

APPROXIMATION TO THE ROOTS OF THE LATERAL EQUATIONS OF MOTION OF AN AIRCRAFT WITH AND WITHOUT A SIMPLE YAW DAMPER

1. INTRODUCTION

The stability of the perturbed lateral motion of an aircraft about a given trimmed flight condition can be determined from the roots of the well-known stability quartic.

This Item examines the relationship between the coefficients of this stability quartic and those of the equivalent quadratic factors. Relationships are shown to exist that, in some cases, allow an approximate solution of the modes of response of the unaugmented aircraft to be obtained directly from the coefficients of the quartic equation, without the need to solve for the quartic roots.

This Item further shows that these same relationships may be used to determine the Dutch roll characteristics of an aircraft equipped with an idealised (*i.e.* no lags) yaw damper.

Finally this Item describes a method for investigating the effects of the dynamic response of a second-order yaw damper on the stability of an aircraft/autostabiliser combination. The method is based on the study of the constant-damping curves obtained in a plane defined by two of the autostabiliser parameters. The Dutch roll approximation is again found to be suitable for simplifying the analysis. With this simplified analysis an optimum combination of values of the autostabiliser natural frequency and damping ratio are shown to exist for any gearing or required damping.

2. NOTATION

Any coherent system of units can be used.

A_5, A_4, A_3, A_2, A_1	coefficients of characteristic equation for aircraft alone
$\bar{A}_5, \bar{A}_4, \bar{A}_3, \bar{A}_2, \bar{A}_1, \bar{A}_0$	coefficients of characteristic equation for aircraft with generalised augmented control
$B_5, B_4, B_3, B_2, B_1, B_0$	coefficients of characteristic equation for aircraft with second-order yaw damper
B	coefficient of linear term in quadratic expression ($= 2\zeta\omega$)
C	coefficient of constant term in quadratic expression ($= \omega^2$)
D	differential operator, dynamic normalised ($= d/d\hat{t}$)
e_x, e_z	constant of inertia ratios $= -I_{xz}/I_x, -I_{xz}/I_z$ (see Reference 2)
F	characteristic equation
G	gearing ratio, roll equation terms (dimensional)
g	as G (nondimensional)
g	acceleration due to gravity

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H	gearing ratio, yaw equation terms (dimensional)
h	as H (nondimensional)
K_1, K_2	constants associated with coefficients of linear term for quadratic factors of aircraft alone characteristic equation
k_1, k_2	damping indices of spiral and roll modes of aircraft alone
$l_p, l_r, l_v, l_\xi, l_\zeta$	dynamic normalised concise rolling moment derivatives (see Appendix B and Reference 2)
$n_p, n_r, n_v, n_\xi, n_\zeta$	dynamic normalised concise yawing moment derivatives (see Appendix B and Reference 2)
p	roll rate
R	Routh's discriminant
R	real part of root
r	yaw rate
t	time
$t_{1/2}$	time to half-amplitude
u	aircraft velocity in "longitudinal" or X -axis direction
v	sideslip velocity
W_1, W_2	constant terms of quadratic factors of aircraft alone characteristic equation
w	velocity in "normal" or Z -axis direction
$y_v, y_p, y_r, y_\xi, y_\zeta$	dynamic normalised concise force derivatives (see Appendix B and Reference 2)
ε	phase angle
ζ	rudder angle
ζ	damping ratio
Θ	inclination angle
μ	relative density parameter (see Appendix B and Reference 2)
ξ	aileron angle
τ	unit of time in dynamic normalised system (see Appendix B and Reference 2)
ϕ	roll angle

ψ	yaw angle
ω	angular frequency
Subscripts	
A	aircraft
DR	Dutch roll
e	initial condition, time zero
o	optimum
N	natural or undamped
S	autostabiliser

A cap (^) above a quantity indicates a dynamic normalised concise quantity.

3. THE LATERAL STABILITY CHARACTERISTICS

3.1 The Equations of Motion and the Stability Quartic

The lateral small disturbance equations of motion³ of a symmetric aircraft from a datum condition of straight symmetric equilibrium flight are given.

These equations use a body-fixed axis system with the X -axis in the plane of symmetry of the aircraft. Using dynamic normalised quantities^{1, 2, 3} and the concise form for the aerodynamic derivatives, the equations of motion are:–

$$\begin{bmatrix} \hat{D} + y_v, & \hat{y}_p - \hat{w}_e, & \hat{y}_r - \hat{u}_e, & -\hat{g} \cos \Theta_e, & 0 \\ \hat{l}_v, & \hat{D} + \hat{l}_p, & e_x \hat{D} + \hat{l}_r, & 0, & 0 \\ \hat{n}_v, & e_z \hat{D} + \hat{n}_p, & \hat{D} + \hat{n}_r, & 0, & 0 \\ 0, & -1, & -\tan \Theta_e, & \hat{D}, & 0 \\ 0, & 0, & -\sec \Theta_e, & 0, & \hat{D} \end{bmatrix} \begin{bmatrix} \hat{v} \\ \hat{p} \\ \hat{r} \\ \hat{\phi} \\ \hat{\psi} \end{bmatrix} = \begin{bmatrix} -\hat{y}_\xi, & -\hat{y}_\zeta \\ -\hat{l}_\xi, & -\hat{l}_\zeta \\ -\hat{n}_\xi, & -\hat{n}_\zeta \\ 0, & 0 \\ 0, & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \quad (3.1)$$

The cap (^) has been used in Equation (3.1) to indicate the use of dynamic normalised quantities but, as is usual practice, this dressing will henceforth be dropped.

The stick-fixed characteristic equation is the determinant of the response matrix of Equation (3.1) and may be written in the form,

$$D(A_5 D^4 + A_4 D^3 + A_3 D^2 + A_2 D + A_1) = 0, \quad (3.2)$$

where

$$\begin{aligned}
 A_5 &= 1 - e_x e_z, \\
 A_4 &= y_v(1 - e_x e_z) + n_r + l_p - (e_x n_p + e_z l_r), \\
 A_3 &= y_v\{n_r + l_p - (e_x n_p + e_z l_r)\} + l_p n_r - l_r n_p \\
 &\quad + l_v\{w_e - y_p + e_z(u_e + y_r)\} + n_v\{-e_x(w_e - y_p) - (u_e + y_r)\}, \\
 A_2 &= -y_v(l_r n_p - l_p n_r) + l_v\{n_r(w_e - y_p) + n_p(u_e + y_r)\} + n_v\{-l_r(w_e - y_p) - l_p(u_e + y_r)\} \\
 &\quad + g \cos \Theta_e \{l_v(-e_z \tan \Theta_e + 1) - n_v(-\tan \Theta_e + e_x)\} \\
 \text{and} \quad A_1 &= g \cos \Theta_e \{l_v(-\tan \Theta_e n_p + n_r) + n_v(l_p \tan \Theta_e - l_r)\}.
 \end{aligned}$$

Restricting attention to a level flight initial condition and choosing aerodynamic body axes³ with the X -axis aligned to the velocity vector,

$$u_e = 1, w_e = 0 \text{ and } \Theta_e = 0.$$

In practice the derivatives y_p and y_r are small compared with other terms and may be neglected.

The coefficient expressions then reduce to,

$$\left. \begin{aligned}
 A_5 &= 1 - e_x e_z, \\
 A_4 &= y_v(1 - e_x e_z) + n_r + l_p - (e_x n_p + e_z l_r), \\
 A_3 &= y_v\{n_r + l_p - (e_x n_p + e_z l_r)\} + l_p n_r - l_r n_p + l_v e_z - n_v, \\
 A_2 &= -y_v(l_r n_p - l_p n_r) + l_v n_p - n_v l_p + g(l_v - e_x n_v) \\
 \text{and} \quad A_1 &= g(l_v n_r - n_v l_r).
 \end{aligned} \right\} \quad (3.3)$$

Appendix A gives the equivalent stick-fixed lateral equations of motion and the coefficient expressions for the characteristic equation in terms of aeronormalised quantities.

Appendix B provides the relationships between the old British, the new ISO and the American notations.

3.2 The Relationship between the Coefficients of the Stability Quartic and those of the Equivalent Quadratic Factors

The lateral stability quartic usually factorises to produce an oscillatory mode, called the Dutch roll oscillation, and two exponential modes, the roll subsidence and the spiral mode. The quartic equation, given by Equation (3.2), can then be expressed as,

$$(D^2 + 2K_1 D + W_1^2)(D + k_1)(D + k_2) = 0, \quad (3.4)$$

from which, if the characteristics are expressed in real time, the following are found.

$$\text{Dutch roll natural frequency} = W_1/\tau \text{ (rad/s)},$$

$$\text{Dutch roll damping ratio, } \zeta_{DR} = K_1/W_1,$$

$$\begin{aligned}\text{Dutch roll damped frequency} &= (W_1^2 - K_1^2)^{1/2}/\tau \text{ (rad/s)}, \\ \text{Dutch roll damping index} &= K_1/\tau \text{ (1/s)}\end{aligned}$$

and the damping indices of the roll and spiral exponential modes are k_1/τ and k_2/τ (1/s).

The roll mode and the spiral mode can combine to give a second oscillation, but this places no restriction on the analysis, and in this case the stability quartic described by Equation (3.4) will take the form,

$$(D^2 + 2K_1D + W_1^2)(D^2 + 2K_2D + W_2^2) = 0, \quad (3.5)$$

where $2K_2$ and W_2^2 simply replace $k_1 + k_2$ and k_1k_2 , respectively.

Equation (3.4) may be expanded and rewritten in the form,

$$\begin{aligned}D^4 + \{2K_1 + k_1 + k_2\}D^3 + \left\{W_1^2 + 2K_1(k_1 + k_2) + k_1k_2\right\}D^2 \\ + \left\{2K_1k_1k_2 + W_1^2(k_1 + k_2)\right\}D + \left\{W_1^2k_1k_2\right\} = 0\end{aligned}$$

and by equating the coefficients of this Equation to the coefficients of Equation (3.2) the following expressions are found,

$$\left. \begin{aligned}2K_1 + k_1 + k_2 &= A_4/A_5, \\ W_1^2 + 2K_1(k_1 + k_2) + k_1k_2 &= A_3/A_5, \\ 2K_1k_1k_2 + W_1^2(k_1 + k_2) &= A_2/A_5, \\ W_1^2k_1k_2 &= A_1/A_5.\end{aligned} \right\} \quad (3.6)$$

A significant advantage is gained by examining the quadratic factor form of the stability equation in that simpler conditions for stability are obtained.

These necessary and sufficient conditions are,

$$K_1 > 0, W_1^2 > 0, k_1 > 0 \text{ and } k_2 > 0.$$

The equivalent conditions in terms of the quartic coefficients are,

$$A_4 > 0, A_2 > 0, A_1 > 0, \text{ and } R > 0,$$

where R (Routh's discriminant) $= A_4A_3A_2 - A_2^2A_5 - A_4^2A_1$.

Using the more general form of Equation (3.4) given by Equation (3.5) the following expressions are obtained,

$$\left. \begin{aligned} 2K_1 + 2K_2 &= A_4/A_5, \\ W_1^2 + (2K_1)(2K_2) + W_2^2 &= A_3/A_5, \\ 2K_1 W_2^2 + 2K_2 W_1^2 &= A_2/A_5, \\ W_1^2 W_2^2 &= A_1/A_5. \end{aligned} \right\} \quad (3.7)$$

An exact algebraic method of reducing the solution of the quartic equation to that of a cubic^{4, 5} has the virtue of not requiring an arbitrary initial guess at the values of the roots and is particularly useful when the quartic has no real roots and cannot be approximated in a standard way. Multiplying the second Equation of (3.7) by $2K_1$ and $2K_2$ in turn, and subtracting from these the third Equation of (3.7) the following two Equations may be found,

$$\begin{aligned} W_1^2(2K_1 - 2K_2) &= 2K_1 \frac{A_3}{A_5} - \frac{A_2}{A_5} - (2K_1)^2 2K_2 \\ \text{and} \quad W_2^2(2K_1 - 2K_2) &= \frac{A_2}{A_5} - 2K_2 \frac{A_3}{A_5} + 2K_1(2K_2)^2. \end{aligned}$$

Multiplication of these and the use of the fourth Equation of (3.7) yields,

$$\frac{A_1}{A_5}(2K_1 - 2K_2)^2 = \left\{ 2K_1 \frac{A_3}{A_5} - \frac{A_2}{A_5} - (2K_1)^2 2K_2 \right\} \left\{ \frac{A_2}{A_5} - 2K_2 \frac{A_3}{A_5} + 2K_1(2K_2)^2 \right\}$$

and using the first Equation of (3.7) the following Equation in $2K_1 2K_2$ can be found,

$$\begin{aligned} A_5^3(2K_1 2K_2)^3 - 2A_3 A_5^2(2K_1 2K_2)^2 + (A_3^2 + A_2 A_4 - 4A_1 A_5)A_5(2K_1 2K_2) \\ - (A_2 A_3 A_4 - A_2^2 A_5 - A_4^2 A_1) = 0. \end{aligned} \quad (3.8)$$

The constant term in Equation (3.8) is seen to be Routh's discriminant and when it is zero the neutral stability boundary for each response mode can be found. Thus,

$$A_1 = \frac{A_2 A_3}{A_4} - \frac{A_2^2 A_5}{A_4^2}$$

and in terms of the original quartic, Equation (3.2), the following factorisation is found,

$$\left\{ D^2 + \frac{A_2}{A_4} \right\} \left\{ A_5 D^2 + A_4 D + \left(A_3 - \frac{A_2 A_5}{A_3} \right) \right\} = 0. \quad (3.9)$$

The roots of this Equation can be immediately evaluated to give the neutral stability boundaries.

3.3 Approximate Solution of the Quartic

When Routh's discriminant is not zero, an approximate root of Equation (3.8) may be found by neglecting the square and cubic terms in $(2K_1 2K_2)$. This gives,

$$2K_1 2K_2 = \frac{A_2 A_3 A_4 - A_2^2 A_5 - A_4^2 A_1}{A_5 (A_3^2 + A_2 A_4 - 4A_1 A_5)}. \quad (3.10)$$

For most present day aircraft which have high wing loading, the terms in Equation (3.10) involving the relative density parameter, μ , are usually sufficiently large to allow a further approximation. Retaining only these dominant terms, the following expression results,

$$2K_1 2K_2 = \frac{A_2 A_3 A_4 - A_2^2 A_5}{A_5 A_3^2} \quad (3.10a)$$

or

$$2K_1 2K_2 = \left(\frac{A_4}{A_5} - \frac{A_2}{A_3} \right) \left(\frac{A_2}{A_3} \right).$$

By inspection,

$$2K_1 = \left(\frac{A_4}{A_5} - \frac{A_2}{A_3} \right)$$

and

$$2K_2 = (A_2/A_3).$$

These are the same relations as those obtained by assuming that W_1 is much larger than W_2 , K_1 and K_2 and using Equation (3.7), the following approximations are found,

$$2K_1 + 2K_2 = A_4/A_5,$$

$$W_1^2 \approx A_3/A_5,$$

$$W_1^2 (2K_2) \approx A_2/A_5,$$

$$W_1^2 W_2^2 = A_1/A_5,$$

which, when rearranged give,

$$\left. \begin{aligned} W_1^2 &\approx A_3/A_5, \\ 2K_2 &\approx A_2/A_3, \\ 2K_1 &\approx \frac{A_4}{A_5} - \frac{A_2}{A_3}, \\ W_2^2 &\approx A_1/A_3. \end{aligned} \right\} \quad (3.11)$$

These approximations cannot be applied to the general case particularly for low values of the relative density parameter, μ , and when R is close to zero.

A quick check on the suitability of the approximation of Equation (3.10a) can be made by examining the size of the discarded terms in both the numerator and the denominator of Equation (3.10). If these discarded terms are found to be small compared with the other terms then it is likely that the approximate relationships of Equation (3.11) will provide a good Dutch roll approximation.

3.4 Example of Approximation

The approximations that have been discussed in the earlier Sections are here used to find the characteristics of two aircraft and these are compared with the full three-degrees-of-freedom solution.

The aircrafts' details are as follows:-

	<i>Aircraft 1</i>	<i>Aircraft 2</i>
Flight Conditions		
Altitude (ft)	30000	30000
Speed (ft/s)	495	795
Air Density (slug/ft ³)	0.000890	0.000890
Mass, Inertia and Geometrical Properties		
Weight (lbf)	38790	8450
Inertia about X-axis (slug ft ²)	25850	1990
Inertia about Z-axis (slug ft ²)	140900	10554
Inertia about the X Z axis (slug ft ²)	2600	298
Characteristic area (ft ²)	538	130
Characteristic length (ft)	38.4	28
Unit of time (s)	10.17	5.71
Inclination angle (°)	10.0	0.0
Relative density parameter	131.2	162.1
e_x	-0.1	-0.15
e_z	-0.0185	-0.0282
$\hat{g}$	0.70	0.23
Dynamic, Normalised Concise Derivatives		
y_v	1.289	1.0
y_ζ	0	0
l_v	1131	2106
n_v	-379	-790
l_p	9.79	20.62
n_p	-0.584	0.195
l_r	-5.905	-4.124
n_r	3.017	3.9
l_ζ	0	0
n_ζ	237	515.1

Tables 3.1 and 3.2 give the characteristics for Aircraft 1 and 2 respectively for both the three-degrees-

of-freedom solution using Equation (3.3), and the approximate solution using Equation (3.11), together with the percentage errors that results by the use of the approximation.

TABLE 3.1

AIRCRAFT 1	$A_5 = 0.998$ $A_4 = 14.02$ $A_3 = 585.8$ $A_2 = 3970$ $A_1 = 509.4$		
	<i>Exact Solution</i>	<i>Approx. Solution</i>	<i>% Error</i>
Dutch Roll – nat. frequency (rad/s)	2.28	2.38	4.4
– damping ratio	0.143	0.150	4.9
– period (s)	2.79	2.67	4.3
– time to ½ amplitude (s)	2.12	1.94	8.5
Roll mode damping index	0.714	0.653	8.5
Spiral mode damping index	0.0129	0.0129	0

TABLE 3.2

AIRCRAFT 2	$A_5 = 0.9958$ $A_4 = 25.43$ $A_3 = 836.3$ $A_2 = 17239$ $A_1 = 1140$		
	<i>Exact Solution</i>	<i>Approx. Solution</i>	<i>% Error</i>
Dutch Roll – nat. frequency (rad/s)	4.87	5.09	4.5
– damping ratio	0.055	0.085	55
– period (s)	1.29	1.24	3.9
– time to ½ amplitude (s)	2.59	1.60	38
Roll mode damping index	3.937	3.609	8.3
Spiral mode damping index	0.0116	0.0116	0

It is seen from the results given in Table 3.1 that the approximate solutions to all the modes of response for Aircraft 1 are good.

For Aircraft 2 it is seen that in percentage terms the approximation of the damping of the Dutch roll mode is poor, though in absolute terms the approximation is fair. The damping of this mode is small and consequently Routh's discriminant will also be small. Examination of the terms in both the numerator and