



An introduction to time-dependent aerodynamics of aircraft response, gusts and active controls

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AN INTRODUCTION TO TIME-DEPENDENT AERODYNAMICS OF AIRCRAFT RESPONSE, GUSTS AND ACTIVE CONTROLS

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AN INTRODUCTION TO TIME-DEPENDENT AERODYNAMICS OF AIRCRAFT RESPONSE, GUSTS AND ACTIVE CONTROLS

1. NOTATION AND UNITS

		<i>SI</i>	<i>British</i>
A	aspect ratio		
[A]	structural inertia matrix	N m s ²	lbf ft s ²
[B]	aerodynamic damping matrix	m ⁴	ft ⁴
[C]	aerodynamic stiffness matrix	m ³	ft ³
<i>c</i>	chord	m	ft
<i>c</i> ₀	centre-line chord	m	ft
$\bar{c}$	geometric mean chord	m	ft
[D]	structural damping matrix	N m s	lbf ft s
[E]	structural stiffness matrix	N m	lbf ft
<i>f</i> (σ_0)	gust profile function		
<i>F</i> _{<i>n</i>}	hereditary functions (<i>n</i> = 0, 1, 2), see Equations (6.1) to (6.8)		
G	column vector of force coefficients due to gust		
<i>H</i> _{<i>n</i>}	hereditary functions (<i>n</i> = 0, 1, 2), see Equations (6.10) to (6.15)		
<i>I</i> _{<i>i</i>}	impulsive force coefficient in mode <i>i</i>		
<i>i</i>	$\sqrt{-1}$		
<i>K</i> _{<i>i</i>}	indicial (Küssner) function due to entry into uniform up-gust, see Equations (3.35) and (3.36)		
$\bar{K}_i$	function due to entry into gust of arbitrary profile, see Equation (3.55)		
<i>L</i> (σ)	instantaneous lift increment	N	lbf
<i>L</i> ₀	reference value for <i>L</i> (σ), steady-state or maximum quasi-static value	N	lbf
<i>l</i>	reference length	m	ft

l_j	time-dependent dimensionless loading, $\Delta p_j / \rho V^2$		
l_g	time-dependent dimensionless loading due to entry into uniform up-gust		
Δp_j	lift per unit area in mode j	N/m ²	lbf/ft ²
M	Mach number		
Q_i	time-dependent force coefficient in mode i		
Q_{ij}	complex generalized airforce coefficient in Equation (3.7), $Q_{ij} = Q'_{ij} + i\nu Q''_{ij}$		
$\underline{\mathbf{q}}$	column vector of generalized displacements		
q_j	complex generalized coordinate in mode j		
$(q_j)_0$	value of q_j at small time $\delta\tau \rightarrow 0$		
$\mathcal{R}$	denotes real part of		
$\underline{\mathbf{R}}$	column vector of force coefficients due to response of aircraft and control inputs		
s	wing semi-span	m	ft
t	time	s	s
W_{ij}	indicial (Wagner) function for mode i due to step change in upwash mode j , see Equations (3.30) and (3.31)		
w_g	upwash velocity due to uniform up-gust	m/s	ft/s
w_j	upwash velocity due to sinusoidal gust	m/s	ft/s
w_j	upwash velocity in mode j	m/s	ft/s
V	free-stream velocity	m/s	ft/s
x, y, z	three-dimensional Cartesian coordinate system with origin at leading edge for aerofoils and at foremost point of planform for wings	m	ft
x_a	distance of pitch axis aft of coordinate origin	m	ft
x_g	gust gradient distance	m	ft
z_i, z_j	upward displacement in mode i or j	m	ft
Z	time-dependent upward displacement of wing	m	ft

$\bar{Z}$	upward displacement, given by $Z = \Re[\bar{Z}(x, y)e^{i\omega t}]$	m	ft
β	$(M^2 - 1)^{1/2}$		
δ	Dirac delta function, see Equations (3.22)		
δ_f	deflection angle of control surface	degree	degree
δ_{f0}	maximum deflection of control surface	degree	degree
ρ	free-stream density	kg/m ³	slug/ft ³
ν	frequency parameter, $\omega l/V$		
ξ	distance from aerofoil leading edge as fraction of chord		
ξ_a	distance from aerofoil leading edge to pitch axis as fraction of chord		
ξ_{cp}	distance from aerofoil leading edge to centre of pressure position, as fraction of chord		
σ, σ_0	dimensionless distance of travel, Vt/l		
σ_1	dimensionless duration of control motion from $\delta_f = 0$ to $\delta_f = \delta_{f0}$		
σ_g	dimensionless gust gradient distance, x_g/l		
σ_m	integration limit in Section (3.3)		
τ, τ_0	dimensionless time, $t/(l/V) = Vt/l$		
τ_u	large value of τ		
Φ	velocity potential	m ² /s	ft ² /s
Φ_0	steady part of velocity potential	m ² /s	ft ² /s
$\bar{\Phi}$	unsteady part of velocity potential	m ² /s	ft ² /s
$\bar{\Phi}_j$	unsteady velocity potential in mode j	m ² /s	ft ² /s
ω	circular frequency	rad/s	rad/s

Subscripts

i	integer denoting force mode 1 for lift 2 for pitching moment (positive nose up)
j	integer denoting upwash mode 1 for heave rate (downwards) 2 for pitch rate (positive nose up) 3 for wing control-surface deflection 4 for pitch attitude

2. INTRODUCTION

2.1 General

The critical loads that determine the necessary structural strength of an aircraft are often those encountered in a gust or atmospheric turbulence. This is particularly true of civil aircraft which have only modest manoeuvring requirements. The prediction of the time history of the loads on the lifting surfaces after entry into a gust is therefore a major application of the theories of unsteady aerodynamics. These loads arise directly from the gust and indirectly from the response of the aircraft. Active control systems designed to alleviate gust loads require rapid control surface movement and the developing aerodynamic loads are highly dependent on the unsteady nature of the aerodynamics.

Aircraft response is traditionally studied by means of ‘indicial’ functions, reviewed in Reference 12. These functions predict the development with time of the aerodynamic loading after an instantaneous step change in any of the variables determining the motion. For example, one such function would describe the growth in lift following a step increase in heave rate, from the instant of the change until the reestablishment of steady values at the increased heave rate. Indicial functions for entry into a uniform vertical gust provide the loading due to the gust itself. Those for step changes in the generalized coordinates in the rigid and flexible modes of the aircraft are needed to predict the loading due to the disturbed motion of the aircraft or the activation of control surfaces. With the indicial functions known, a convolution or superposition integral allows the aerodynamic forces from a gust of arbitrary profile or from arbitrary aircraft motion to be calculated in a standard manner, see Reference 6, 9 or 12 for example. For studies with active controls a convolution integral is unwieldy for practical use and recourse is usually made to one or other of the available approximations. These may be as simple as using quasi-steady aerodynamics corresponding to infinitesimal frequency parameter, Reference 21, or may be based on a fuller simulation of each indicial function by a linear transfer function, Reference 14.

The calculation of indicial functions has been performed by many authors over a number of years. The classical work was carried out by Wagner, Reference 1, who considered the case of an aerofoil subjected to a step change in heave rate, and Küssner, Reference 2, who predicted the lift growth on an aerofoil entering a stationary step gust. Although this work was two-dimensional and incompressible, it is common practice and convenient to use the terms Wagner and Küssner loosely to describe any indicial functions of these types, even though they may be for a wing of finite aspect ratio in compressible flow. It may be noted that the term ‘indicial aerodynamics’ is of uncertain origin, but in the literature of unsteady aerodynamics it has for many years been widely used to describe the response to step changes.

The numerical calculation of indicial functions can be approached in two ways. For supersonic flow it is possible to obtain exact analytical solutions for the complete indicial responses of aerofoils and certain simple classes of wing planform by solving the potential flow equation for the unsteady motion in a direct manner. For subsonic speeds this approach remains important for predicting the indicial response during the very early stages of an unsteady motion, but such calculations become increasingly laborious or impracticable as the time period is extended. This direct approach is often referred to as the 'supersonic flow analogy' because a physical reinterpretation of the coordinate system of the potential flow equation allows the results of calculations in steady-state supersonic motion to be employed as a means of solution, see Reference 4, 5, 9 or 12.

A more general way of calculating indicial functions is through the long established reciprocal relationship between oscillatory and indicial forces. Fourier integrals of the frequency-dependent generalized airforce coefficients due to sinusoidally oscillating wings and controls or due to immersion in a sinusoidal gust field yield expressions for Wagner or Küssner functions respectively. The evaluation of these integrals is then dependent only on a knowledge of the airforce coefficients over a sufficiently wide range of frequency.

For Wagner functions in two dimensions this presents no problem and computer programs written at RAE Farnborough, References 16 and 18, readily provide subsonic and supersonic airforce coefficients for oscillating aerofoils and controls throughout the frequency range. The situation is less satisfactory for Küssner functions as there are no corresponding facilities for sinusoidal gust fields. In three dimensions, advances in lifting-surface theory and computer programming techniques now permit oscillatory airforce coefficients on wings of general planform, alone and in tandem, to be calculated more easily than in the past, see References 10, 13 and 15 for example. These in turn lead to a more reliable prediction of indicial functions. However, because the available three-dimensional theories are restricted to frequencies up to some fairly large value, an approximate numerical treatment is necessary to cover the higher frequency parameters. Reference 17 gives an example of how this may be done. Overall, the full three-dimensional calculations remain time consuming and are therefore unsuitable for parametric studies. However, for individual aircraft such calculations have provided a successful framework for obtaining indicial responses and are used as illustrative examples herein.

2.2 Scope of Item

Item No. 82020 (Reference 20) already provides an introduction to oscillatory aerodynamics with special reference to flutter. The purpose of the present Item is to introduce the subject of indicial aerodynamics, from the development of the basic linearized equations to a description of current 'state of the art' applications to gust encounter and active controls.

Section 3 considers a general expression for the time-dependent aerodynamic forces. Fourier integral expressions for the Wagner and Küssner functions follow and the implications for calculations in two and three dimensions are discussed. Section 3 concludes by considering gust profiles of arbitrary shapes and transient control motion.

Section 4 gives a systematic coverage of indicial functions in subsonic and supersonic two-dimensional flows without separation. It combines calculations made from oscillatory data specially prepared by the programs of References 16 and 18 with existing data collected from other sources, References 7, 8, 9 and 12.

Indicial functions for attached flow in three dimensions are presented in Section 5. The results have been compiled from existing published and unpublished sources of data. The most commonly available information relates to Wagner and Küssner functions for rectangular and delta planforms. Section 5.1 concentrates mainly on these using References 3, 11, 12 and 14. It covers as wide a range of wing aspect ratio as possible in incompressible flow and the effect of Mach number for selected aspect ratios. The influence of sweep angle is shown, and the Wagner and Küssner functions for typical aircraft wings are compared with the corresponding two-dimensional functions from Section 4. Section 5.2 considers wing-tailplane combinations and illustrates the Wagner and Küssner functions for two aircraft, References 19 and 22. Section 5.3 deals with control surface motion and gives Wagner functions to demonstrate the different characteristics of leading-edge and trailing-edge controls.

The response of a wing is also usefully studied through a number of hereditary functions that embody the history of the wing motion. Alternative expressions for the time-dependent aerodynamic force are introduced in Section 6 where their merits and defects are discussed in practical terms. The hereditary functions are illustrated for a particular aircraft.

Section 7, which concludes the main body of the Item, reviews the application of the techniques of indicial aerodynamics. It describes the relationship between design gust profiles and structural loads, giving examples from Reference 22 of the variation of loads due to discrete gusts of different lengths. The treatment and effectiveness of active controls is then illustrated from References 17 and 21. Finally, the savings in structural loadings that can be achieved through active control systems are demonstrated, together with examples from References 21 and 22 to show how a quasi-steady simplification of the unsteady aerodynamics can often prove satisfactory.

3. BASIC EQUATIONS

For many decades the principal motivation for theoretical developments in unsteady aerodynamics has been the prevention of wing or control-surface flutter. The necessary confidence and caution in demonstrating freedom from flutter both derive in large measure from aeroelastic calculations of critical flutter speed involving small-perturbation theories for wings in simple harmonic motion of moderately high frequencies. Subcritical response calculations call for general time-dependent aerodynamics, but the needs to calculate aircraft response to gust encounter and to incorporate active controls have provided the primary stimulus in this field.

The theoretical methods start from the concept of a uniform flow of velocity V in the x -direction subject to small perturbations with velocity potential $\Phi(x, y, z, t)$. The linearized form of the governing equation past a wing in the plane $z = 0$ is then

$$(1 - M^2) \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = \frac{M^2}{V^2} \frac{\partial^2 \Phi}{\partial t^2} + \frac{2M^2}{V} \frac{\partial^2 \Phi}{\partial x \partial t} \quad (3.1)$$

with a boundary condition on the upward velocity over the wing planform

$$\left(\frac{\partial \Phi}{\partial z} \right)_{z=0} = V \frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial t} \quad (3.2)$$

where $Z(x, y, t)$ is the time-dependent upward displacement of the wing from the plane $z = 0$. A general mathematical solution of these equations can be formulated, but it is too complicated for numerical treatment. In the special case of two-dimensional incompressible flow, with $\Phi(x, z, t)$ and $M = 0$, there are two classical solutions in References 1 and 2 for a step change in heave velocity and for entry into a step up-gust respectively. The review in Reference 12 discusses these and some exact solutions for two-dimensional and three-dimensional supersonic flows.

In general, numerical methods of solution are first developed for oscillatory flow where $Z(x, y, t)$ has simple harmonic time-dependence and can represent a sinusoidal gust. For a circular frequency ω , the velocity potential is written as

$$\Phi(x, y, z, t) = \Phi_0(x, y, z) + \Re[\bar{\Phi}(x, y, z)e^{i\omega t}] \quad (3.3)$$

where under the linearized conditions the steady and unsteady parts Φ_0 and $\bar{\Phi}$ are independent. Equations (3.1) and (3.2) then give

$$(1 - M^2) \frac{\partial^2 \bar{\Phi}}{\partial x^2} + \frac{\partial^2 \bar{\Phi}}{\partial y^2} + \frac{\partial^2 \bar{\Phi}}{\partial z^2} = -\frac{\omega^2 M^2 \bar{\Phi}}{V^2} + \frac{2i\omega M^2}{V} \frac{\partial \bar{\Phi}}{\partial x} \quad (3.4)$$

$$\text{and} \quad \left(\frac{\partial \bar{\Phi}}{\partial z} \right)_{z=0} = V \frac{\partial \bar{Z}}{\partial x} + i\omega \bar{Z} \quad (3.5)$$

where $\Re[\bar{Z}(x, y)e^{i\omega t}]$ is the upward displacement. A reference length l is used to define a frequency parameter

$$v = \omega l / V. \quad (3.6)$$

The lifting-surface methods, such as Reference 15 for subsonic flow and Reference 10 for supersonic flow, yield generalized force coefficients from the integrals

$$Q_{ij}(v) = \frac{1}{l^3} \iint z_i(x, y) l_j(v, x, y) dx dy \quad (3.7)$$

taken over the planform. Here $z_i(x, y)$ denotes the displacement $\bar{Z}(x, y)$ corresponding to the desired force and the dimensionless loading is

$$l_j(v, x, y) = \frac{\Delta p_j}{\rho V^2} = \frac{2}{V} \left(\frac{\partial \bar{\Phi}_j}{\partial x} + \frac{iv \bar{\Phi}_j}{l} \right) \quad (3.8)$$

where the complex velocity potential $\bar{\Phi}_j$ corresponds to a complex upwash distribution $w_j(v, x, y)$ equivalent to Equation (3.5). Thus the upwash angle is

$$\frac{w_j}{V} = \frac{\partial z_j}{\partial x} + \frac{iv z_j}{l} \quad (3.9)$$

or it may be defined directly; for example, for a sinusoidal gust

$$w_j / V = \exp(-ivx/l) . \quad (3.10)$$

Consider arbitrary wing motion in mode j

$$z(x, y, t) = z_j(x, y)q_j(\tau) \quad (3.11)$$

where $\tau = Vt/l$ (3.12)

is dimensionless time. If the motion starts with initial conditions

$$q_j(\tau) = \frac{dq_j(\tau)}{d\tau} = 0 \text{ at } \tau = 0, \quad (3.13)$$

then the analysis in Appendix A of Reference 17 leads to expressions for $Q_i(\tau)$ requiring the complete knowledge of either the real or the imaginary part of the generalized force coefficient

$$Q_{ij}(\nu) = Q'_{ij}(\nu) + i\nu Q''_{ij}(\nu). \quad (3.14)$$

The basic principle is that of linear superposition whereby $q_j(\tau)$ is built up from elementary sinusoidal motions represented in Equation (3.9), and correspondingly the time-dependent force $Q_i(\tau)$ in mode i is built up from the elementary quantities $Q''_{ij}(\nu)$ or $Q'_{ij}(\nu)$ over the whole frequency spectrum.

The alternative expressions for $\tau \geq 0$ from Reference 17 are

$$\begin{aligned} Q_i(\tau) = & Q''_{ij}(\infty) \frac{dq_j(\tau)}{d\tau} + Q'_{ij}(0) q_j(\tau) \\ & + \frac{2}{\pi} \int_0^\tau \int_0^\infty \frac{dq_j(\tau_0)}{d\tau_0} \{Q''_{ij}(\nu) - Q''_{ij}(\infty)\} \cos[\nu(\tau - \tau_0)] d\nu d\tau_0 \end{aligned} \quad (3.15)$$

and
$$Q_i(\tau) = Q''_{ij}(\infty) \frac{dq_j(\tau)}{d\tau} + \frac{2}{\pi} \int_0^\tau \int_0^\infty \frac{dq_j(\tau_0)}{d\tau_0} \frac{Q'_{ij}(\nu)}{\nu} \sin[\nu(\tau - \tau_0)] d\nu d\tau_0. \quad (3.16)$$

Both Equations (3.15) and (3.16) involve double integration with respect to ν and τ_0 . The road to indicial aerodynamics is to integrate first with respect to ν , after which the problem is treated in the time domain. On the other hand, integration first with respect to τ_0 leads to numerical treatment in the frequency domain. This has the advantage that the frequency range can be truncated at a finite upper limit and the disadvantage that it sacrifices the physical understanding provided by the Wagner and Küssner functions and by other so-called hereditary functions used in the time domain, (see Section 6).

3.1 Wagner and Küssner Functions

With linearized aerodynamics the principle of superposition ensures a reciprocal relationship between the time-dependent force coefficients Q_i for simple harmonic motions and for step changes in q_j . Consider a rapid smooth change from $q_j = 0$ at time $\tau = 0$ to $q_j = (q_j)_0$ at small time $\delta\tau$. As $\delta\tau \rightarrow 0$ the first term on the right-hand side of Equation (3.15) or (3.16) produces an impulsive force coefficient

$$I_i = Q''_{ij}(\infty) (q_j)_0 \quad (3.17)$$

at $\tau = 0$.

This combines with alternative expressions $Q_i(\tau)$ for $\tau \geq 0$

$$\frac{Q_i(\tau)}{(q_j)_0} = Q'_{ij}(0) + \frac{2}{\pi} \int_0^\infty \{Q''_{ij}(v) - Q''_{ij}(\infty)\} \cos(v\tau) dv \quad (3.18)$$

and

$$\frac{Q_i(\tau)}{(q_j)_0} = \frac{2}{\pi} \int_0^\infty \frac{Q'_{ij}(v)}{v} \sin(v\tau) dv. \quad (3.19)$$

The quantity τ may also be thought of as a dimensionless distance of travel σ relative to an initial event at $\tau = \tau_0$, i.e. $\sigma = \tau - \tau_0$. For any planform of wing or wing-tailplane configuration at some arbitrary subsonic or supersonic Mach number these equations then define a generalized Wagner function, $W_{ij}(\sigma)$, which is normalized with respect to $Q'_{ij}(0)$ when this is non-zero. Thus for unit $(q_j)_0$ Equations (3.17) and (3.18) together produce

$$W_{ij}(\sigma) = \frac{Q''_{ij}(\infty)}{Q'_{ij}(0)} \delta(\sigma) + 1 + \frac{2}{\pi} \int_0^\infty \frac{Q''_{ij}(v) - Q''_{ij}(\infty)}{Q'_{ij}(0)} \cos(v\sigma) dv. \quad (3.20)$$

Alternatively from Equations (3.17) and (3.19) the generalized Wagner function is

$$W_{ij}(\sigma) = \frac{Q''_{ij}(\infty)}{Q'_{ij}(0)} \delta(\sigma) + \frac{2}{\pi} \int_0^\infty \frac{Q'_{ij}(v)}{Q'_{ij}(0)} \frac{\sin(v\sigma)}{v} dv \quad (3.21)$$

where $\delta(\sigma)$ is the Dirac delta function defined by the equations

$$\left. \begin{aligned} \delta(\sigma) &= 0 \text{ for } \sigma \neq 0 \\ \delta(\sigma) &= \infty \text{ for } \sigma = 0 \\ \int_{-\infty}^{\infty} \delta(\sigma) d\sigma &= 1. \end{aligned} \right\} \quad (3.22)$$

The classical Wagner function of Reference 1 relates to heaving motion ($j = 1$) with

$$w_1/V = -1 \quad (3.23)$$

in two-dimensional incompressible flow. For lift ($i = 1$)

$$z_1 = l \quad (3.24)$$

and the representative length l is taken as the chord c , so that σ is the distance of travel in chord lengths. In this special case the impulsive term at $\sigma = 0$ is

$$\frac{Q''_{11}(\infty)}{Q'_{11}(0)} \delta(\sigma) = \frac{1}{4} \delta(\sigma) \quad (3.25)$$

and, with normalizing factor $Q'_{11}(0) = \pi$, the Wagner function $W_{11}(\sigma)$ for $\sigma > 0$ is shown in Figure 1.

Note that in the limit as $\sigma \rightarrow 0$

$$W_{11}(0+) = \frac{Q'_{11}(\infty)}{Q'_{11}(0)} = \frac{1}{2} \quad (3.26)$$

in this special two-dimensional case at $M = 0$; moreover, the curve $W_{11}(\sigma)$ for $\sigma > 0$ applies to the force for any displacement function $z_i(x, y)$ and indeed to the growth of loading at every chordwise position. However, it is not true to say that the complete Wagner function is unique for heaving motion, because the impulsive term in Equation (3.20) depends on $z_i(x, y)$. For pitching moment about the leading edge

$$z_2 = -x \quad (3.27)$$

the impulsive force is given by

$$\frac{Q''_{21}(\infty)}{Q'_{21}(0)} \delta(\sigma) = \frac{1}{2} \delta(\sigma) \quad (3.28)$$

for the mid-chord pitching moment it is zero, and for the quarter-chord axis the normalization is inappropriate because $Q'_{21}(0) = 0$.

The derivation of the Küssner functions for a step gust impinging on the leading edge at $\sigma = 0$ is somewhat different. The stationary step gust

$$\left. \begin{aligned} w &= w_g \text{ for } x < \sigma l \\ w &= 0 \text{ for } x > \sigma l \end{aligned} \right\} \quad (3.29)$$

is obtained from the sinusoidal gusts w_J in Equation (3.10) of wavelengths $2\pi l/v$ by means of alternative integrals

$$\frac{w}{w_g} = \frac{1}{\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{ivx}{l}\right) \frac{\sin(v\sigma)}{v} dv \quad (3.30)$$

$$\text{or} \quad \frac{w}{w_g} = 1 - \frac{i}{\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{ivx}{l}\right) \frac{\cos(v\sigma)}{v} dv. \quad (3.31)$$

For $\sigma \geq 0$, the generalized force coefficients corresponding to Equation (3.29) are then given by

$$\frac{Q_i(\sigma)}{(w_g/V)} = \frac{2}{\pi} \int_0^{\infty} Q'_{iJ}(v) \frac{\sin(v\sigma)}{v} dv \quad (3.32)$$

or

$$\frac{Q_i(\sigma)}{(w_g/V)} = Q'_{i1}(0) + \frac{2}{\pi} \int_0^{\infty} Q''_{iJ}(v) \cos(v\sigma) dv. \quad (3.33)$$

The generalized Küssner function for a unit step gust $w_g = V$ is simply

$$K_i(\sigma) = \frac{Q_i(\sigma)}{Q'_{i1}(0)} \quad (3.34)$$

where it is conditional that $Q'_{i1}(0) \neq 0$. Thus there are alternative expressions

$$K_i(\sigma) = \frac{2}{\pi} \int_0^\infty \frac{Q'_{ij}(\nu)}{Q'_{i1}(0)} \frac{\sin(\nu\sigma)}{\nu} d\nu \quad (3.35)$$

and

$$K_i(\sigma) = 1 + \frac{2}{\pi} \int_0^\infty \frac{Q''_{ij}(\nu)}{Q'_{i1}(0)} \cos(\nu\sigma) d\nu \quad (3.36)$$

which closely resemble the generalized Wagner functions in Equations (3.30) and (3.31). Essential differences occur because no forces arise from a sinusoidal gust of infinitesimal wave length ($\nu = \infty$). Thus $Q''_{ij}(\infty) = 0$, so that there is no impulsive force at $\sigma = 0$; moreover $Q'_{ij}(\infty) = 0$, so that with reference to Equation (3.26) the initial value $K_i(0)$ is zero.

The classical Küssner function of Reference 2 for the growth of lift in two dimensions after meeting the step gust at low speed is shown as $K_1(\sigma)$ in Figure 1. The values of $W_{11}(\sigma)$ and $K_1(\sigma)$ are taken respectively from Tables 9.1 and 9.5 of Reference 12, where the dependent variable must be halved to convert the reference length from $\frac{1}{2}c$ to c . Although $K_1(\sigma)$ is always less than $W_{11}(\sigma)$, they are found to have the same asymptotic form when σ is large.

3.2 Limiting Forms at Large and Small Time

The asymptotic expansions of hereditary functions in three-dimensional subsonic flow are discussed in Section 3.4 of Reference 17. The behaviour at large time is governed by the low-frequency characteristics of the generalized force coefficient $Q_{ij}(\nu)$. In the present notation, Equation (3-30) of Reference 17 becomes

$$W_{ij}(\sigma) = 1 + \frac{Q'_{1j}(0)Q'_{il}(0)}{8\pi Q'_{ij}(0)\sigma^2} + O(\sigma^{-3}). \quad (3.37)$$

If the reference length l is taken to be some chordwise dimension of the wing, $Q_{ij}(\nu)$ in Equation (3.7) is roughly proportional to wing span or aspect ratio. Thus the Wagner functions approach their asymptotic limits more slowly as aspect ratio increases. The second term on the right-hand side of Equation (3.37) is associated with the low-frequency expansion

$$Q''_{ij}(\nu) = Q''_{ij}(0) - \frac{1}{16} Q'_{1j}(0) Q'_{i1}(0) \nu + O(\nu^2 \log \nu). \quad (3.38)$$

In two-dimensional subsonic flow, however, $Q''_{ij}(\nu)$ generally has a term proportional to $\log \nu$ at small frequency, and in consequence the asymptotic expansion becomes

$$W_{ij}(\sigma) = 1 + O(\sigma^{-1}) \quad (3.39)$$

and in particular for heaving motion at $M = 0$

$$W_{i1}(\sigma) = 1 - \frac{1}{2\sigma} + \dots \quad (3.40)$$

The approach to the steady state $W_{ij}(\infty) = 1$ in Figure 1 for the two-dimensional case is therefore much slower than that for a finite wing in subsonic flow.

Different principles apply in supersonic flow, which is treated thoroughly in Reference 12. In two-dimensional flow the steady state is reached once the initial disturbance at the leading edge has cleared the trailing edge. Thus

$$W_{ij}(\sigma) = 1 \text{ when } \sigma \geq M/(M-1). \quad (3.41)$$

Similarly in three-dimensional flow the steady state is reached at a finite value of σ .

The behaviour of $W_{ij}(\sigma)$ at small time is governed by the high-frequency characteristics of $Q_{ij}(v)$. The analysis in Section 3.2 of Reference 17 shows that for large v

$$Q'_{ij}(v) = Q'_{ij}(\infty) + O(v^{-2}) \quad (3.42)$$

and
$$Q''_{ij}(v) = Q''_{ij}(\infty) + O(v^{-2}). \quad (3.43)$$

In consequence Equations (3.20) and (3.21) both give the expansion

$$W_{ij}(\sigma) = \frac{Q''_{ij}(\infty)}{Q'_{ij}(0)} \delta(\sigma) + \frac{Q'_{ij}(\infty)}{Q'_{ij}(0)} + O(\sigma). \quad (3.44)$$

The evaluation of $Q'_{ij}(\infty)$ and $Q''_{ij}(\infty)$ needs clarification. For compressible flow the formulation depends on the nature of w_j/V . If this is real, as for heaving motion in Equation (3.23) or for pitching motion about the axis $x = x_a$

$$w_2/V = (x_a - x)/l, \quad (3.45)$$

then from piston theory

$$Q'_{ij}(\infty) = -\frac{2}{M} \int \int \frac{z_i}{l} \frac{w_j}{V} \frac{dx}{l} \frac{dy}{l} \quad (3.46)$$

and
$$Q''_{ij}(\infty) = 0 \quad (3.47)$$

where the integral is taken over the planform. In such cases there is no impulsive force and $W_{ij}(0)$ is easy to evaluate. If, on the other hand, w_j/V is defined by Equation (3.9) in terms of $z_j(x, y)$ and the flow is subsonic, then $Q'_{ij}(\infty)$ is not generally obtainable from piston theory and, as explained in Section 3.2 of Reference 17, it is necessary to calculate

$$Q''_{ij}(\infty) = -\frac{2}{M} \int \int \frac{z_i}{l} \frac{z_j}{l} \frac{dx}{l} \frac{dy}{l} \quad (3.48)$$

and
$$Q'_{ij}(\infty) = Q'_{ij}(0) + \frac{2}{\pi} \int_0^\infty [Q''_{ij}(v) - Q''_{ij}(\infty)] dv. \quad (3.49)$$

Unless $Q'_{ij}(\infty)$ happens to be zero, there is an impulsive force and $W_{ij}(0+)$ from the second term on the right-hand side of Equation (3.44) needs careful integration of Equation (3.49). The situation in fully supersonic flow is easier because $Q'_{ij}(\infty)$ can then be obtained directly from piston theory which gives the double integral

$$Q'_{ij}(\infty) = -\frac{2}{M} \iint \frac{z_i}{l} \frac{\partial z_j}{\partial x} \frac{dx}{l} \frac{dy}{l}. \quad (3.50)$$

Different principles apply in incompressible flow, which is well covered by References 11 and 12. Equations (3.46) and (3.48) become inapplicable. In particular for heaving or pitching motion, as defined by

$$w(x, y, t) = w_j(x, y)q_j(\tau) \quad (3.51)$$

in conjunction with Equation (3.23) or (3.45), instead of Equation (3.47) for compressible flow $Q'_{ij}(\infty)$ is generally non-zero for $M = 0$. This has already been illustrated in Equations (3.25) and (3.28). Furthermore, for step changes in displacement in Equation (3.11), the impulsive force would normally be infinite so as to invalidate Equation (3.44). For $M \neq 0$, however, Equation (3.44) applies to both two-dimensional and three-dimensional flows.

3.3 Range of Input Forces

There is considerable variety in the time-dependent aerodynamic quantities required to define the equations to be solved in response calculations such as those in References 21 and 22. For a given wing-tailplane configuration and Mach number there are four distinct groups:

- (1) Wagner functions corresponding to rigid motion of the configuration which is conveniently treated in terms of rate modes through Equations (3.23), (3.45) and (3.51),
- (2) Wagner functions corresponding to elastic motion of the configuration, which must be treated in terms of displacement modes through Equations (3.9) and (3.11),
- (3) input forces derived from Küssner functions for a gust of particular profile, '1-cosine' for example, which is to disturb the equilibrium in level flight,
- (4) Wagner functions for the active aileron and elevator, which are akin to those in group (2) but involve discontinuities in $z_i(x, y)$ and have characteristics associated with the location of the control.

There is a fifth group which is not strictly necessary, but which offers considerable insight:

- (5) Wagner functions associated with interfering surfaces, for example, concerning the influence of the presence of the tailplane on the lift or pitching moment due to rigid wing motion.

Figure 2 gives two examples of group (1), namely the two-dimensional lift at $M = 0.8$ due to step changes in heave rate and pitch rate ($j = 1$ and 2) and distinguishes between these and a step change in pitch attitude which, being a displacement mode, corresponds to group (2). The displacement in pitch attitude ($j = 4$) is defined by

$$z_4 = x_a - x \quad (3.52)$$

and hence

$$\frac{w_4}{V} = -1 + \frac{iv(x_a - x)}{l}. \quad (3.53)$$

Although for a given frequency parameter this is a linear combination of w_1/V and w_2/V in Equations (3.23) and (3.45), a step in q_4 of Equation (3.11) for $j = 4$ causes an impulsive initial disturbance which is not found with the rate modes at $M = 0.8$. Figure 2 shows that the subsequent behaviour at low σ is more violent for $W_{14}(x_a = 0)$ than for W_{11} or $W_{12}(x_a = 0.25c)$. Each growth of lift has the same asymptotic behaviour for large σ but the approach to the steady state is more gradual than for $M = 0$ in Figure 1.

Given the Küssner function $K_i(\sigma)$ for a particular configuration and Mach number, it is a simple matter to formulate the corresponding function $\bar{K}_i(\sigma)$ in group (3) for an arbitrary gust profile

$$w = w_g f(\sigma_0) \quad (3.54)$$

where $f(\sigma_0)$ increases from 0 to 1 over the length of the gust gradient from $\sigma_0 = 0$ to $\sigma_0 = \sigma_g$. The convolution integral gives

$$\bar{K}_i(\sigma) = \int_0^{\sigma_m} \frac{df(\sigma_0)}{d\sigma_0} K_i(\sigma - \sigma_0) d\sigma_0 \quad (3.55)$$

where σ_m is σ or σ_g , whichever is the smaller. The corresponding force coefficients are

$$Q_i(\sigma) = (w_g/V) Q'_{i1}(0) \bar{K}_i(\sigma). \quad (3.56)$$

For a linear-ramp gust profile with

$$\left. \begin{aligned} f(\sigma_0) &= 0 && \text{for } \sigma_0 < 0 \\ f(\sigma_0) &= \sigma_0/\sigma_g && \text{for } 0 \leq \sigma_0 \leq \sigma_g \\ f(\sigma_0) &= 1 && \text{for } \sigma_0 > \sigma_g \end{aligned} \right\} \quad (3.57)$$

substitution into Equation (3.55) gives

$$\bar{K}_i(\sigma) = \frac{1}{\sigma_g} \int_0^{\sigma_m} K_i(\sigma - \sigma_0) d\sigma_0. \quad (3.58)$$

For an on-off '1 - cosine' gust with

$$\left. \begin{aligned} f(\sigma_0) &= 0 && \text{for } \sigma_0 < 0 \text{ and } \sigma_0 > 2\sigma_g \\ \text{and} &&& \\ f(\sigma_0) &= \frac{1}{2}[1 - \cos(\pi\sigma_0/\sigma_g)] && \text{for } 0 \leq \sigma_0 \leq 2\sigma_g. \end{aligned} \right\} \quad (3.59)$$

Equation (3.55) becomes

$$\bar{K}_i(\sigma) = \frac{\pi}{2\sigma_g} \int_0^{\sigma_m} \sin(\pi\sigma_0/\sigma_g) K_i(\sigma - \sigma_0) d\sigma_0 \quad (3.60)$$

where σ_m is σ or $2\sigma_g$, whichever is the smaller. Figure 3 illustrates the growth of lift on a wing-tailplane configuration at $M = 0.7$ for a range of on-off '1 - cosine' gust profiles with $\sigma_g = 1, 3$ and 5 . The gust profiles $f(\sigma_0)$ are shown in Figure 3a and in Figure 3b the lift is normalized with respect to the steady-state lift due to an on gust. For the shortest gust the maximum lift experienced by the aircraft is only about 65 per cent of the steady-state value and occurs at a distance $\sigma \approx 0.6$ behind the maximum in the gust forcing function. The two longer gusts allow sufficient time for the aircraft to approach within 10 per cent of the quasi-static lift, with the lag in the occurrence of the maximum increasing to $\sigma \approx 1.0$.

The Wagner functions in group (4) for different types of control will be discussed in Sections 5.3 and 7.2. A trailing-edge control exhibits the slowest growth rate for lift, especially at high subsonic speeds. Figure 4 illustrates the lift at $M = 0.8$ due to transient motion of an outboard 25 per cent chord trailing-edge control with deflection angle

$$\left. \begin{aligned} q_j(= \delta_f) &= 0 \text{ for } \sigma < 0 \text{ and } \sigma > 2\sigma_1 \\ q_j(= \delta_f) &= \delta_{f0}(\sigma/\sigma_1)^3(2 - \sigma/\sigma_1)^3 \text{ for } 0 \leq \sigma \leq 2\sigma_1 \end{aligned} \right\} \quad (3.61)$$

The ratio of the instantaneous lift L to the quasi-static lift L_0 corresponding to the maximum control deflection δ_{f0} has been calculated by means of Equation (3.16). The mean wing chord $\bar{c}$ is used as the reference length, and $2\sigma_1$ therefore denotes the distance travelled in units of $\bar{c}$ during the transient control motion. In Figure 4, L/L_0 is plotted against σ/σ_1 ; the main features are the progressive reduction and delay in the peak lift as σ_1 , the duration of increasing δ_f , is reduced and the accompanying increase in residual lift beyond $\sigma = 2\sigma_1$ which marks the end of the excursion in control angle.

Some examples from group (5) for wing-tailplane combinations are discussed in Section 5.2.

Functions for elastic motion from group (2) are used in the examples of aircraft response in Sections 7.1, 7.3 and 7.4.

4. NUMERICAL RESULTS IN TWO DIMENSIONS

4.1 Wagner Functions for $M < 1$

Wagner functions for a thin aerofoil experiencing a step change in heave rate, pitch rate or deflection of a trailing-edge flap of chord $0.2c$ have all been calculated numerically for subsonic flow from the Fourier integral of Equation (3.21). For this purpose airforce coefficients $Q'_{ij}(\nu)$ in the range $0.05 \leq \nu \leq 10$ were obtained by running the program of Reference 16 and theoretical, steady-state values $Q'_{ij}(0)$ were determined analytically. For the plain aerofoil analytical values of $Q'_{ij}(\infty)$ are also known, and the infinite frequency control force coefficients were determined by assuming the asymptotic behaviour of Equation (3.42). The integration was carried out in three sections. Simpson's rule was used over a small initial interval from $\nu = 0$ to $\nu = 0.04$, Filon's rule (see Reference 17) was used from $\nu = 0.04$ to $\nu = 10$, and analytical integration was performed between $\nu = 10$ and $\nu = \infty$ by invoking the asymptotic behaviour. The results are presented with σ based on the chord length. The functions W_{11} , W_{12} and W_{22} for the aerofoil are given in Tables 9.1 to 9.3, where the exact solutions from Reference 9 have been taken for $\sigma \leq M/(M+1)$. For flap motion the functions W_{13} and W_{23} are given in Tables 9.4 and 9.5. The pitch axis has been taken at the quarter-chord point of the aerofoil. Consequently $Q'_{21}(0) = 0$ and normalized values of W_{21} are unobtainable. Figures 5a, 6a and 7a illustrate W_{11} , W_{12} and W_{13} respectively. It can be seen that with increasing Mach number the early variations of these functions are steadily reduced, and that their approach to the steady state becomes progressively slower. The values of $W_{ij}(0)$ in compressible flow are

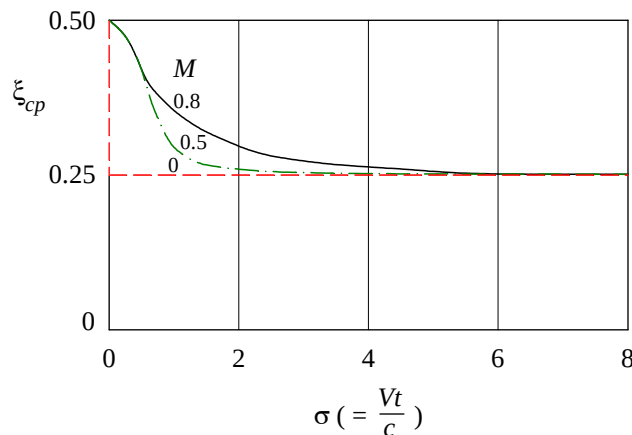
$$W_{11} = 2\beta/(\pi M), \quad (4.1)$$

$$W_{12} = \beta/(\pi M) \quad (4.2)$$

$$\text{and} \quad W_{22} = 14\beta/(3\pi M). \quad (4.3)$$

With regard to the integration process employed it should be noted that in compressible flow the variation of $Q'_{ij}(\nu)$ at high ν consists of decaying undulations superimposed on the asymptotic approach to $Q'_{ij}(\infty)$. As Mach number decreases the undulatory contribution becomes larger and remains significant up to higher values of ν . The consequence is that if reliable values of $Q'_{ij}(\nu)$ are not calculated to a sufficiently high value of ν , then the analytical integration above this introduces spurious waviness in $W_{ij}(\sigma)$ at low values of σ . With precise values of $Q'_{ij}(\nu)$ taken only up to $\nu = 10$ and the asymptotic variation assumed thereafter, this spurious behaviour was found in some cases for Mach numbers up to 0.75. This has been overcome in the presented results by interpolating a smooth variation between the exact values of $W_{ij}(\sigma)$ known for $\sigma \leq M/(M+1)$ and the reliable values obtained from the Fourier integration for $\sigma \geq 3$. Such regions of interpolation are indicated by dashed lines in Figures 5a, 6a and 7a and the corresponding numerical values are given in brackets in Tables 9.1 to 9.5.

The pitching moment on a heaving wing is studied in Reference 4 in terms of the centre of pressure position. In compressible flow this moves rapidly forward from the mid-chord position at $\sigma = 0+$ and approaches asymptotically its steady-state position at the quarter chord. The distance of travel for the movement to be essentially completed increases with Mach number, from $\sigma \approx 2$ at $M = 0.5$ to $\sigma \approx 5$ at $M = 0.8$, see Sketch 4.1. In incompressible flow the forward movement is accomplished by an instantaneous jump in ξ_{cp} from 0.5 to 0.25 at $\sigma = 0+$.



Sketch 4.1 Variation of centre of pressure position on heaving aerofoil

In References 4 and 12 an exact solution of the potential flow equation for the unsteady motion has also been used to study the early growth of the chordwise loading on an aerofoil following a step change in heave rate $q_1 V$ or pitch rate $q_2 V/l$. The results for $M = 0.8$ are reproduced in Figures 8a and 8b which show how l_1/q_1 and l_2/q_2 develop over the first four chord lengths of travel. Immediately after $\sigma = 0$ the loading l_1/q_1 is constant and l_2/q_2 increases linearly with ξ . The case shown in Figure 8b is for a pitch axis at the leading edge so that $l_2/q_2 = 0$ at $\xi = 0$ when $\sigma = 0+$. As σ increases the pressure disturbances emanating from the leading and trailing edges increasingly modify the initial loadings towards their steady-state values although there is an unaffected mid-region until $\sigma = M/(M+1)$. The singularity apparent at $\xi = 1 - \sigma(1-M)/M$ persists until it clears the leading edge at $\sigma = M/(M-1) = 4$. At subsonic speeds the influence of the initial starting motion is never completely escaped, but its effect diminishes with time and the steady state is approached asymptotically after the singularity disappears. Table 9.1 shows that at $\sigma = 4$ the integrated lift forces are only at about 65 per cent of their steady values. However, Figure 8c compares the loadings at $\sigma = 4$ with the steady-state loadings multiplied by the lift ratio $L(4)/L_0$, and reveals that after four chord lengths of travel l_1/q_1 has almost reached its final shape, although l_2/q_2 has some way to go.

4.2 Wagner Functions for $M > 1$

For supersonic flow a direct solution of the potential flow equation for the unsteady motion yields analytical solutions for the complete response of an aerofoil to a step change in heave rate or pitch rate. These equations are given in References 6 and 12 and are set out in Table 9.6 for a pitch axis at the leading edge, together with equations for converting to a general reference axis. Note that for a mid-chord axis ($\xi_a = 0.5$), W_{12} and W_{21} are unobtainable since their respective denominators are both zero. Unlike the response in subsonic flow, in supersonic flow steady-state conditions are reached after a finite time, $\sigma = M/(M-1)$ when the influence of the initial disturbance at the leading edge is beyond the trailing edge. For example, Figures 8d and 8e, reproduced from Reference 4, show the exact solution for the chordwise loadings l_1/q_1 and l_2/q_2 over the first six chord lengths of travel for $M = 1.2$. Immediately after $\sigma = 0$ the loading l_1/q_1 is constant and l_2/q_2 increases linearly with ξ , as in subsonic flow. As the aerofoil moves forward the front of the aerofoil escapes from the effect of the starting disturbance and steady-state values are progressively established along the chord to $\xi = \sigma(M-1)/M$ until they exist over the entire chord when $\sigma = 6$, with l_1/q_1 attaining its final constant value and l_2/q_2 its final linear variation with ξ .

Values of W_{13} and W_{23} for a step change in control deflection have been calculated for an aerofoil with a trailing-edge flap of chord $0.2c$ using Equation (3.21). For this purpose airforce coefficients in the range $0 \leq v \leq 10$ were obtained by running the program of Reference 18 and piston theory was used to determine $Q'_{ij}(\infty)$. The integration was carried out over three ranges of v , as described in Section 4.1. The numerical values of W_{13} and W_{23} for a reference axis $\xi_a = 0.5$ are given in Tables 9.7 and 9.8 respectively.

The variation of W_{11} , W_{12} and W_{13} with Mach number is illustrated in Figures 5b, 6b and 7b. The remarkable feature of Figure 5b is that constant values of W_{11} exist until $\sigma = M/(M+1)$, when the trailing edge meets the initial leading-edge disturbance. The extent of this region increases with Mach number, but it should be remembered that σ is dimensionless and that in real time t , the duration of the constant value, is proportional to $1/(M+1)$ and becomes shorter as Mach number increases. For W_{12} in Figure 6b the pitch axis has been taken at the leading edge; the curves show a similar pattern to those in Figure 5b except that there is no constant value region at low σ . Because of the small size of the trailing-edge control the function W_{13} in Figure 7b completes its response very rapidly, in the time $\sigma = 0.2M/(M-1)$.

4.3 Küssner Functions for All M

Values of the Küssner function K_1 for thin aerofoils entering a uniform stationary up-gust at subsonic Mach numbers are shown in Figure 9a and have been compiled from the calculated results presented in Reference 12 for $M = 0.8$ and in Reference 7 for $M = 0.5, 0.6$ and 0.7 . The data in Reference 12 were obtained by exact solution of the potential flow equation for the first four chords of travel, $\sigma \leq 4$, followed by an asymptotic extrapolation to the steady-state value. In Reference 7 a Fourier integration of frequency response coefficients was used for all but a short initial interval for which analytical solutions exist. As noted in Reference 12 there are some inconsistencies between the two sets of results, with the Fourier integral data being considered the more reliable at high σ . In Figure 9a the data for $M = 0.8$ have been slightly adjusted to give a smooth family of asymptotic curves. The incompressible flow data from Figure 1 are repeated for completeness. Figure 8f, reproduced from Reference 12, shows exact calculations of the early history of the chordwise loading $l_g/(w_g/V)$ for $M = 0.8$ and $\sigma \leq 4.5$ following entry into a step gust. The loading rises rapidly from zero over the forward part of the aerofoil but remains zero in the region $\xi \geq (M+1)/M$ until the trailing edge meets the initial leading-edge disturbance at $\sigma \approx 0.44$. At $\sigma = 4.5$ Figure 9a shows that the integrated lift force is only about 65 per cent of its steady value. However, Figure 8f includes a dashed curve showing the steady-state distribution of $l_g/(w_g/V)$ multiplied by the lift ratio $L(4.5)/L_0$. It is clear that by the time $\sigma = 4.5$ the loading is close to its final shape.

There are no corresponding published data for K_2 but Reference 6 examines the exact solutions that exist to describe the centre of pressure position of the indicial lift for subsonic flow in the initial stages after gust entry, $0 \leq \sigma \leq M/(1+M)$. During this time the centre of pressure is a distance $\sigma c(1+M)/4M$ aft of the leading edge. At the end of this initial period the centre of pressure has reached the quarter-chord point of the aerofoil. As this is its position in the steady state, Reference 6 suggests that it is reasonable to assume that it remains close to the quarter-chord point for $\sigma \geq M/(1+M)$.

For supersonic speeds Reference 6 gives exact solutions for the indicial lift and pitching moment on an aerofoil entering a uniform up-gust. The expressions for K_1 and K_2 for the leading-edge axis $\xi_a = 0$ are given in Table 9.6 together with the equations for conversion to a general reference axis. The function K_2 is unobtainable for the axis $\xi_a = 0.5$ because the steady-state pitching moment is zero. Figure 9b illustrates the behaviour of K_1 for a range of supersonic Mach numbers. Again, in contrast to the subsonic results, the response is accomplished in a finite time, $\sigma = M/(M-1)$, and this produces a very steep rise in K_1 for the higher Mach numbers, although the curves at low supersonic Mach numbers have some similarity to the subsonic curves in Figures 9a.

The history of the chordwise loading $l_g/(w_g/V)$ in Figure 8g for $M = 1.2$ has been calculated from the exact equation given in Reference 12. Following gust encounter at $\sigma = 0$ the constant steady-state loading is established on the forward part of the aerofoil as it escapes from the initial pressure disturbance originating at the leading edge. The loading remains zero in the region $\xi \geq \sigma(M+1)/M$ until the initial leading-edge disturbance first affects the trailing edge. As σ increases further the transient part of the loading becomes steadily smaller until at $\sigma = M/(M-1) = 6$ the initial leading-edge disturbance fully clears the trailing edge and the entire loading becomes steady, as was the case for the supersonic Wagner functions.

5. NUMERICAL RESULTS IN THREE DIMENSIONS

5.1 Wings

5.1.1 $M < 1$

Wagner functions W_{11} and Küssner functions K_1 for rectangular wings in incompressible flow are given for $A = 1, 2$ and 4 in Reference 11 and for $A = 4$ and 6 in Reference 3. In both references the functions have been calculated by Fourier integration of frequency response characteristics, but the later work is probably more accurate. Values of W_{11} and K_1 for $A = 1, 2, 4$ and 6 at $M = 0$ are shown in Figures 10a and 11a respectively with added two-dimensional results labelled $A = \infty$. At low σ the curves for $A = 6$ have been adjusted to give a smooth family of curves and should be regarded as approximate. It can be seen that the effect of decreasing aspect ratio is to increase the initial value of W_{11} and to hasten the approach of both indicial functions to their steady-state values.

Reference 11 also provides W_{11} and K_1 for delta wings in incompressible flow with $A = 1, 2$ and 4 . These functions are shown in Figures 10b and 11b, respectively, with $\sigma = Vt/c_0$. The Wagner functions are primarily governed by the ratio of spanwise to chordwise extent, and so W_{11} for a delta wing resembles that for a rectangular wing of half the aspect ratio. The initial parabolic rise in K_1 for the delta wings is typical of swept wings.

The effects of finite aspect ratio are illustrated further in Figure 12 for two swept wings, with $\sigma = Vt/\bar{c}$. Figure 12a shows W_{11} at $M = 0.4$ and 0.8 for an aspect ratio of 6 and the corresponding two-dimensional results for an aerofoil are also shown. After the initial variation in the first fraction of a chord of travel the function for the wing soon nears its asymptote. The function for the aerofoil is always much smaller and is still far from its steady-state value when $\sigma = 5$. The differences between the two- and three-dimensional results are greater than those due to the Mach number variation. Figure 12b presents K_1 at $M = 0.8$ for an aspect ratio of 8 (Reference 17). It shows that although this function has a slower initial climb when, because of sweep, only a small part of the wing has encountered the gust, it soon rises much more steeply and approaches its asymptote far more quickly than in the case of the aerofoil. These data provide a clear warning not to ignore the differences between two- and three-dimensional subsonic flows.

It is important to remember that the indicial functions are normalized with respect to their steady-state values. Figure 13a compares W_{11} for untapered swept and unswept wings of aspect ratio 6 at $M = 0.7$ (Reference 14). Figure 13b compares the corresponding non-normalized functions $W_{11}Q'_{11}(0)$. For W_{11} the effect of sweep is apparent only over the first chord length of travel. In contrast, the initial development of the non-normalized function is little influenced by sweep, which primarily affects the steady-state values.

5.1.2 $M > 1$

For supersonic speeds Reference 12 gives exact solutions for W_{11} and K_1 for rectangular wings with $A \geq 1/\beta$ and for delta wings with supersonic leading edges, $A \geq 4/\beta$. Figures 14a and 14b demonstrate the effect of Mach number for a rectangular wing of $A = 6$ and Figures 14c and 14d show the effect of aspect