



ESDU 82037

Issued December 1982
With Amendments A to E
June 2010

The response of two-degree-of-freedom systems (computer program)

Associated software: ESDUpac A8237
VIEWpac 8237A

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THE PREPARATION OF THIS DATA ITEM

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The Dynamics Committee, which first met in January 1962, has the co-operation of many engineers and scientists in industry, research establishments and universities from whom much assistance and unpublished information is being received.

THE RESPONSE OF TWO-DEGREE-OF-FREEDOM SYSTEMS (COMPUTER PROGRAM)

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THE RESPONSE OF TWO-DEGREE-OF-FREEDOM SYSTEMS (COMPUTER PROGRAM)

1. INTRODUCTION

The Fortran computer program presented in this Item may be used to determine the response of any two-degree-of-freedom system with constant coefficients to any known applied forcing functions.

The program is capable of dealing with any two-degree-of-freedom system the description of which requires two independent coordinates and which can be described by two second-order constant coefficient differential equations. The independent coordinates are the dependent variables of the system.

An applied input forcing function is allowed for each second-order equation and to ease program input one of six possible types of input function can be specified: zero, impulsive, step, cosine, sine and an arbitrary input.

The program evaluates the characteristic polynomial of the system together with the roots of this equation from which the modes of response are found. For systems involving two second-order equations the eigenvalues and corresponding eigenvectors are also found. For second-order (oscillatory) modes the natural frequency, damping ratio, period and time to half-amplitude are found and for first-order modes only the time to half-amplitude is found. The dependent variables of the system are then found for the specified input forcing function(s) using a step-by-step integration routine.

2. NOTATION

Any coherent system of units can be used.

		Program Input and Output Symbols
c_{11}, c_{21}	coefficients of dy_1/dx	C(1, 1), C(2, 1)
c_{12}, c_{22}	coefficients of dy_2/dx	C(1, 2), C(2, 2)
F_1, F_2	external forcing functions	F1, F2
k_{11}, k_{21}	coefficients of y_1	K(1, 1), K(2, 1)
k_{12}, k_{22}	coefficients of y_2	K(1, 2), K(2, 2)
m_{11}, m_{21}	coefficients of d^2y_1/dx^2	M(1, 1), M(2, 1)
m_{12}, m_{22}	coefficients of d^2y_2/dx^2	M(1, 2), M(2, 2)
x	independent variable of equations	X

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y_1, y_2	dependent variables of equations (independent coordinates)	Y1, Y2
ζ	damping ratio [†]	ZETA
—	flag for order of equations	ORDER
—	case numbers for F_1 and F_2 , respectively	IF1, IF2
—	magnitude of step or harmonic input of F_1 and F_2 , respectively	P1, P2
—	frequency of harmonic inputs of F_1 and F_2 , respectively	O1, O2
—	phase angle of harmonic inputs of F_1 and F_2 , respectively	Q1, Q2
—	values of x for impulse, step or harmonic input	X1, X2
—	x output step	OUTSTP
—	value of 'decimal accuracy' for integration	DACC
—	starting value of x for integration	XSTART
—	end value of x for integration	XEND
—	dy_1/dx and dy_2/dx , respectively	D(Y1)/DX, D(Y2)/DX
—	d^2y_1/dx^2 and d^2y_2/dx^2 , respectively	D2(Y1)/DX2, D2(Y2)/DX2
—	values of x and arbitrary function values of F_1 , $J = 1$ to $N1$	X3(1, J), P(1, J)
—	values of x and arbitrary function values of F_2 , $J = 1$ to $N2$	X3(2, J), P(2, J)
—	total number of values of x for an arbitrary function input for F_1 and F_2 , respectively	N1, N2
—	time to half (or double) amplitude [†]	T(1/2)
—	period of oscillation [†]	PERIOD
—	natural frequency of oscillation [‡]	WN
—	output symbol for number of subdivisions of output step used for convergence	ITER

[†] Value associated with WN and obtained from the roots of the characteristic equation (see Section 4.4).

[‡] When the damping of a system is zero, WN is the natural frequency of oscillation. When the system damping is non-zero, WN is the equivalent natural frequency that with the equivalent viscous damping ratio, ZETA, produces the damped natural frequency.

3. SYNTHESIS OF TWO-DEGREE-OF-FREEDOM SYSTEMS

3.1 General Description of Second-order Equations

A system has two-degrees-of-freedom if its configuration can be represented by two independent coordinates. The second-order differential equations which describe such a system are in general coupled in that both coordinates appear in each equation.

In the most general case for systems with constant coefficients, where the coordinates and their first and second derivatives are coupled, the two equations have the form

$$m_{11} \frac{d^2 y_1}{dx^2} + m_{12} \frac{d^2 y_2}{dx^2} + c_{11} \frac{dy_1}{dx} + c_{12} \frac{dy_2}{dx} + k_{11} y_1 + k_{12} y_2 = F_1(x) \quad (3.1a)$$

and

$$m_{21} \frac{d^2 y_1}{dx^2} + m_{22} \frac{d^2 y_2}{dx^2} + c_{21} \frac{dy_1}{dx} + c_{22} \frac{dy_2}{dx} + k_{21} y_1 + k_{22} y_2 = F_2(x) , \quad (3.1b)$$

or in matrix form,

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} d^2 y_1 / dx^2 \\ d^2 y_2 / dx^2 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} dy_1 / dx \\ dy_2 / dx \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} . \quad (3.2)$$

The two independent coordinates are y_1 and y_2 which are the dependent variables of the system equations with the independent variable of the system chosen as x . The coefficients m , c and k are constants.

The notation that has been chosen for the coefficients of y_1 and y_2 and their first and second derivatives is most descriptive of a mechanical translation system, with m denoting the mass, c the damping, and k the spring stiffness. However, such equations can equally represent an electrical, a rotational mechanical, a thermal or an acoustical system. Table 3.1 shows the relationship between some of the parameters of equivalent systems.

TABLE 3.1 Equivalent system parameters

ELECTRICAL	MECHANICAL TRANSLATIONAL	MECHANICAL ROTATIONAL	ACOUSTICAL	THERMAL
inductance	mass	moment of inertia	acoustical inertance	—
resistance	damping constant	rotational damping	acoustical resistance	thermal resistance
$\frac{1}{\text{capacitance}}$	spring stiffness	rotational stiffness	$\frac{1}{\text{acousticalcompliance}}$	$\frac{1}{\text{thermalcapacitance}}$
current	displacement	rotational angle	volume velocity	rate of heat flow
e.m.f.	force	torque	pressure	temperature change

The matrix form of Equation (3.2) immediately indicates the type of coupling that exists between the coordinates of the system. Mass or dynamical coupling exists if the mass matrix is nondiagonal and stiffness or static coupling exists if the stiffness matrix is nondiagonal. The damping matrix is said to be proportional if it is diagonal. The two equations become decoupled when all the coefficient matrices are diagonal. The equations may then be solved independently.

The systems described by Equation (3.2) include all cases in which the coefficient matrices are not symmetric, although for physical systems it is common for the mass and stiffness terms to be symmetric. Such systems include those with negative damping whose responses do not decay with time.

3.2 Example on the Formulation of the Equations of Motion

Consider the mechanical system of Sketch 3.1 with masses of m_1 and m_2 connected by springs, of stiffness k , and dashpots of damping constant c .

Using the coordinates y_1 and y_2 as the distances moved by the masses m_1 and m_2 measured from their rest positions, the differential equations that describe the system are

$$m_1 \ddot{y}_1 = -k(y_1 - y_2) - ky_1 - c(\dot{y}_1 - \dot{y}_2) - c\dot{y}_1 + F_1 \quad (3.3a)$$

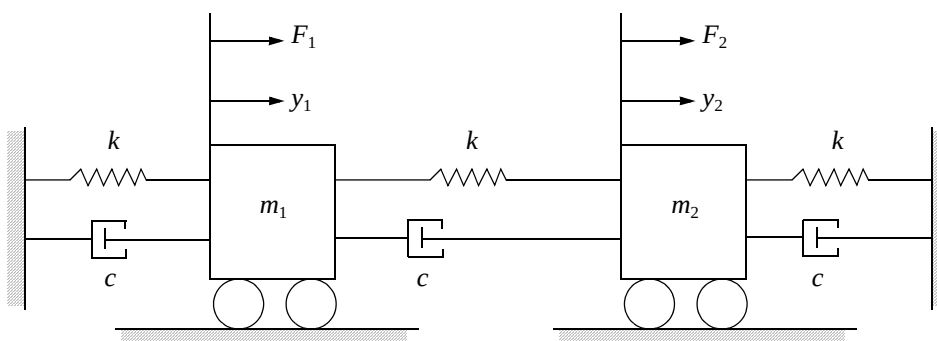
and
$$m_2 \ddot{y}_2 = k(y_1 - y_2) - ky_2 + c(\dot{y}_1 - \dot{y}_2) - c\dot{y}_2 + F_2, \quad (3.3b)$$

where F_1 and F_2 represent the external forces applied to the masses m_1 and m_2 , respectively.

Rearranging these equations into the matrix form of Equation (3.2)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} 2c & -c \\ -c & 2c \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \quad (3.4)$$

This system is seen to contain coupling in the damping and stiffness matrices only.



Sketch 3.1 Schematic of multiple-element two-degree-of-freedom system

3.3 Two-degree-of-freedom Systems Involving First- and Second-order Equations

Although all two-degree-of-freedom systems require two independent coordinates to describe their behaviour, not all involve two second-order equations. The variables of interest can for certain systems be expressed in terms of one second-order and one first-order equation, or in some cases two first-order equations, respectively described by the following:

$$\begin{bmatrix} m_{11} & 0 \\ m_{21} & 0 \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad (3.5)$$

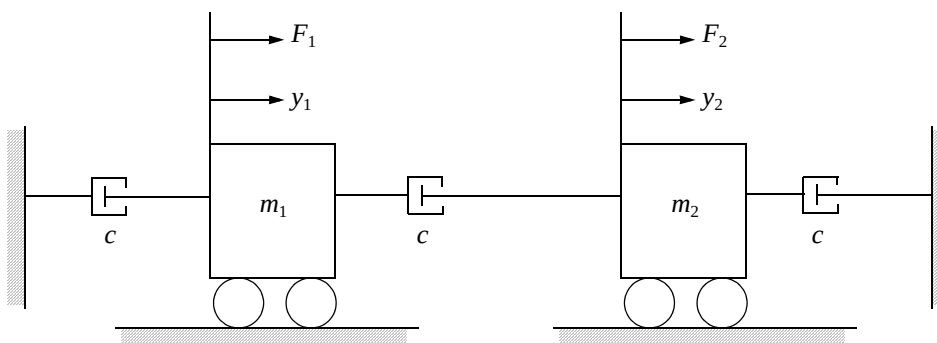
and

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \quad (3.6)$$

It should be noted that the y_2 coordinate in Equation (3.5) and both coordinates in Equation (3.6) cannot be directly associated with displacement when the systems these equations represent are viewed as 'spring/mass/damper' systems, nor will the terms m , c and k necessarily correspond to mass, damping and stiffness when associated with the coordinates used. (The symbols m , c and k have been retained, however, for continuity reasons associated with the input and output for the computer program.)

As an example consider the 'mass/damper' system illustrated in Sketch 3.2. In terms of the displacement coordinates y_1 and y_2 , the system of Sketch 3.2 is represented by two second-order equations, with coupling in the 'damping' matrix, as detailed in the following matrix equation:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} 2c & -c \\ -c & 2c \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \quad (3.7)$$



Sketch 3.2 Two-degree-of-freedom mass/damper system

The system of Equation (3.7) can be transposed by one, or both, of the simple substitutions,

$$y_3 = \dot{y}_1 \quad \text{and} \quad y_4 = \dot{y}_2 ,$$

which respectively give

$$\begin{bmatrix} m_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_4 \end{bmatrix} + \begin{bmatrix} 2c & 0 \\ -c & m_2 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_4 \end{bmatrix} + \begin{bmatrix} 0 & -c \\ 0 & 2c \end{bmatrix} \begin{bmatrix} y_1 \\ y_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad (3.8)$$

or

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} + \begin{bmatrix} 2c & -c \\ -c & 2c \end{bmatrix} \begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} . \quad (3.9)$$

Equation (3.8) involves the coordinates y_1 and y_4 and Equation (3.9) involves the coordinates y_3 and y_4 .

These systems are 'special cases' of the general system of Equation (3.2) and only deserve particular mention because the computer program associated with this Item has been written to accept these cases.

A practical example, taken from Example B of Data Item 81040^[3] in which the variation of velocity $\bar{u}$ has been ignored, expressed in the notation of that Item, is the system

$$\begin{pmatrix} \left(1 - \frac{Z\dot{w}}{\mu}\right)s - \frac{Z_w}{\tau} & -\left(\frac{Z_q}{\mu} + 1\right)U_0 \\ -\frac{M\dot{w}}{i_y \tau U_0}s - \frac{\mu M_w}{i_y \tau^2 U_0} & s - \frac{M_q}{i_y \tau} \end{pmatrix} \begin{pmatrix} \bar{w} \\ \bar{q} \end{pmatrix} = \begin{pmatrix} \frac{U_0 Z_\eta}{\tau} \\ \frac{\mu M_\eta}{i_y \tau^2} \end{pmatrix} (\bar{\eta}) , \quad (3.10)$$

which is first-order in $\bar{w}$ and $\bar{q}$, or if $\bar{q}$ is replaced by $d\bar{\theta}/dt$ then it is second-order in $\bar{\theta}$.

4. DESCRIPTION OF COMPUTER PROGRAMS

The computer program associated with this Data Item is supplied in two forms:

- (i) an executable VIEWpac[†] program, 8237Av##[‡]
- and (ii) a Fortran executable program, A8237v##.EXE[‡]

4.1 Executable VIEWpac Program

This version of the program, named 8237Av##, contains the Fortran executable code, A8237v##.EXE, wrapped within an interface called ESDUview and is for use on computers running Microsoft Windows 95, or later, operating systems. The main purpose of ESDUview is to allow:

- (i) input files to be created following descriptive screen prompts,
- (ii) embedded help to be available for every input quantity,
- (iii) program runs to be launched within the interface
- and (iv) automatic view of the main output file.

Other features of ESDUview include an editor, that may be used to view all files, and a calculator. A guide to the use of ESDUview (see ESDU 00009^[4]) is also available. The ESDUview interface is included within the ESDU web site, www.esdu.com.

4.2 Fortran Executable Program

This version of the program, named A8237v##.EXE, is a 32-bit executable code compiled with Salford Fortran 77, Version 4.x, for use on a PC running MS-DOS (Microsoft® - Disk Operating System) Windows 95 (or later) operating system.

The program accepts the main input file as an “externally directed” file, *i.e.* its name is allocated at run time externally to the program (see Section 4.2.1). The input data are entered into the named input file with the use of a text editor in the data entry order as detailed in Section 4.3.

Running instructions for the program are given in Section 4.2.1.

[†] The term VIEWpac is here used to describe an interface program that runs within ESDUview (see ESDU 00009^[4] and ESDU 04013^[5])

[‡] The symbols ## refer to an encoded version number of the program: for example if the version number is 2.2 then ## is equal to 22. Thus the full name of Version 2.2 of the executable program A8237 will be A8237v22.EXE and this name must be used when accessing the file.

4.2.1 Running Fortran program

The program accepts the input file as an “externally directed” file, *i.e.* its name is allocated at run time externally to the program. The input data are entered into the named input data file with the use of a text editor in the data entry order as detailed in Section 4.3. The output file is named within the main input file (see Section 4.3).

The program is run by entering the following command at the DOS prompt

```
A8237v##.EXE < MAIN.IN
```

where MAIN.IN is an example name of a main input file. (The inclusion of the extension.EXE is optional.)

If the main input file name is excluded from the command line (undirected input), *i.e.* the command

```
A8237v##.EXE
```

is entered, then the program will prompt the user to enter the name of the input file from the keyboard.

4.3 Program Input Data

Data are entered into the input files as detailed in the following tables. The tables specify where appropriate the input symbols used by the program which are described in the Notation of Section 2.

Any coherent system of units may be used. All data are entered in free format.

Entry Number	Entry Description	Variable Type
1	Enter program identifier: ESDU82037 (This identifier must be the only string other than spaces in the first 256 characters of the first line of the file)	CHARACTER
2, 3, 4	Enter descriptive run title – 1st line Enter descriptive run title – 2nd line Enter descriptive run title – 3rd line (The first 75 characters of each line, including spaces, are read)	CHARACTER
5	Enter file name for output data (The name of the file, including the path, are read from the first 256 characters of the line)	CHARACTER
6	Enter flag for order of equations: 'ORDER' = 4 two second-order equations = 3 one second-order equation and one first-order equation = 2 two first-order equations	INTEGER
7=	If 'ORDER' = 4, then enter coefficients: 'M(1,1), M(1,2), M(2,1), M(2,2)'	REAL
7=	If 'ORDER' = 3, then enter coefficients: 'M(1,1), M(2,1)'	REAL
8	Enter coefficients: 'C(1,1), C(1,2), C(2,1), C(2,2)'	REAL

Entry Number	Entry Description		Variable Type
9	Enter coefficients: 'K(1,1), K(1,2), K(2,1), K(2,2)'		REAL
10	Enter case numbers for forcing function inputs: 'IF1, IF2' = 0 ZERO = 1 IMPULSE = 2 STEP = 3 COSINE = 4 SINE = 5 ARBITRARY		INTEGER
11	IF1 = 0	No input required (do not insert a blank line)	-
	IF1 = 1	Enter value of independent variable at which input starts: 'X1'	REAL
	IF1 = 2	Enter value of independent variable at which input starts and value of magnitude of input: 'X1, P1'	REAL
	IF1 = 3 and IF1 = 4	Enter value of independent variable at which input starts, value of magnitude of input, value of frequency of input and value of phase of input: 'X1, P1, O1, Q1'	REAL
	IF1 = 5	Enter number of function values: 'N1'	INTEGER
11 (cont.)	IF1 = 5	Enter N1 pairs of values of independent variable (at which input starts) and magnitude of input: 'X3(1,1), P(1,1) = X3(1,2), P(1,2) = ... X3(1,N1), P(1,N1) = '	REAL
12	IF2 = 0	No input required (do not insert a blank line)	-
	IF2 = 1	Enter value of independent variable at which input starts: 'X2'	REAL
	IF2 = 2	Enter value of independent variable at which input starts and value of magnitude of input: 'X2, P2'	REAL
	IF2 = 3 and IF2 = 4	Enter value of independent variable at which input starts, value of magnitude of input, value of frequency of input and value of phase of input: 'X2, P2, O2, Q2'	REAL
	IF2 = 5	Enter number of function values: 'N2'	INTEGER
		Enter N2 pairs of values of independent variable (at which input starts) and magnitude of input: 'X3(2,1), P(2,1) = X3(2,2), P(2,2) = ... X3(2,N2), P(2,N2) = '	REAL