

SUMMARY OF THE CONCEPTS RELATING TO RANDOM PROCESSES

1. NOTATION

$A(if)$	Fourier transform of $x(t)$
a_n, b_n, c_n	Fourier series coefficients
$C_{xx}(\tau)$	autovariance function of a non-stationary or stationary random process $\{k_x(t)\}$ at time difference τ
$C_{xy}(\tau)$	covariance function of two non-stationary or stationary random processes $\{k_x(t)\}$ and $\{k_y(t)\}$ at time difference τ
f	frequency
$K_{xx}(\tau)$	autocorrelation function of a non-stationary or stationary random process $\{k_x(t)\}$ at time difference τ
$K_{xy}(\tau)$	cross-correlation function of two non-stationary or stationary random processes $\{k_x(t)\}$ and $\{k_y(t)\}$ at time difference τ
k	a variable denoting a member of an ensemble
N	the number of members of an ensemble
n	the number of coefficients of a Fourier Series
P_N	N 'th order probability distribution function
$R_{xx}(\tau)$	autocorrelation or autovariance function of an ergodic random process $\{k_x(t)\}$ at time difference τ
$R_{xy}(\tau)$	cross-correlation or covariance function of two ergodic random processes $\{k_x(t)\}$ and $\{k_y(t)\}$ at time difference τ
$S(f)$	spectral density function
$-T, T$	time limits of a member function
t	time
$t_1, t_2, \dots$	time values of a member
W_N	N 'th order probability density function
$W_N(\mid)$	N 'th order conditional probability density function
$k_x(t), k_y(t), k_z(t)$	member variables of an ensemble
τ	a time difference, <i>e.g.</i> equal to $(t_1 - t_2)$

ϕ	phase angle
ω	angular frequency of a function

Special symbols

$\langle F[k_X(t_1)] \rangle$	an ensemble average of the function $F[k_X(t)]$ at $t = t_1$
$\bar{F}[k_X(t)]$	a time average of the function $F[k_X(t)]$
$\{k_X(t)\}$	an ensemble of time functions of a random process consisting of the members functions $k_X(t)$, $k = 1, 2, \dots, N$, for $N \rightarrow \infty$
$\langle \rangle$	denotes the ensemble average of a quantity
Pr	stands for “the probability that”

2. INTRODUCTION

This Item presents a summary of random processes and is intended for use by those who require a reference to the basic concepts and definitions. It is suited to those who have not previously encountered the subject or only have a nodding acquaintance with it but will also act as a reference guide to those familiar with the subject. No previous knowledge of statistics is assumed.

There are numerous examples of random processes such as the fluctuation of noise currents in an electrical circuit, fluid motion turbulence or the vibration response of structures due to wind loading. Each of these cases is characterised by an inability to describe precisely the future behaviour of a quantity from past knowledge of that quantity. The need exists however to describe the general character of a “randomness” so that a degree of prediction can be achieved.

Random processes are therefore quantified using statistical properties which describe in the amplitude, time and frequency domains quantities representative of the “level distribution” and the “shape characteristics” of the process.

This Item adopts a strictly mathematical approach to the presentation of the concepts relating to random processes. This is necessary so that it is clear what limitations and assumptions apply to each of the given relationships.

3. DETERMINISTIC PROCESSES

A process is called deterministic when a known past record of a quantity, which belongs to the process, allows all future values of the quantity to be determined.

As an example a process which is characterised by the time-dependent periodic function,

$$x(t) = A \sin(\omega t + \phi), \quad (3.1)$$

where A and ϕ are constants, for $t < T$. Then Equation (3.1) is taken to hold for $t > T$.

4. RANDOM PROCESSES

There are many processes of interest which are not deterministic, that is, processes in which future values cannot be completely determined from past knowledge of the process. Such processes are random.

4.1 Definition

It is necessary to describe random processes in terms of a mathematical formulation. Suppose that N identical experiments (experiments in which identical apparatus is used in identical situations) are conducted simultaneously and that a quantity, $^k x(t)$, is recorded, where k refers to the specific experiment. The “ensemble” of records $\{^k x(t)\}$ for $-\infty < t < \infty$ and $k = 1, N$ where $N \rightarrow \infty$, defines the random process. It is usual to use curly brackets around the record, $\{^k x(t)\}$, to denote the ensemble of all possible time functions and keep the unbracketed quantity, $^k x(t)$, to denote a single representative record, or “member” record, of the ensemble.

The above definition of a random process has time as the independent variable and furthermore each record, $^k x(t)$, is taken to be a continuous function of time. The independent variable need not be time, *e.g.* $\{^k x(d)\}$ might represent the vibration response of vehicles to road roughnesses where the independent variable, d , is distance.

It is possible to have discrete random processes, for example the throwing of dice where the independent variable would be the number of times the dice are thrown.

4.2 Ensemble Averages

Consider the ensemble $\{^k x(t)\}$ illustrated in Sketch 4.1, where member records $^1 x(t)$, $^2 x(t)$ and $^N x(t)$ have been plotted as a function of t . Each member extends mathematically from $t = -\infty$ to $t = \infty$ and the complete ensemble or random process is given when $N \rightarrow \infty$.

The properties of the random process are expressed in terms of ensemble averages.

An ensemble average is found by summing in a ‘vertical’ direction over Sketch 4.1 for all k at a chosen time, say $t = t_1$, and dividing this sum by the total number of members. This ensemble average is dependent upon t_1 and independent of k .

The ensemble average of $\{^k x(t)\}$ at $t = t_1$ is mathematically defined as,

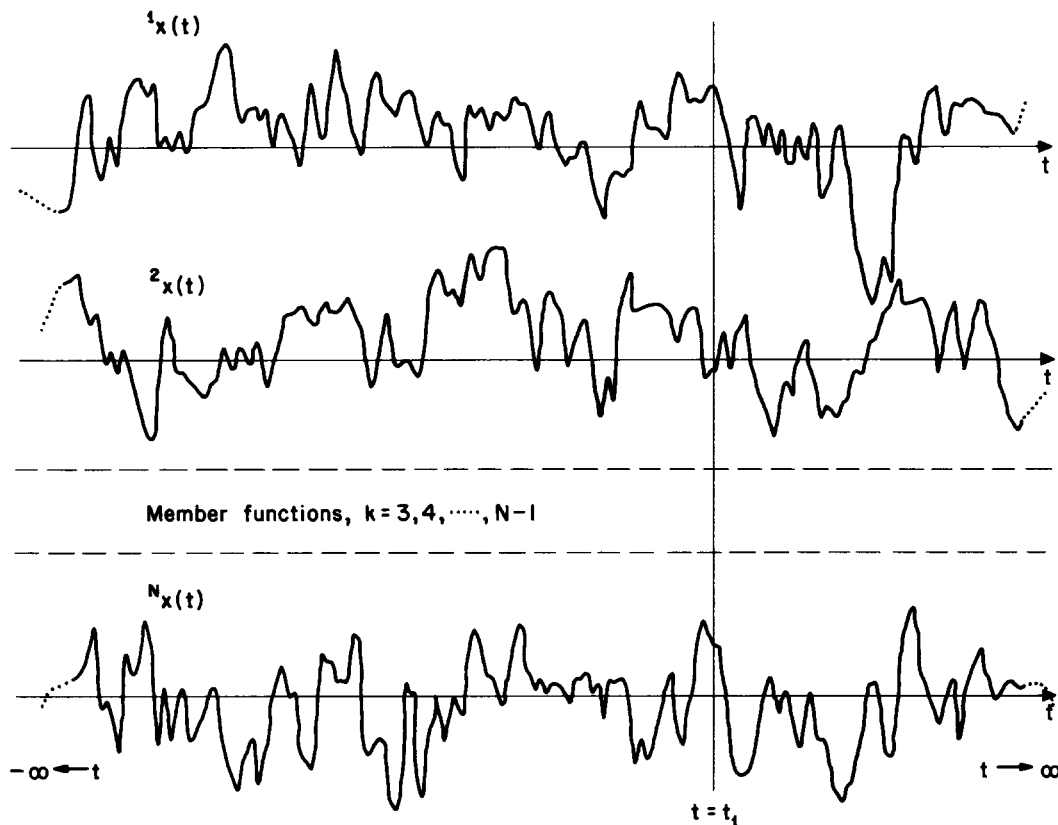
$$\langle ^k x(t_1) \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{k=1}^N ^k x(t_1) \right), \quad (4.1)$$

where the angular brackets “ $\langle \rangle$ ” around the quantity of interest indicates an ensemble average.

Equation (4.1) is known as the *ensemble mean average* value of the process at time t_1 .

The *ensemble mean square* value of the process at time t_1 is similarly defined as,

$$\langle (^k x(t_1))^2 \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{k=1}^N (^k x(t_1))^2 \right). \quad (4.2)$$



Sketch 4.1 Ensemble member records

If a quantity of interest is a particular function of each record, $k_X(t)$, at t_1 , say $F[k_X(t)]$, then the ensemble average is given by,

$$\langle F[k_X(t_1)] \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{k=1}^N F[k_X(t_1)] \right) \quad \dagger \quad (4.3)$$

More general averages, other than those which are simply functions of one value of time can be formulated, for example the ensemble average of $\langle k_X(t_1) \cdot k_X(t_2) \rangle$, yields the *autocorrelation function* of the random process $\{k_X(t)\}$ at times t_1 and t_2 , which is given by,

$$\langle k_X(t_1) \cdot k_X(t_2) \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{k=1}^N k_X(t_1) \cdot k_X(t_2) \right). \quad (4.4)$$

As $t_2 \rightarrow t_1$ the autocorrelation function reduces to the ensemble mean square value of the process at t_1 .

[†] An alternative notation for the ensemble average, which some authors adopt, is to use a curly bar ($\sim$) over the function $\tilde{F}[k_X(t_1)]$, in which the suffix k does not appear, indicating that the sum has been taken over all k .

Extending Equation (4.4) to ensemble averages, involving $t_1, t_2, \dots, t_M$, the following expression is obtained,

$$\langle k_X(t_1) \cdot k_X(t_2) \dots k_X(t_M) \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N (k_X(t_1) \cdot k_X(t_2) \dots k_X(t_M)). \quad (4.5)$$

A random process is completely defined from a knowledge of all the ensemble averages for $M = 1$ for all t_1 , for $M = 2$ for all t_1 and t_2 , up to $M \rightarrow \infty$ for all $t_1, t_2, t_3, \dots, t_\infty$. Successive knowledge of all these ensemble averages given by Equation (4.5) describes the random process in more and more detail.

4.3 Probability

Probability is a measure of the ratio of the probable occurrence of a particular event to the occurrence of all possible events. An event which cannot possibly occur has zero probability of occurrence and an event which is certain to occur has a probability of unity. Any other event will therefore have a probability of occurrence of between zero and unity.

The probability that $k_X(t) \leq X_1$ at $t = t_1$ for any member k , is usually written as $Pr[k_X(t_1) \leq X_1]$ and referred to as the first-order *probability distribution function*, $P_1(X_1, t_1)$, and so,

$$P_1(X_1, t_1) = Pr[k_X(t_1) \leq X_1], \quad \text{at time } t = t_1 \text{ for any } k. \quad (4.6)$$

By definition $P_1(\infty, t_1) = 1$

and $P_1(-\infty, t_1) = 0$.

There is a relationship between $P_1(X_1, t_1)$ and the ensemble $\{k_X(t)\}$. If a function $F[k_X(t_1)]$ is defined by

$$\begin{aligned} F[k_X(t_1)] &= 1, \quad \text{when } k_X(t_1) < X_1 \\ &= 0, \quad \text{when } k_X(t_1) > X_1 \end{aligned} \quad (4.7)$$

then,

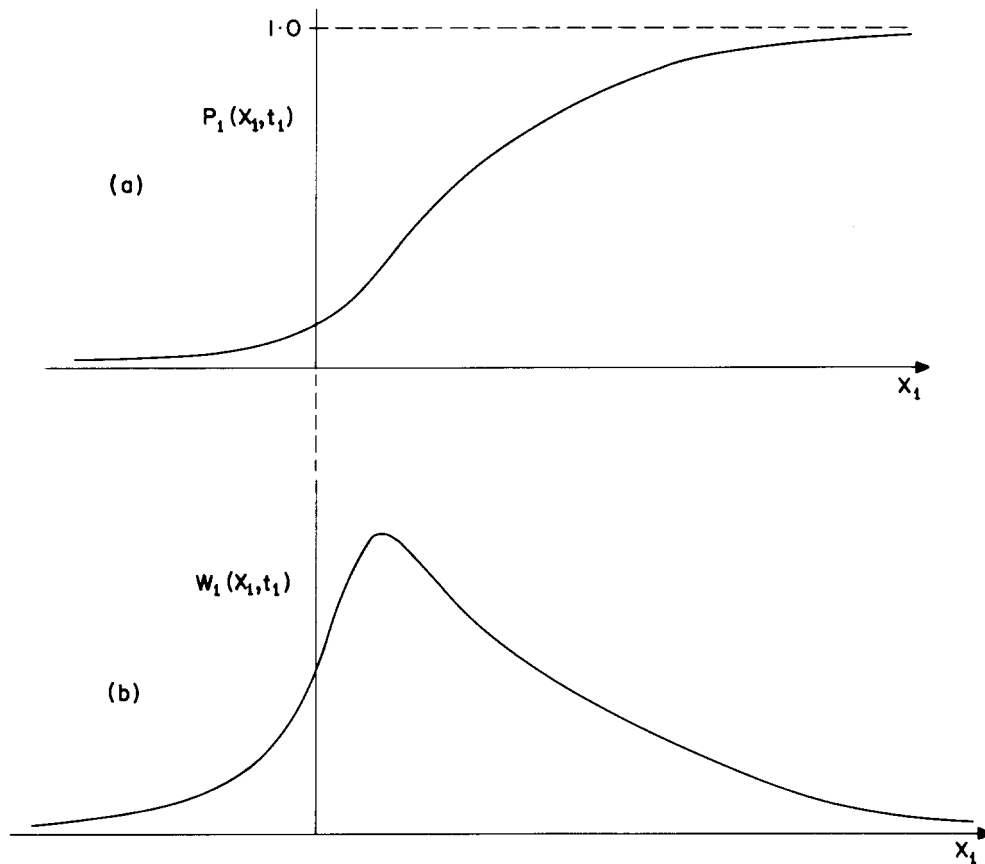
$$P_1[X_1, t_1] = \langle F[k_X(t_1)] \rangle = \frac{1}{N} \sum_{k=1}^N F[k_X(t_1)]. \quad (4.8)$$

The probability distribution function associated with the ensemble $\{k_X(t_1)\}$ at time t_1 , for all values of X_1 between $-\infty$ and ∞ , is illustrated in Sketch 4.2(a).

The probability that $k_X(t)$ at $t = t_1$ lies between X_1 and $X_1 + \Delta X_1$ is defined as

$$W_1(X_1, t_1) \cdot \Delta X_1 = Pr[X_1 < k_X(t_1) \leq X_1 + \Delta X_1], \quad (4.9)$$

where, $W_1(X_1, t_1)$ is the first-order *probability density function* and is illustrated in Sketch 4.2(b).



Sketch 4.2 (a) Probability distribution (b) Probability density

Thus, by definition

$$P_1(X_1, t_1) = \int_{-\infty}^{X_1} w_1(X_1, t_1) dX_1, \quad (4.10)$$

and

$$w_1(X_1, t_1) = \frac{dP_1(X_1, t_1)}{dX_1}. \quad (4.11)$$

The probability that $k_X(t) \leq X_1$ at $t = t_1$ and simultaneously that $k_X(t) \leq X_2$ at $t = t_2$ is given by the second-order probability distribution function, $P_2(X_1, t_1; X_2, t_2)$.

The probability that $k_X(t)$ lies between X_1 and $X_1 + \Delta X_1$ at time t_1 and simultaneously that $k_X(t)$ lies between X_2 and $X_2 + \Delta X_2$ at time t_2 is given by $W_2(X_1, t_1; X_2, t_2) \Delta X_1 \Delta X_2$ where $W_2(X_1, t_1; X_2, t_2)$ is the second-order probability density function. The second-order probability distribution and density function are related by,

$$P_2(X_1, t_1; X_2, t_2) = \int_{-\infty}^{X_1} \int_{-\infty}^{X_2} W_2(X_1, t_1; X_2, t_2) dX_1 dX_2. \quad (4.12)$$

In terms of the ensemble, if

$$\begin{aligned} F[k_X(t)] &= 1, \text{ when } k_X(t) < X_1 \text{ at time } t_1 \\ &\text{and when } k_X(t) < X_2 \text{ at time } t_2 \\ &= 0, \text{ otherwise} \end{aligned} \quad (4.13)$$

$$\text{then } P_2(X_1, t_1; X_2, t_2) = \langle F[k_X(t)] \rangle. \quad (4.14)$$

Third-order and higher-order probability functions can be defined. Knowledge of all the W_N , or P_N , functions as $N \rightarrow \infty$ for $-\infty < X_N < \infty$ completely determines all the statistical properties of the random process, and is an alternative to the descriptions of Equation (4.5).

Knowledge of W_N (or P_N) implies knowledges of all the lower order functions (W_{N-1} , W_{N-2} , etc.) since,

$$W_{N-1}(X_1, t_1; \dots; X_{N-1}, t_{N-1}) = \int_{-\infty}^{+\infty} W_N(X_1, t_1; \dots; X_N, t_N) dX_N. \quad (4.15)$$

The basis of Equation (4.15) is that W_{N-1} is independent of the values of $\{k_X(t)\}$ at time t_N , while the integration of W_N with respect to X_N covers all possible values of $\{k_X(t)\}$ at time t_N .

Now,

$$\begin{aligned} \langle k_X(t_1) \rangle &= \frac{1}{N} \sum_{k=1}^N k_X(t_1), \\ &= \sum_{X_i=-\infty}^{X_i=+\infty} X_i \left[\frac{\text{no. of times } X_i < k_X(t_1) < X_i + \Delta X_i}{N} \right], \end{aligned} \quad (4.16)$$

$$\langle k_X(t_1) \rangle = \int_{-\infty}^{+\infty} X_1 \cdot W_1(X_1, t_1) dX_1 \quad (4.17)$$

Similarly for any real single-valued continuous function $F[k_X(t_1)]$,

$$\langle F[k_X(t_1)] \rangle = \int_{-\infty}^{+\infty} F(X_1) \cdot W_1(X_1, t_1) dX_1. \quad (4.18)$$

Higher order averages (or higher moments) can be similarly defined; thus,

$$\langle k_X(t_1) \cdot k_X(t_2) \dots k_X(t_N) \rangle = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} (X_1 X_2 \dots X_N) \cdot W_N(X_1, t_1; X_2, t_2; \dots; X_N, t_N) dX_1 \dots dX_N. \quad (4.19)$$

This equation shows how the ensemble averages and the probability density functions are related.

4.4 Conditional Probability

To formulate the concept of the degree of randomness of a random process it is necessary to define the *conditional probability distribution*.