

NONLINEAR SYSTEMS: AN INTRODUCTION TO DESCRIBING FUNCTIONS

1. NOTATION

A_n, B_n	coefficients of terms in Fourier series
C	magnitude of output signal from a system
$c(\omega t)$	output signal from a system [$= C \sin(\omega t)$]
$G(s)$	open loop transfer function
K	gain
N	Describing Function of a nonlinear element
n	integer variable
M	magnitude of output from ideal relay
R	magnitude of input signal to a system
$r(\omega t)$	input signal to a system [$= R \sin(\omega t)$]
s	Laplace operator
t	normalised time
X	magnitude of input signal to a nonlinear element
$x(\omega t)$	input signal to a nonlinear element [$= X \sin(\omega t)$]
x_b	backlash parameter (see Section A8)
x_d	deadzone parameter (see Section A1)
x_h	hysteresis parameter (see Section A6)
x_s	saturation parameter (see Section A2)
Y	magnitude of chosen output sine wave from a nonlinear element
$y^*(\omega t)$	chosen output sine wave from a nonlinear element [$= Y \sin(\omega t + \phi)$]
$y(\omega t)$	non-sinusoidal output signal from a nonlinear element
α_b	$\cos^{-1}[(X - 2x_b)/X]$
α_d	$\sin^{-1}(x_d/X)$
α_h	$\sin^{-1}(x_h/X)$

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α_s	$\sin^{-1}(x_s / X)$
γ	normalised pole of first-order factor [see Equation (8.2)]
ζ	damping ratio
ϕ	phase angle
σ	pole of first-order factor [see Equation (8.1)]
ω	frequency
ω_N	natural frequency

2. INTRODUCTION

Many systems, encountered in engineering, are described by nonlinear differential equations, the nonlinear characteristics arising from either the plant dynamics or the control elements. When the nonlinear effects are small, or when the parameters of a system vary slowly with time, it is often possible to reduce the system equations to a linear form and so use the powerful techniques of linear constant-parameter analysis. In such cases the predicted results are adequate for engineering purposes. But often it is not possible to linearise the system in this way and so nonlinear equations must be used.

By definition a linear system is one which gives a sine wave output for a sine wave input. The output sine wave will have the same frequency as the input frequency but there will be a change in amplitude and phase angle. The response of the system to one type of input can be inferred from knowledge of the response to another type of input.

A nonlinear system can be defined as a system which gives a periodic non-sinusoidal output for a sinusoidal input. Such a system can exhibit certain phenomena such as self-sustaining oscillations, jump resonance, the generation of sub-harmonic and harmonic frequencies, the dependence of the response and stability on input amplitude and initial conditions, and the existence of multiple equilibrium points. None of these phenomena can exist in linear systems.

This Item presents an introduction to Describing Functions which are used to represent the part of a system that contains the nonlinear features of the system. Typical examples of such parts, or elements, are given in Section 3. The Describing Function uses a linearisation technique which approximates the nonlinear element by a transfer function that is valid only for a sinusoidal input. It is a method which is adequate for design purposes and enables an estimate to be made of the stability and the steady-state response of nonlinear systems.

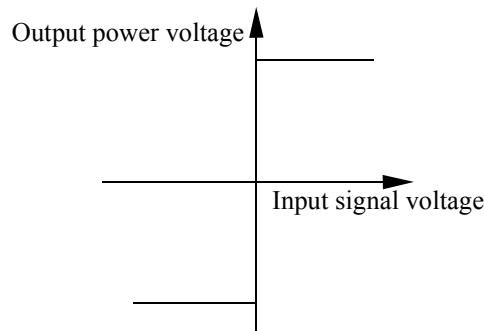
3. SOME COMMON PHYSICAL NONLINEAR ELEMENTS

A control system may contain elements which exhibit nonlinear behaviour. Such elements may be inherent in the plant dynamics, such as the backlash in the gears of a motor or the saturation effects of the limited travel of an actuator; or be controllers, typical examples being on-off temperature controllers or the jet thrusters of space vehicles.

Typical examples of nonlinear elements are listed below.

(i) Ideal Relay

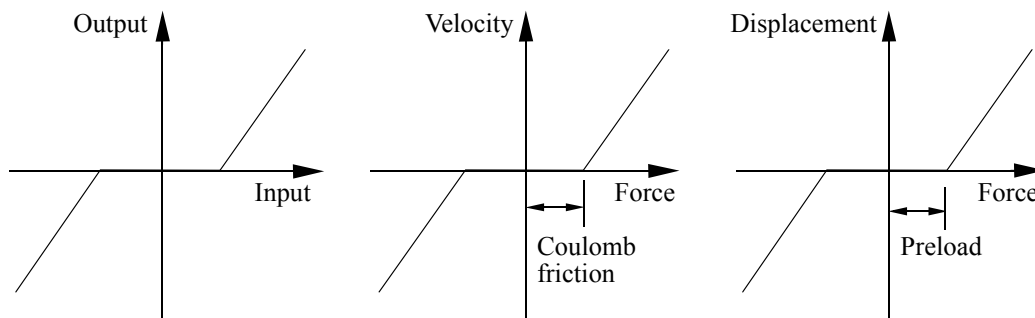
The ideal relay, illustrated in Sketch 3.1, is characterised by a step discontinuity of the output signal. The relay is the simplest possible amplifier and advantages in terms of reliability, weight and space savings, with a resultant saving in cost can be achieved by the use of this device in place of proportional amplifiers.



Sketch 3.1 Ideal relay characteristics

(ii) Deadzone

Deadzone, alternatively known as deadspace, threshold or flat spot, is a feature of many mechanical components and gives zero output over a range of input. It is illustrated in Sketch 3.2.



Sketch 3.2 Deadzone characteristics

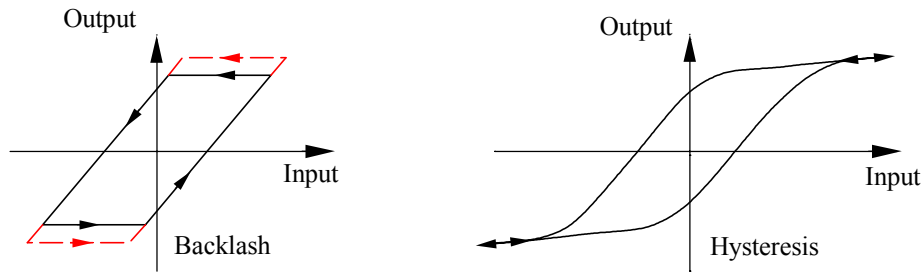
Coulomb friction and preload are normally treated as a deadzone where the input to the nonlinear element is force and the output is velocity and displacement respectively. Deadzone is often added to an ideal relay to reduce the oscillations, and wear, of the relay contacts.

(iii) Backlash and Hysteresis

Backlash is the looseness or play between the input and output of a mechanical component, such as the play between the gears of a motor. It is the variation in the input to the component which causes no change in the output and occurs each time there is a reversal of the input.

The element of a component which characterises backlash is illustrated in Sketch 3.3, where the solid line shows the closed-path variation for a particular cyclic variation of the input. If the maximum value of the input is increased then the output will increase and this is illustrated in the sketch by the dotted line. Backlash

is seen to be multi-valued and is known as a memory-type nonlinear characteristic. Backlash can be friction-controlled, which is illustrated in the sketch, or inertia-controlled. The former type occurs when the output member remains in contact with the input member until the velocity is zero. The output then remains constant until the backlash is taken up, at which point the output is assumed instantaneously to take on the velocity of the input. Inertia-controlled backlash occurs when the inertia dominates the friction and the output member remains in contact with the input member as long as the acceleration is in a direction to keep the backlash spacing closed. When the acceleration becomes zero, the output member is assumed to coast at a constant velocity.



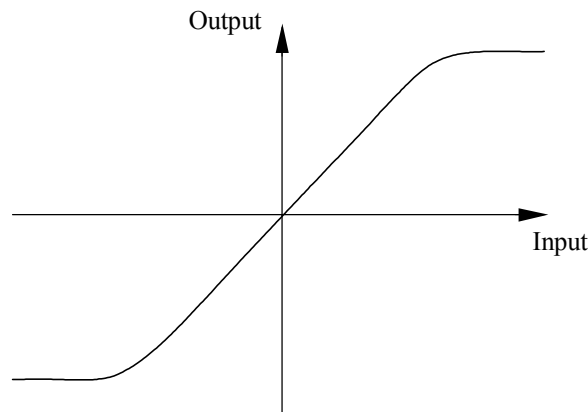
Sketch 3.3 Backlash and hysteresis characteristics

Hysteresis is similar in form to backlash and is commonly associated with magnetic circuits. It is illustrated in Sketch 3.3. It often occurs only over a limited range of inputs and is single valued outside this range. Practical relays display the hysteresis characteristic where a larger value of signal voltage is required to close the relay than that required as a minimum to hold in the relay.

(iv) Saturation

Many components, such as amplifiers, have an output which is proportional to input for a limited range of input signals and when the input exceeds this range the output tends to become constant. This effect is known as saturation and is illustrated in Sketch 3.4.

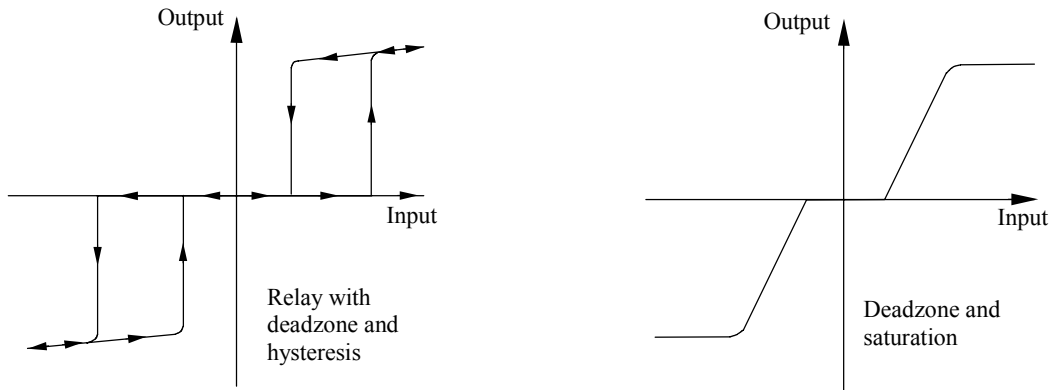
Saturation is inevitable and all components, if driven by sufficiently large signals, exhibit this phenomenon due to the limitation of the physical capabilities of their components.



Sketch 3.4 Saturation characteristics

(v) Combined Nonlinear Elements

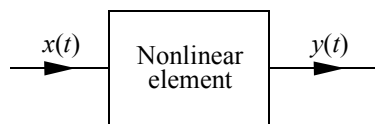
Systems often contain a combination of the above nonlinear elements, for example a relay servo-system may contain both deadzone and hysteresis or an electric motor may contain both deadzone and saturation. Sketch 3.5 illustrates these two combinations.



Sketch 3.5 Combined nonlinear characteristics

4. DESCRIPTION AND ASSUMPTIONS

A Describing Function is used to represent an element of a system that contains only the nonlinear characteristics of the system.



Sketch 4.1 Nonlinear element

Consider the nonlinear element depicted in Sketch 4.1, which has a sine wave input and a non-sinusoidal periodic output, where the output has the same period as the input. The Describing Function method approximates the output wave by a sine wave that in some way best describes the output. Thus this method provides a linearisation of the nonlinear element by replacing it by a transfer function relating input and output sine waves.

Suppose that the input to the nonlinear element is given by

$$x(t) = X \sin(\omega t), \quad (4.1)$$

then the output, $y(t)$, will generally contain a transient response term (which will die out for stable elements), a $d-c$ term, and periodic output. The periodic output, in the Describing Function method, is represented by the sine wave,

$$y^*(t) = Y(X) \sin[\omega t + \phi(X)], \quad (4.2)$$

(Note: $y(t)$ is the actual output and $y^*(t)$ is the Describing Function representation of the actual output).

The amplitude, Y , and phase angle, ϕ , of Equation (4.2) are both in general functions of input magnitude.

The Describing Function is denoted by $N(X)$ and given by,

$$N(X) = \frac{Y(X) e^{i\phi(X)}}{X} \quad (4.3)$$

The Describing Function approximation is classified as a frequency response method since the system is excited by a sinusoidal input and provides information on the steady-state response of the system.

The Describing Function has made use of two basic assumptions:

- (i) the input to the nonlinear element is a single frequency sinusoid,
- (ii) sub-harmonics and the $d-c$ component of the output are negligible, while harmonics are attenuated by the rest of the system.

A consequence of assumption (i), inherent in Equation (4.3), is that a range of calculations is needed to cover the range of X of interest.

Various Describing Function schemes have been proposed in the general literature and this Item will define the five simplest and most well known.

5. DESCRIBING FUNCTION SCHEMES

The five most well known Describing Functions all have the same definition of phase angle and differ only in the definition of the amplitude of the output sine wave. They are commonly referred to as^{*}

- (a) Conventional^{1, 2, 3, 4}
 - (b) Corrected-conventional⁵
 - (c) New RMS⁶ (or Unconventional RMS)
 - (d) Minimum Average Error⁷
 - (e) Equivalent Gain⁸.
- (a) The Conventional Describing Function expands the output of a nonlinear element in a Fourier series and uses the first Fourier harmonic as the chosen sinusoidal output. If the input to the element is given by $x(t) = X\sin(\omega t)$, then the output, as a full Fourier series, is given by,

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \{A_n \cos(n\omega t) + B_n \sin(n\omega t)\} \quad (5.1)$$

where $A_n = \frac{1}{\pi} \int_0^{2\pi} \{y(t) \cos(n\omega t)\} d(\omega t) \quad (5.2)$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} \{y(t) \sin(n\omega t)\} d(\omega t) \quad (5.3)$$

* The superscripts refer to cited References given in Section 10.

Hence neglecting the *d-c* output component, A_0 , and all harmonics of the output, the Describing Function becomes,

$$N(X) = \frac{[A_1^2 + B_1^2]^{1/2} \angle \tan^{-1}(A_1/B_1)}{X} \quad (5.4)$$

- (b) The Corrected-conventional Describing Function includes the effects of the first and third* Fourier harmonics for the amplitude of the chosen output sine wave. The Describing Function is given by,

$$N(X) = \frac{[A_1^2 + B_1^2 + A_3^2 + B_3^2]^{1/2} \angle \tan^{-1}(A_1/B_1)}{X} \quad (5.5)$$

where A_1 , B_1 , A_3 , and B_3 are given by Equations (5.2) and (5.3).

- (c) The New RMS Describing Function equates the rms value of the amplitude of the actual nonsinusoidal output of the nonlinear element to the rms value of the chosen sinusoidal output. The Describing Function is given by,

$$N(X) = \frac{\sqrt{2} \left\{ (1/2\pi) \int_0^{2\pi} [y(t)]^2 d(\omega t) \right\}^{1/2} \angle \tan^{-1}(A_1/B_1)}{X} \quad (5.6)$$

- (d) The Minimum Average Error Describing Function minimises the average error between the chosen output sine wave and the actual nonsinusoidal output of the nonlinear element. For the positive half cycle of the actual and chosen output sinusoids the minimum error can be found from,

$$\begin{aligned} \text{average error} &= 0 = \int_{-\phi}^{\pi-\phi} y(t) d(\omega t) - \int_{-\phi}^{\pi-\phi} \{Y \sin(\omega t + \phi)\} d(\omega t) \\ &= \int_{-\phi}^{\pi-\phi} y(t) d(\omega t) - 2Y \end{aligned}$$

where $Y \sin(\omega t + \phi)$ is the chosen output sinusoid, and ϕ is the phase angle which is equal to $\tan^{-1}(A_1/B_1)$. The Describing Function is given by,

$$N(X) = \frac{(1/2) \int_{-\phi}^{\pi-\phi} y(t) d(\omega t) \angle \phi}{X} \quad (5.7)$$

*The output of the nonlinear element is assumed odd harmonic *i.e.* $y(t) = -y(t + T/2)$, where T is the period), in which case the third harmonic is the first harmonic above the fundamental.

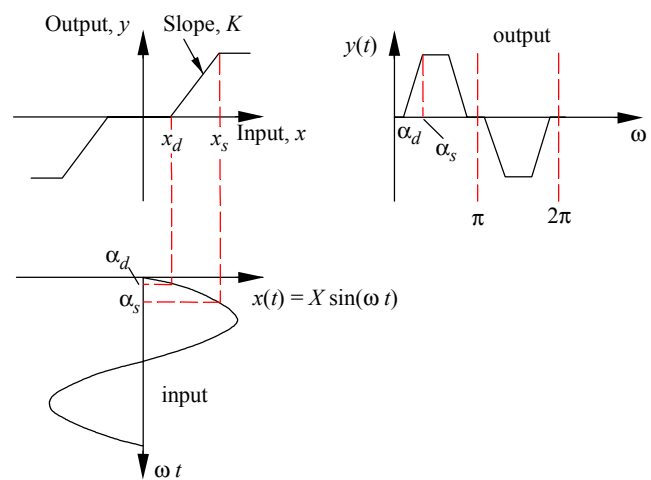
- (e) The Equivalent Gain Describing Function uses the peak value of the actual nonsinusoidal output of the nonlinear element as the amplitude of the chosen output sine wave. The Describing Function is given by,

$$N(X) = \frac{y(t)_{peak} / \tan^{-1}(A_1/B_1)}{X} \quad (5.8)$$

The Describing Function method is approximate and so the accuracy of the amplitude and phase angle prediction of the output sine wave will depend on the particular nonlinear system that is under analysis. Examples in Section 8 show how these Describing Functions are used to estimate the gain and phase of a nonlinear system and the results are compared with those found by numerical integration.

6. ILLUSTRATIVE DERIVATION OF CONVENTIONAL DESCRIBING FUNCTION FOR COMBINED DEADZONE AND SATURATION

The idealised characteristic of the nonlinear features of combined deadzone and saturation and its response to a sinusoidal input is shown in Sketch 6.1.



Sketch 6.1 Sinusoidal response to combined deadzone and saturation

When the peak value of the input sine wave, X , is larger than the saturation point, x_s , then the output wave may be described as,

$$y(t) = \begin{cases} 0 & , & 0 < \omega t < \alpha_d & , \\ K[x(t) - x_d] & , & \alpha_d < \omega t < \alpha_s & , \\ K[x_s - x_d] & , & \alpha_s < \omega t < \pi - \alpha_s & , \\ K[x(t) - x_d] & , & \pi - \alpha_s < \omega t < \pi - \alpha_d & , \\ 0 & , & \pi - \alpha_d < \omega t < \pi & , \end{cases} \quad (6.1)$$

where

$$\alpha_d = \sin^{-1}(x_d/X) \quad (6.2)$$

$$\alpha_s = \sin^{-1}(x_s/X) .$$

The output is seen in Sketch 6.1 to have half-wave and quarter-wave symmetry and so, using Equations (5.2), (5.3) and (6.1),

$$\begin{aligned}
 A_1 &= 0 \\
 B_1 &= \frac{4}{\pi} \int_0^{\pi/2} \{y(t) \sin(\omega t)\} d(\omega t) \\
 &= \frac{4}{\pi} \left\{ \int_{\alpha_d}^{\alpha_s} \{K[x(t) - x_d] \sin(\omega t)\} d(\omega t) + \int_{\alpha_s}^{\pi/2} \{K(x_s - x_d) \sin(\omega t)\} d(\omega t) \right\}.
 \end{aligned}$$

Evaluating this expression and using Equations (6.2) the following expression is obtained.

$$B_1 = \frac{KX}{\pi} \left(2(\alpha_s - \alpha_d) + \sin(2\alpha_s) - \sin(2\alpha_d) \right).$$

Using Equation (5.4), the Describing Function for combined deadzone and saturation, is given by,

$$N(X) = \frac{K}{\pi} \left(2(\alpha_s - \alpha_d) + \sin(2\alpha_s) - \sin(2\alpha_d) \right) \quad \underline{0}. \quad (6.3)$$

This equation has been derived for $X > x_s$, but can be modified for solutions for all X .

For $X < x_d$, the Describing Function can be found from Equation (6.3) by substituting $\alpha_d = \alpha_s = \pi/2$, which can be seen by examining the output curve of Sketch 6.1. Hence

$$N(X) = 0. \quad (6.4)$$

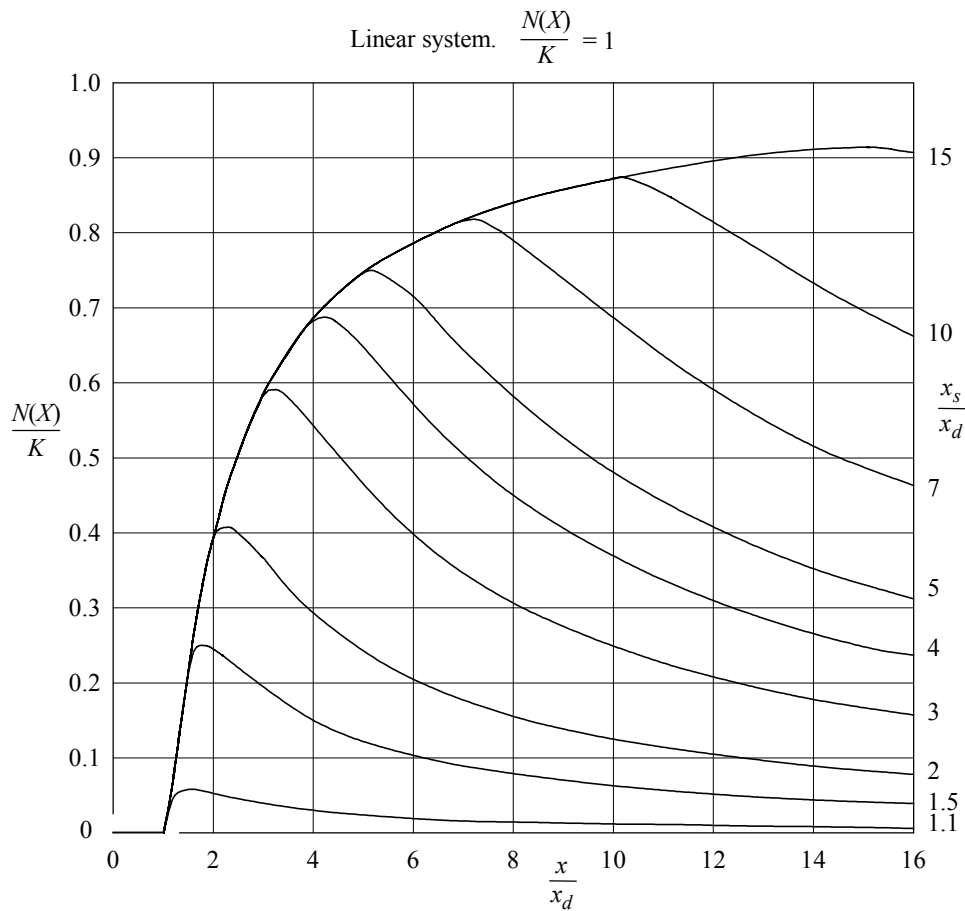
For $x_d < X < x_s$, $\alpha_s = \pi/2$ and the Describing Function for this range becomes,

$$N(X) = \frac{K}{\pi} \left(\pi - 2\alpha_d - \sin(2\alpha_d) \right). \quad (6.5)$$

The non-dimensional forms of these equations, $N(X)/K$, are illustrated in Sketch 6.2 where they are plotted against X/x_d for various values of x_s/x_d .

$N(X)/K$ is seen to be zero when $X < x_d$, i.e. when the input sine wave is always inside the deadband region and consequently the output sine wave, $y^*(t)$, is zero. When X is greater than the deadband limit but less than the saturation limit, $N(X)/K$ is seen to rise rapidly and forms the common curve of the x_s/x_d variation.

When the magnitude of the input sine wave reaches the saturation limit, $N(X)/K$ reaches a maximum and then decreases with increasing X . Here the peak value of the output sine wave is constant with increasing X and this is reflected in the decreasing $N(X)/K$ variation with increasing X . For $x_s/x_d = 1$ the function $N(X)/K$ is always zero. This can best be seen by examining the input/output curve of Sketch 6.1. If the saturation limit, x_s , is reduced so that it approaches the deadband limit, x_d , in such a manner that the slope between the two points, K , is kept constant, then when $x_s = x_d$ the output is zero for all X .



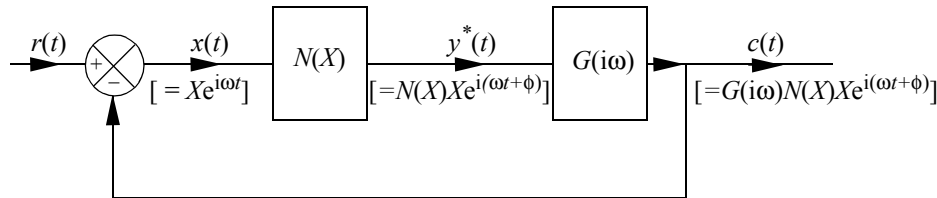
Sketch 6.2 Conventional describing function characteristics for combined deadzone and saturation

Appendix A contains the Describing Function derivations corresponding to the five different schemes for the following commonly encountered nonlinear elements.

- A1. Deadzone
- A2. Saturation
- A3. Deadzone and Saturation
- A4. Ideal Relay
- A5. Deadzone and Ideal Relay
- A6. Ideal Relay with Hysteresis
- A7. Ideal Relay, Deadzone and Hysteresis
- A8. Backlash (friction-controlled)

7. STABILITY ANALYSIS

Describing Functions are used in stability investigations and in particular for the prediction of limit cycles in feedback system. Consider a nonlinear system with unity feedback, illustrated in Sketch 7.1, which comprises two elements, one containing the nonlinear part, $N(X)$, and the other the linear part, $G(i\omega)$.



Sketch 7.1 Nonlinear unity feedback system

The characteristic equation of the system is given by,

$$1 + G(i\omega)N(X) = 0. \quad (7.1)$$

Equation (7.1) is similar to a linear characteristic equation and the stability of this system may be determined by studying the zeros of this equation or by using any of the techniques that have been developed for dealing with linear systems.

Alternatively by invoking the Nyquist criterion the system will be neutrally stable and exhibit sustained oscillations when

$$G(i\omega)N(X) = -1. \quad (7.2)$$

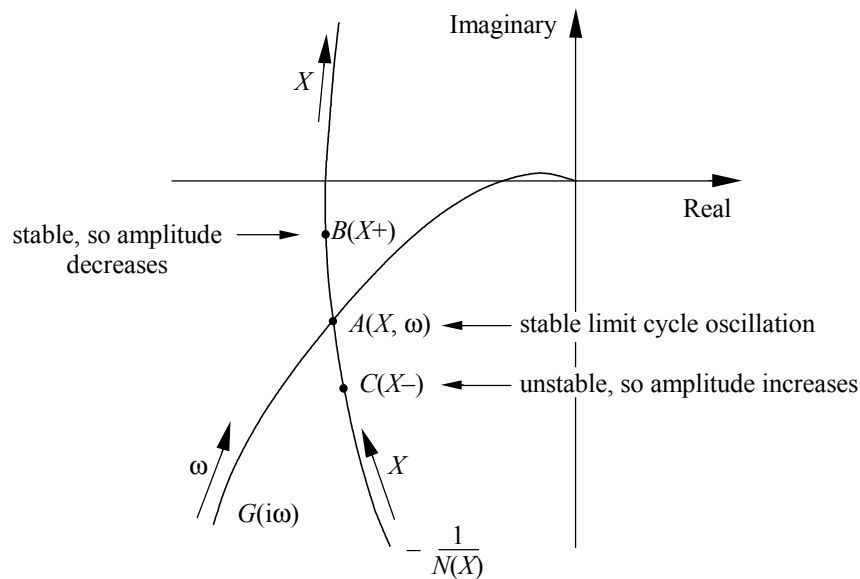
This equation may be rewritten in the form,

$$G(i\omega) = -\frac{1}{N(X)}. \quad (7.3)$$

The critical point -1 of Equation (7.2) has now become the critical locus, $-1/N(X)$. If $G(i\omega)$ and $-1/N(X)$ are plotted separately on a Nyquist diagram then the crossing point will indicate the possible existence of a self-sustaining, or limit cycle, oscillation, which will have a frequency and amplitude derived from the values at the intersection. The limit cycle oscillation may be stable or unstable (and hence not exist). Stability may be checked by considering oscillations in the neighbourhood of the intersection point on the $-1/N(X)$ locus.

To illustrate this consider the Nyquist plots of Sketch 7.2, where the linear portion of a system, $G(i\omega)$, and a nonlinear describing function, $-1/N(X)$ are shown to intersect at $A(X, \omega)$. Suppose that the system is operating at point A under the state of a limit cycle of amplitude X and at a frequency of ω . If a slight perturbation is given to the system so that the input to the nonlinear element increases to $X+$ (i.e. to point B) then B is in the range of stable operation and so the perturbation amplitude will decrease towards A.

A similar, but converse, argument can be constructed about a point C which has an amplitude $X-$, where $X- < X$. The system has, therefore, a stable limit cycle oscillation at point A. If the loci of $G(i\omega)$ and $-1/N(X)$ cross at two points then there will be two limit cycles, one stable and the other unstable.



Sketch 7.2 Nyquist plots of $G(i\omega)$ and $-\frac{1}{N(X)}$

A nonlinear system can be considered stable, or not, in the same way as a linear system; that is, when the critical locus lies to the left of $G(i\omega)$, or is not enclosed by it. In this case the poles of the system will have negative real parts, disturbances will decay and the system will be stable. When there are no intersections the system is stable and the relative proximity to a possible intersection is an indication of relative stability.

The above description has used the Nyquist diagram to illustrate an evaluation of the stability of a system but any of the methods of linear analysis, such as Bode diagrams or the Nicholls chart, may be used.

8. EXAMPLES

The examples in this section illustrate how the Describing Function (DF) technique is used to evaluate the gain of a nonlinear open-loop system and further how it is used to investigate the stability of an autonomous closed-loop system, particularly in the estimation of the amplitude and phase of a limit cycle oscillation.

The results obtained using the approximate DF method are compared with values obtained using numerical integration.

8.1 Example 1. Systems with Saturation

(a) Open-loop System

Consider a system which, for small input signals, is linear and given by a transfer function of the form:

$$\frac{c(s)}{x(s)} = \frac{\sigma \omega_N^2}{(s^2 + 2\zeta \omega_N s + \omega_N^2)(s + \sigma)} \quad (8.1)$$

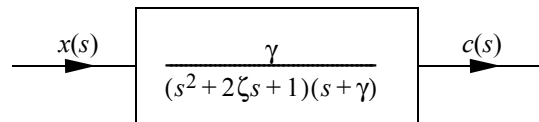
This equation contains three unknowns and can be normalised and simplified by making the substitutions $s_1 + s/\omega_N$ and $\gamma = \sigma/\omega_N$.

Thus,
$$\frac{c(s)}{x(s)} = \frac{\gamma}{(s_1^2 + 2\zeta s_1 + 1)(s_1 + \gamma)} \quad (8.2)$$

(The suffix 1 attached to s may be dropped since this new Laplace operator will be used throughout the remainder of the Example.)

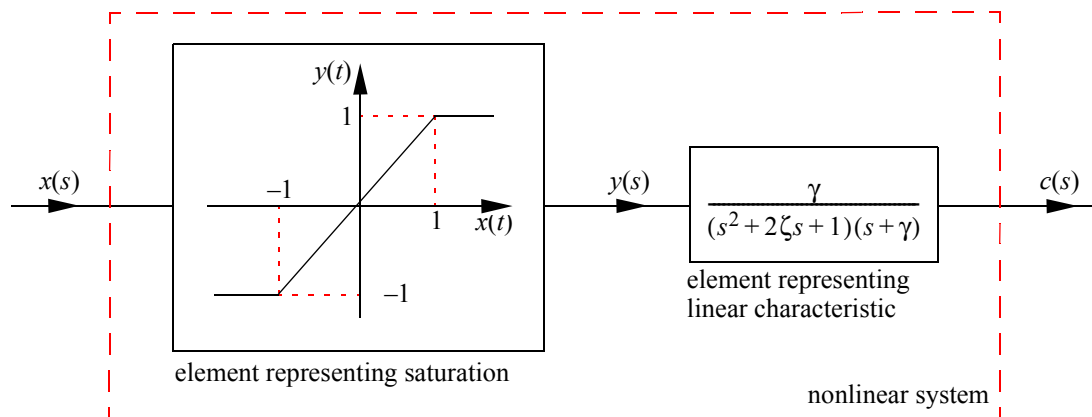
This new expression involves only two variables; γ , the normalised pole of the first-order factor and ζ the damping ratio of the second-order factor.

A block diagram of Equation (8.2) is shown in Sketch 8.1.



Sketch 8.1 Linear open-loop system

The frequency response, for small signals, of this system for γ values of 0.9 and 3.0 and for ζ values of 0.1, 0.3, 0.5 and 0.7 is shown plotted in Figure 1. The range of values of γ and ζ used are seen to represent a number of systems with different response characteristics.



Sketch 8.2 Nonlinear open-loop system

Consider now the same system subject to larger input signals which are saturated for inputs below -1 and above $+1$. The system, in block diagram form, is shown in Sketch 8.2. The element representing the saturation characteristic in equation form assuming a sinusoidal input ($x(t) = X\sin(\omega t)$) may be represented by,

$$\frac{y(s)}{x(s)} = N(X), \quad (8.3)$$

where $N(X)$ represents the Describing Function for saturation.

The magnitude of the output of the nonlinear open-loop system can be found from

$$N(X) \left| \frac{\gamma}{(s^2 + 2\zeta s + 1)(s + \gamma)} \right|,$$

evaluated for chosen values of X and ω . Table 11.1 presents these values for X values of 1 and 2 at an input frequency, ω , of unity, for the five Describing Function schemes given in Section 5. These schemes are designated in the Tables as:

- (a) Conventional
- (b) Corrected-conventional
- (c) New RMS
- (d) Minimum Average Error
- (e) Equivalent Gain.

For an X value of 1, $N(X) = 1$ and the system is linear, since the input sine wave is never large enough to saturate.

To find how well these Describing Functions estimate the amplitude of the exact output a Runge-Kutta-Gill integration routine (see ESDU Data Item No .82037) has been completed for the system. The equation for the system is of the form,

$$\frac{d^3 c}{dt^3} + (2\zeta + \gamma) \frac{d^2 c}{dt^2} + (2\zeta\gamma + \gamma) \frac{dc}{dt} + \gamma c = \gamma F(t), \quad (8.4)$$

where the forcing function, $F(t) = X \sin(\omega t)$, for $|X \sin(\omega t)| < 1$
 $= 1$, for $X \sin(\omega t) > 1$
 $= -1$, for $X \sin(\omega t) < -1$

Equation (8.4) is converted to three first-order differential equations for the integration process. The results of this integration are also included in Table 11.1 designated as "Numerical Solution".

The results of the amplitude predictions from Table 11.1 show at the lower damping ratio values that the Conventional Describing Function (DF) is the most accurate and overestimates the "Numerical Solution" by differences that are less than 1.1%. The Corrected-conventional and the New RMS DF's give differences of the order of 1 to 3%. At the higher damping ratios there is little to choose between these three DF's and they all produce differences between 1 and 3%. The Minimum Average Error DF overestimates in all cases and give differences between 6 and 9%. The Equivalent Gain DF is by far the worst and underestimates the gain, giving differences in the range 17 to 20%.

(b) Closed-loop system

To illustrate how the Describing Function technique can be used to examine the stability of a closed-loop nonlinear system, consider that the previously used transfer function now represents a closed-loop unity feedback system, given by

$$\frac{c(s)}{r(s)} = \frac{G(s)}{1 + G(s)} = \frac{\gamma}{(s^2 + 2\zeta s + 1)(s + \gamma)}. \quad (8.5)$$