses per unit length, beam flexural rigidities, masses per unit length and lengths, and the lumped floor mass, including the total mass of the beams, for each storey in the building. For the ground level, an equivalent rotational spring stiffness is required to simulate the foundation effects. For a built-in base, this spring stiffness is infinite and for a pinned base zero. The increase in the effective column stiffness due to the interaction between the cladding and the columns and the increase in the effective beam stiffness due to the attachment of the floor slabs may be accounted for by appropriate modifications in the beam and column stiffness terms. Further guidance on this matter is given in Derivation 5. Neglecting these composite action effects leads to an underestimate of the frequency, and the mode shapes and the generalised masses and stiffnesses may also be modified slightly.

To facilitate a graphical presentation, the sway frame building model has been further idealised to include

two non-dimensional ratios describing the mass and stiffness characteristics of the sway frame building. The mass ratio is considered to be constant throughout the building and is equal to the ratio of the total mass acting at a particular floor level, as discussed in Section 3.3, to the total mass of the supporting columns at that level. The stiffness ratio is equal to the ratio of the sum of the flexural rigidities of all columns at a floor level divided by the storey height (k_r^c , see Equation (3.3)), to the sum of the flexural rigidities of all the beams parallel to the sway plane each divided by its own length (k_r^b , see Equation (3.8)). It is important to note that the graphical presentation is independent of the number of bays in a particular storey.

If it is felt that this further idealisation imposed by the graphical presentation is not justified in the case of the particular sway frame building under consideration, the more general computer program described in Appendix A may be used for the solutions.

3.5 Idealisation Imposed on the Sway Frame Building Model to Allow a Graphical Presentation

For the graphical presentation, it is necessary to use a constant non-dimensional mass and stiffness ratio in the analysis. Three values of each of these non-dimensional ratios are considered in the graphical presentation and vibration parameters for other combinations of the ratios may be found by graphical interpolation. The storey spacing is assumed to be uniform throughout the building and the foundations are considered to be completely rigid.

3.5.1 Idealised substitute sway frame column characteristics

The mass and flexural rigidities of the substitute frame columns for a given storey are defined by Equations (3.1) and (3.2) to be

$$m_r^c = \sum_{u=1}^v (m_r^c)_u, \quad (3.1)$$

$$\text{and} \quad (EI)_r^c = \sum_{u=1}^v [(EI)_r^c]_u, \quad (3.2)$$

where the summation extends to all the columns in the r th storey of the frame building.

We may then define the stiffness term, k_r^c , for each storey of the substitute frame to be

$$k_r^c = \frac{(EI)_r^c}{l_r^c}. \quad (3.3)$$

The uniform substitute frame column mass, flexural rigidity and storey spacing are given by the following equations,

$$m^c = \frac{1}{n} \sum_{r=1}^n m_r^c, \quad (3.4)$$

$$(EI)^c = \frac{1}{n} \sum_{r=1}^n (EI)_r^c, \quad (3.5)$$

$$\text{and} \quad l^c = \frac{1}{n} \sum_{r=1}^n l_r^c. \quad (3.6)$$

In addition, we may define the overall substitute column stiffness term, k^c , to be

$$k^c = \frac{1}{n} \sum_{r=1}^n k_r^c. \quad (3.7)$$

3.5.2 Idealised substitute sway frame beam characteristics

A beam stiffness term for a given storey may be defined as the summation of the flexural rigidities of each beam in the sway plane divided by the length of that beam, expressed by Equation (3.8) as,

$$k_r^b = \sum_{w=1}^x \left(\left(\frac{EI}{l} \right)_r \right)_w. \quad (3.8)$$

In addition, we may define the overall substitute beam stiffness term, k^b , similar to that in Equation (3.7) for the substitute column as

$$k^b = \frac{1}{n} \sum_{r=1}^n k_r^b. \quad (3.9)$$

3.5.3 Lumped floor mass characteristics

The uniform lumped floor mass characteristics can be taken as

$$m = \frac{1}{n} \sum_{r=1}^n m_r, \quad (3.10)$$

where the m_r are the lumped floor masses of the real building, previously discussed in Section 3.3.

3.5.4 Uniformly varying beam stiffness to column stiffness idealisation

Data for real buildings have shown a typical tapering of the ratio of the beam stiffness term, k_r^b , to the column stiffness term, k_r^c , increasing from the bottom storey to the top of the building. Where this beam to column stiffness ratio taper may be idealised by a uniformly varying beam to column stiffness ratio, the non-dimensional stepped change in the beam to column stiffness ratio can be taken as,

$$\Delta(k^b/k^c) = \frac{2}{n(n-1)} \sum_{r=1}^n \{k_r^b/k_r^c - k_1^b/k_1^c\}, \quad (3.11)$$

where the k_r^b and k_r^c are the beam and column stiffness terms for the r th storey of the idealised sway frame building.

The non-dimensional beam to column stiffness ratio taper factor D may be expressed as,

$$D = \frac{\Delta(k^b/k^c)}{k_1^b/k_1^c}. \quad (3.12)$$

3.6 Range of Graphical Data Provided

For the idealised building with constant storey spacing, graphical data are provided for the first four natural frequencies, corresponding mode shapes and normal-mode generalised masses for sway frame buildings with 1* to 20 storeys.

Three values of the stiffness ratio, k^b/k^c , of 0.1, 1 and 10 and three values of the mass ratio, m/m^c , of 1, 3 or 10 and 20, are considered for each vibrational parameter and an appropriate correction factor may be used to account for the value of the stiffness taper factor, D . The value $D = 0$ applies to buildings with a uniform ratio of the beam to column stiffness ratio throughout the building (*i.e.* no stiffness taper).

3.6.1 Natural frequencies

For the fundamental frequency, the frequency parameter, ϕ_1 , is plotted against the number of storeys for the range of values of k^b/k^c and m/m^c in Figure 1a for $D = 0$. The fundamental frequency correction factor, $\delta(\phi_1)$, for various values of D is plotted in Figure 1b for $k^b/k^c = 0.1$, Figure 1c for $k^b/k^c = 1$ and Figure 1d for $k^b/k^c = 10$. The value of the fundamental frequency may be obtained using the expression,

$$f_1 = \delta(\phi_1) \times \phi_1 \sqrt{\frac{(EI)^c}{m^c (l^c)^3}}. \quad (3.13)$$

The frequencies of the higher modes are given by

$$f_j = \delta(v_j) \times v_j f_1, \quad (3.14)$$

where v_j is the frequency ratio, f_j/f_1 , for mode j , and $\delta(v_j)$ is the correction factor for that frequency ratio. Values of v_2 are presented for the range of values of m/m^c and k^b/k^c in Figure 2a for $D = 0$. The correction factor for the second natural frequency is presented in Figures 2b, 2c and 2d for values of the stiffness ratio, k^b/k^c , of 0.1, 1 and 10 respectively. Similar data are presented for v_3 and $\delta(v_3)$ in Figure 3, and for v_4 and $\delta(v_4)$ in Figure 4.

3.6.2 Mode shapes

The normalised sway mode shapes, $\mu_j(\xi_r)$ for the first four modes are presented in Figures 5, 6, 7 and 8 respectively. Each of these Figures covers the full range of k^b/k^c considered. A uniform value of $D = 0$ (*i.e.* no stiffness taper) has been used for the plots. The variation of mode shape with the mass ratio, m/m^c , is generally small and a constant value of 10 has been used for the mass ratio in the graphical presentation for the mode shapes considered. Each mode is normalised with respect to the lateral deflection of the top storey, so that

$$\mu_j(n) = 1. \quad (3.15)$$

The lateral deflection of the base is assumed to be zero, so that

$$\mu_j(0) = 0. \quad (3.16)$$

For each mode, the substitute frame method of analysis gives n discrete values of the deflection $\mu_j(\xi_r)$ for an n -storey building. However, in order to cover the range of storeys considered in this Item, mode

* Lower values of n of two, three and four respectively have been taken for the second, third and fourth vibration parameters.

shapes are presented as continuous curves of normalised displacements plotted against r/n , where,

$$\xi_r = r/n. \quad (3.17)$$

The curves are calculated for a 10-storey building with no stiffness taper but may provide a reasonable approximation to the mode shapes for values of n above 4.

3.6.3 Normal-mode generalised masses

For a sway frame building, M_j , the normal-mode generalised mass for mode j , may be approximated by*,

$$M_j = \sum_{r=1}^n \{ (m_r + m_r^c) \mu_j^2(\xi_r) \}. \quad (3.18)$$

In terms of the non-dimensional generalised mass, ρ_1 , for the first mode

$$M_1 = \delta(\rho_1) \times \rho_1 \sum_{r=1}^n (m_r + m_r^c), \quad (3.19)$$

where the summation on the right-hand side is the total mass of the building and $\delta(\rho_1)$ is the non-dimensional generalised mass correction factor for D . Values of ρ_1 are plotted against n in Figure 9a for $D = 0$ and the range of values of k^b/k^c and m/m^c considered. Values of $\delta(\rho_1)$ are plotted for the range of values of D for $k^b/k^c = 0.1, 1$ and 10 in Figures 9b, 9c and 9d respectively.

The normal-mode generalised masses for the higher modes are given by

$$M_j = \delta(\theta_j) \times \theta_j M_1. \quad (3.20)$$

Values of θ_2 , θ_3 and θ_4 for the range of values of m/m^c and k^b/k^c considered, and the appropriate correction factors $\delta(\theta_2)$, $\delta(\theta_3)$ and $\delta(\theta_4)$ for the range of values of k^b/k^c and D considered, are given in Figures 10, 11 and 12 respectively.

3.6.4 Normal-mode generalised stiffnesses

The normal-mode generalised stiffness, K_j , for mode j of the idealised sway frame building is determined from the corresponding normal-mode generalised mass value by the equation,

$$K_j = (2\pi f_j)^2 M_j. \quad (3.21)$$

* The normal-mode generalised mass may be derived using the kinetic energy equation of the form

$$\text{total kinetic energy} = \frac{1}{2} M_j \omega^2.$$

The approximate expression in Equation (3.18) does not correctly account for the column kinetic energy but the simplification is acceptable for most engineering applications and is amendable to formal graphical presentation. A precise value of M_j may be found using the computer program.

4. UNDAMPED NATURAL VIBRATION OF FRAME STRUCTURES USING AN EXACT STIFFNESS METHOD

4.1 Notes

A simple substitute method of analysis for estimating the natural frequency characteristics of multistorey, multibay sway frame buildings for which the idealisations imposed by the substitute sway frame method described in Section 3.3 may be applied is presented in Section 3 of this Item. Using the further idealisations described in Section 3.4, a graphical presentation for the natural vibration characteristics of the idealised sway frame building model may be used for the solution. In this Section of the Item, a method is presented which enables the solution for a frame structure which may not satisfy the idealisations imposed by the substitute sway frame method. The method allows the solution for any chosen mode of undamped natural vibration for an N -node, rigidly jointed plane frame structure with members at any orientation. The method of analysis allows the application of externally acting elastic restraints at any node of the rigidly jointed plane frame structure. These external restraints are applied by the restriction of the degrees of freedom at a particular node. Either of the translational, or rotational, degrees of freedom may be restrained by applying an effective spring stiffness at the node. For the translational degrees of freedom, the springs are assumed to be acting along the global axes (see Sketch C2.2, Appendix C). These elastic restraints allow the simulation of built-in, sprung, or pinned support conditions. The effects of axial loading, shear deformation and rotatory inertia in the members and lumped masses at the nodes may be included in the analysis if required. The method will allow frame structures such as lattice towers, cranes and bridges to be considered. A typical frame structure is shown in Sketch 4.1.

An idealised model of the frame structure is given in Sketch 4.2. Discussions of the node and member numbering are given in Section 4.3 and Appendix D. A computer program is described in Appendix C to allow the solution of the model, using the assumptions detailed in Section 4.2.

4.2 Frame Structure Model

A typical model of a frame structure is shown in Sketch 4.2. In this model, the following assumptions are made:

- (i) each joint is rigid,
- (ii) the support conditions at the nodes may be modelled by simple springs corresponding to each degree of freedom at the supported node,

and

- (iii) each member of the frame is continuous and uniform, with finite mass, flexural, shear and extensional rigidities.

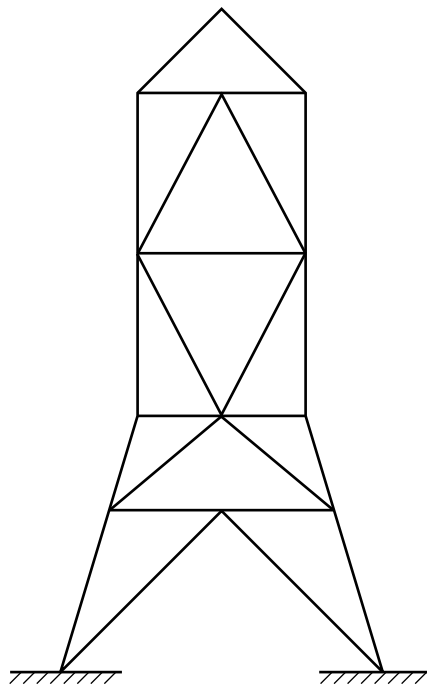
The frame structure is analysed as a system with an infinite number of degrees of freedom. The method allows any natural frequency to be located although in many cases only the first few modes of vibration are of interest for response calculations.

4.3 Method of Solution

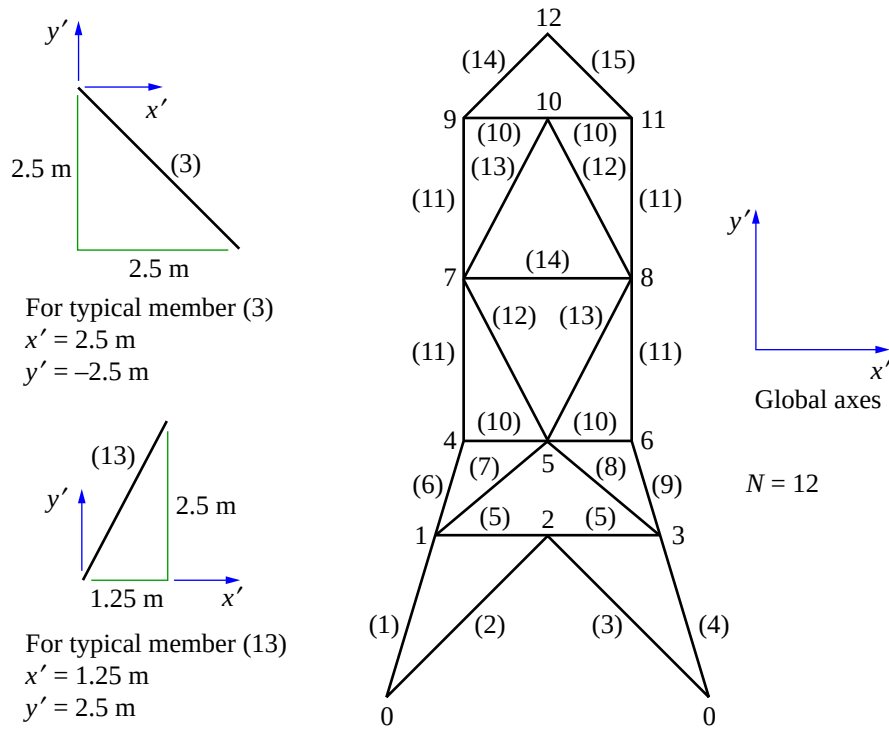
An exact stiffness method (Derivations 1, 2 and 4) has been used to calculate the vibrational properties of frame structures. The theory for this method is outlined in Appendix D. The computer program described in Appendix C requires the node connection information, describing the topology of the frame, the member properties, the node restriction information and the lumped nodal mass information for the solution. The

input data have been minimised by adopting certain rules for data preparation. Built-in nodes must be numbered zero and the other nodes must be numbered sequentially from one. It is likely that the frame may contain two or more members that are identical both in relative orientation to the set of global axes and in all other member properties, therefore provision is made to allow such members to be referred to as identical member types.

The program requires the values of the elastic modulus (which may be given the same value for all members of the frame), the second moment of area, the cross-sectional area, the mass per unit length and the member projections on the global x' and y' axes for each member type. If axial loading is to be considered, the axial load on the member is required and if shear deflections are to be considered the section shape factor and the shear modulus (which may be given the same value for all members of the frame) are required. The connection information is input in the form of a connection list comprising three integers for each connection. These are the numbers of the connected nodes followed by the member type reference number. The node restriction information is input in the form of a restraints list, detailing the restraining spring rate corresponding to each degree of freedom as the detailed node. In addition, the lumped nodal masses and radii of gyration are input as a list, detailed for each node. It is important to note that a system of node numbering should be chosen which minimises the greatest difference between the node numbers at the ends of any member (excluding members which are built-in at one end) since the resulting computation will be faster and require less computer storage space. The data input scheme is described in detail in Appendix C.



Sketch 4.1 Typical frame structure



Sketch 4.2 N-node frame structure model illustrating node-numbering and member type numbering

5. DERIVATION AND REFERENCES

5.1 Derivation

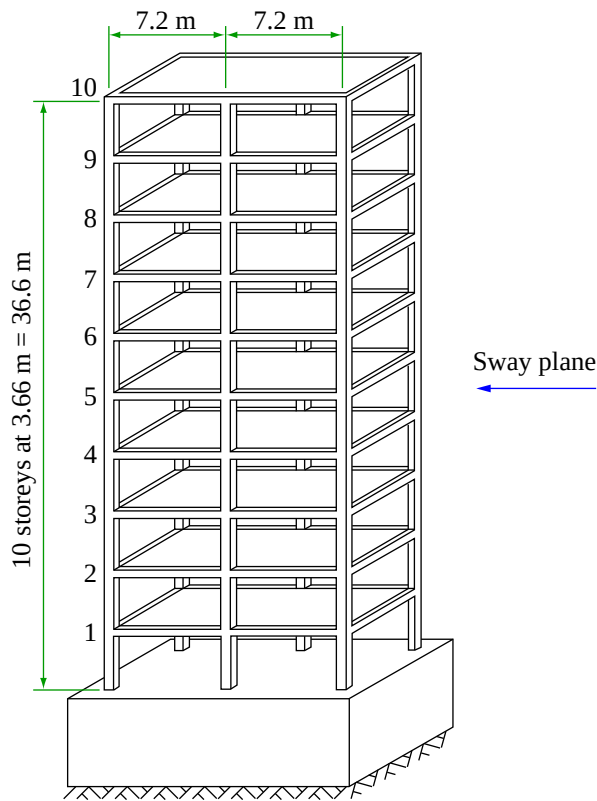
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5.2 References

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6. EXAMPLE

It is required to estimate the natural frequencies, mode shapes, normal-mode generalised masses and normal-mode generalised stiffnesses for the first four sway modes of the ten-storey sway frame building shown in Sketch 6.1 in the sway plane indicated. The mass, flexural rigidity and dimensional characteristics are tabulated below.



r	m_r (kg)	$(m_r^c)_u$ (kg)	$[(ED)_r^c]_u$ (N m ²)	$[(ED)_r^b]_w$ (N m ²)
10	100 000	1120	5.52×10^7	2.21×10^8
9	100 000	1120	5.52×10^7	2.21×10^8
8	100 000	1120	5.52×10^7	2.21×10^8
7	100 000	1370	8.28×10^7	2.76×10^8
6	100 000	1370	8.28×10^7	2.76×10^8
5	100 000	1370	8.28×10^7	2.76×10^8
4	100 000	1900	1.10×10^8	3.31×10^8
3	100 000	1900	1.10×10^8	3.31×10^8
2	100 000	1900	1.10×10^8	3.31×10^8
1	100 000	1900	1.10×10^8	3.31×10^8

Sketch 6.1 Example frame building

6.1 Building Idealisation

For this Example, it is assumed that the foundations are built-in and may be considered completely rigid. There are six columns at each storey level and four beams parallel to the sway plane at each storey level. Firstly, the substitute frame column and substitute frame beam characteristics are calculated using the following procedure.

6.1.1 Substitute frame column characteristics

- (i) The value of m_r^c , the total mass of the columns at a given storey, is given by Equation (3.1) as,

$$m_r^c = \sum_{u=1}^v (m_r^c)_u.$$

As there are six columns in each storey, the total mass of the columns for each storey is as follows,

$$\text{storeys 8 – 10} \quad m_r^c = \sum_{u=1}^6 1120 = 6720 \text{ kg},$$

$$\text{storeys 5 – 7} \quad m_r^c = \sum_{u=1}^6 1370 = 8220 \text{ kg},$$

$$\text{and} \quad \text{storeys 1 – 4} \quad m_r^c = \sum_{u=1}^6 1900 = 11\,400 \text{ kg}.$$

(ii) The substitute frame column flexural rigidity, $(EI)_r^c$, is given by Equation (3.2) as,

$$(EI)_r^c = \sum_{u=1}^v [(EI)_r^c]_u.$$

For each storey in the building this gives the following flexural rigidity characteristics

$$\text{storeys 8 – 10} \quad (EI)_r^c = \sum_{u=1}^6 5.52 \times 10^7 = 3.31 \times 10^8 \text{ N m}^2,$$

$$\text{storeys 5 – 7} \quad (EI)_r^c = \sum_{u=1}^6 8.28 \times 10^7 = 4.97 \times 10^8 \text{ N m}^2,$$

$$\text{and} \quad \text{storeys 1 – 4} \quad (EI)_r^c = \sum_{u=1}^6 1.10 \times 10^8 = 6.60 \times 10^8 \text{ N m}^2.$$

(iii) The substitute frame column stiffness term, k_r^c , for each storey in the substitute frame is given by Equation (3.3) as,

$$k_r^c = \frac{(EI)_r^c}{l_r^c}$$

giving the values,

$$\text{storeys 8 – 10} \quad k_r^c = \frac{3.31 \times 10^8}{3.66} = 9.04 \times 10^7 \text{ N m},$$

$$\text{storeys 5 – 7} \quad k_r^c = \frac{4.97 \times 10^8}{3.66} = 1.36 \times 10^8 \text{ N m},$$

$$\text{and} \quad \text{storeys 1 – 4} \quad k_r^c = \frac{6.60 \times 10^8}{3.66} = 1.80 \times 10^8 \text{ N m}.$$

- (iv) The uniform substitute frame column mass term, m^c , is given by Equation (3.4) as,

$$\begin{aligned} m^c &= \frac{1}{n} \sum_{r=1}^n m_r^c \\ &= \frac{1}{10} 90\,420 = 9042 \text{ kg} . \end{aligned}$$

- (v) The uniform substitute frame column flexural rigidity term, $(EI)^c$, is given by the Equation (3.5) as,

$$\begin{aligned} (EI)^c &= \frac{1}{n} \sum_{r=1}^n (EI)_r^c \\ &= \frac{1}{10} 5.12 \times 10^9 = 5.12 \times 10^8 \text{ N m}^2 . \end{aligned}$$

- (vi) The uniform substitute frame column storey spacing, l^c , is given by Equation (3.6) as,

$$\begin{aligned} l^c &= \frac{1}{n} \sum_{r=1}^n l_r^c \\ &= \frac{1}{10} 36.6 = 3.66 \text{ m} . \end{aligned}$$

- (vii) The overall substitute frame stiffness term, k^c , is given by Equation (3.7) as,

$$\begin{aligned} k^c &= \frac{1}{n} \sum_{r=1}^n k_r^c \\ &= \frac{1}{10} 1.40 \times 10^9 = 1.40 \times 10^8 \text{ N m} . \end{aligned}$$

6.1.2 Substitute frame beam characteristics

- (i) The substitute frame beam stiffness term is the summation of the flexural rigidities of each beam divided by the length of that beam for each storey in the building and is given by Equation (3.8) to be,

$$k_r^b = \sum_{w=1}^x \left(\left(\frac{EI}{l} \right)_r^b \right)_w$$

where x , equal to four for each storey in this particular example, is the number of beams lying parallel to the sway plane at storey r . For each storey in the building, this gives the following stiffness term values,

$$\text{storeys 8 – 10} \quad k_r^b = \sum_{w=1}^4 \frac{2.21 \times 10^8}{7.2} = 1.23 \times 10^8 \text{ N m} ,$$

$$\text{storeys 5 - 7} \quad k_r^b = \sum_{w=1}^4 \frac{2.76 \times 10^8}{7.2} = 1.53 \times 10^8 \text{ N m},$$

$$\text{and} \quad \text{storeys 1 - 4} \quad k_r^b = \sum_{w=1}^4 \frac{3.31 \times 10^8}{7.2} = 1.84 \times 10^8 \text{ N m}.$$

(ii) The overall substitute frame beam stiffness term k^b is given by Equation (3.9) as,

$$\begin{aligned} k^b &= \frac{1}{n} \sum_{r=1}^n k_r^b \\ &= \frac{1}{10} 1.56 \times 10^9 = 1.56 \times 10^8 \text{ N m}. \end{aligned}$$

6.1.3 Lumped floor mass characteristics

(i) The uniform lumped floor mass characteristics are given by Equations (3.10) as,

$$\begin{aligned} m &= \frac{1}{n} \sum_{r=1}^n m_r, \\ &= \frac{1}{10} 1\,000\,000, \\ &= 100\,000 \text{ kg}. \end{aligned}$$

6.1.4 Idealisation of the sway frame building model to the uniformly varying beam to column stiffness ratio type

(i) The value of $\Delta(k^b/k^c)$, the constant stepped change in the beam to column stiffness ratio between storeys is given by Equation (3.11) as,

$$\begin{aligned} \Delta(k^b/k^c) &= \frac{2}{n(n-1)} \sum_{r=1}^n \{k_r^b/k_r^c - k_1^b/k_1^c\}, \\ &= \frac{2}{10 \times 9} \times 1.32 = 0.029. \end{aligned}$$

(ii) The value of D , the non-dimensional beam to column stiffness ratio taper, is given by Equation (3.12) as,

$$\begin{aligned} D &= \frac{\Delta(k^b/k^c)}{k_1^b/k_1^c}, \\ &= \frac{0.029}{1.02} = 0.028. \end{aligned}$$

6.2 Natural Frequencies

- (i) We may obtain the value of the fundamental frequency parameter ϕ_1 for $n = 10$, $k^b/k^c = 1.1$ and $m/m^c = 11$ for $D = 0$ by interpolation from Figure 1a. The values used in the interpolation, and the interpolated values for $k^b/k^c = 1.1$ are tabulated below.

$D = 0$	ϕ_1			
$k^b/k^c \backslash m/m^c$	0.1	1	10	1.1
1	0.0200	0.0435	0.0572	0.0445
10	0.00835	0.0182	0.0239	0.0186
20	0.00603	0.0131	0.0173	0.0134

By graphical interpolation between the values of ϕ_1 , at $k^b/k^c = 1.1$, for $m/m^c = 11$, we obtain the value,

$$\phi_1 = 0.0178.$$

We may estimate the value of the fundamental frequency correction factor, $\delta(\phi_1)$ for this particular example by graphical interpolation between the values of D and k^b/k^c presented in Figures 1b, 1c and 1d. The values used in the interpolation, and the interpolated values for $k^b/k^c = 1.1$ are tabulated below.

		$\delta(\phi_1)$			
$k^b/k^c \backslash D$		0	0.05	0.1	0.3
0.1		1.000	1.006	1.001	0.968
1		1.000	1.012	1.012	0.987
10		1.000	1.028	1.040	1.034
1.1		1.000	1.013	1.014	0.990

By graphical interpolation between the values of $\delta(\phi_1)$, at $D = 0.028$ and $k^b/k^c = 1.1$, we obtain the value,

$$\delta(\phi_1) = 1.011.$$

- (ii) The value of the fundamental frequency may be obtained from Equation (3.13),

$$f_1 = \delta(\phi_1) \times \phi_1 \sqrt{\frac{(EI)^c}{m^c (l^c)^3}}$$

$$= 1.011 \times 0.0178 \times \sqrt{\frac{5.12 \times 10^8}{9042 \times (3.66)^3}} = 0.612 \text{ Hz}.$$

- (iii) We obtain the value of the second frequency ratio, ν_2 , for $n = 10$, $k^b/k^c = 1.1$ and $m/m^c = 11$ for $D = 0$ by graphical interpolation from Figure 2a. The values used in the interpolation, and the interpolated values at $k^b/k^c = 1.1$, are tabulated below.

$D = 0$	ν_2			
$k^b/k^c \backslash m/m^c$	0.1	1	10	1.1
1	3.19	3.03	3.00	3.02
3	3.19	3.02	2.99	3.02
20	3.19	3.02	2.98	3.02

Interpolating between the values of ν_2 at $k^b/k^c = 1.1$, for $m/m^c = 11$, we obtain the value,

$$\nu_2 = 3.02.$$

We may estimate the value of the second frequency ratio correction factor $\delta(\nu_2)$ by interpolation between the values of D and k^b/k^c presented in Figures 2b, 2c and 2d. The values used in the interpolation, and the interpolated values for $k^b/k^c = 1.1$ are tabulated below.

		$\delta(\nu_2)$			
$D \backslash k^b/k^c$		0	0.05	0.1	0.3
0.1		1.000	0.997	0.994	0.988
1		1.000	0.982	0.970	0.948
10		1.000	0.967	0.946	0.912
1.1		1.000	0.981	0.969	0.946

By interpolating between these values, for $D = 0.028$ at $k^b/k^c = 1.1$, we obtain the value,

$$\delta(\nu_2) = 0.988.$$

Similarly, by graphical interpolation from Figure 3, we obtain the frequency ratio for the third mode

$$\nu_3 = 5.08.$$

By graphical interpolation from Figures 3b, 3c and 3d, we obtain the value of the correction factor for the third frequency ratio,

$$\delta(v_3) = 0.987.$$

From Figure 4, for the fourth mode,

$$v_4 = 7.20,$$

and from Figures 4b, 4c and 4d, we obtain the correction factor

$$\delta(v_4) = 0.987.$$

- (iv) The frequencies of the higher modes are given by Equation (3.14),

$$f_j = \delta(v_j) \times v_j f_1$$

to give the results,

$$f_2 = 0.988 \times 3.02 \times 0.612 = 1.83 \text{ Hz},$$

$$f_3 = 0.987 \times 5.08 \times 0.612 = 3.07 \text{ Hz},$$

and $f_4 = 0.987 \times 7.20 \times 0.612 = 4.35 \text{ Hz}.$

6.3 Mode Shapes

The normalised displacements for the first mode, $\mu_1(\xi_r)$, for $n = 10$ and $k^b/k^c = 1.1$ may be estimated by graphical interpolation. From Figure 5, the interpolated values for $k^b/k^c = 1.1$ are tabulated below.

		Mode 1, $\mu_1(\xi_r)$			
r	$\xi_r \backslash k^b/k^c$	0.1	1	10	1.1
10	1.0	1.00	1.00	1.00	1.00
9	0.9	0.96	0.98	0.98	0.98
8	0.8	0.91	0.93	0.93	0.93
7	0.7	0.83	0.86	0.87	0.86
6	0.6	0.73	0.77	0.78	0.77
5	0.5	0.61	0.67	0.68	0.67
4	0.4	0.48	0.54	0.56	0.54
3	0.3	0.34	0.41	0.43	0.41
2	0.2	0.19	0.26	0.29	0.26
1	0.1	0.06	0.11	0.14	0.11
0	0	0	0	0	0

Similarly, from Figures 6, 7 and 8, we may estimate the mode shapes for the second, third and fourth modes respectively for $k^b/k^c = 1.1$. These values are tabulated in the following table.