

NONLINEAR SYSTEMS: THE DESCRIBING FUNCTION METHOD FOR CLOSED-LOOP SYSTEM RESPONSE

1. NOTATION

a, b, c, d	arbitrary constants
C	magnitude of output signal from a system
$c(t)$	output signal from a system
E	magnitude of error signal of a system
$e(t)$	error signal of a system
$G_1(i\omega), G_2(i\omega)$	transfer functions of linear elements
N	Describing Function of nonlinear element
n_1, n_2	arbitrary constants
R	magnitude of input signal to a system
$r(t)$	input signal to a system
s	Laplace operator
t	time
X	magnitude of input signal to nonlinear element
$x(t)$	input signal to nonlinear element
Y	magnitude of chosen output sine wave from a nonlinear element
$y^*(t)$	chosen output sine wave from a nonlinear element
$y(t)$	non-sinusoidal output wave from a nonlinear element
α, β	arbitrary angles
ϕ	phase angle
ϕ_1, ϕ_2	phase angles of linear elements $G_1(i\omega)$ and $G_2(i\omega)$ respectively
ϕ_C	phase angle of closed-loop response
ϕ_L	total phase angle around loop
$\phi_{L1}, \phi_{L2}, \dots$	loop phase angles at frequencies $\omega_1, \omega_2, \dots$

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ϕ_N	phase angle of nonlinear element
ω	frequency
$\omega_1, \omega_2, \dots$	frequency at points $P_1, P_2, \dots$

2. INTRODUCTION

This document is the second Item that deals with the use of Describing Function techniques for the design and analysis of nonlinear systems.

An earlier Item No. 84009 (Reference 1) introduced the concept and definition of a Describing Function (DF) and showed how it could be used to represent only that part, or element, of a system that contained the nonlinear features of a system. It detailed how the DF uses a linearisation procedure which approximates a nonlinear element by a transfer function that is valid for a sinusoidal input. Further it showed how the stability of an autonomous closed-loop system can be assessed using DF techniques.

This Item extends the DF method to include analysis of the closed-loop steady-state response of a nonlinear system that is subject to a sinusoidal forcing function. Two methods are presented; one is essentially an Amplitude Response Method and the second is a Graphical Method. Examples illustrate how both of these methods are used to find the closed-loop frequency response of a nonlinear system.

3. DESCRIPTION

The closed-loop steady-state response of a nonlinear system may be found by representing the nonlinear element of the system by a Describing Function.

Several modes of anomalous behaviour are possible with forced closed-loop nonlinear systems subject to sinusoidal inputs. These may be classified as:

- (i) the system undergoes self-oscillation regardless of the existence of the forcing function,
- (ii) the system output is a periodic function whose period is a multiple or sub-multiple of the input period, *i.e.* the system undergoes sub-harmonic or harmonic oscillations respectively,
- (iii) the system output is 'almost periodic' or even 'chaotic',
- (iv) the system output is a periodic function whose period is the same as that of its input.

Generally speaking, the only desirable type of response is that characterised by (iv), that is where the system is unconditionally stable, and it is this type of system that is considered in this Item.

Categories (i) to (iii) represent anomalous system behaviour that is, in general, to be avoided.

The self-sustaining, or limit cycle, oscillation of Category (i) has been described in Item No. 84009.

The system behaviour of Category (ii) can occur, for example, when a system's possible limit cycle oscillation is close to a multiple or sub-multiple of the input period. Such a response can only be examined by extending the Describing Function approach still further using an input composed of two harmonic components. This approach is commonly referred to as the Dual Input Describing Function (DIDF). This and other modified inputs are discussed in References 2 to 5.

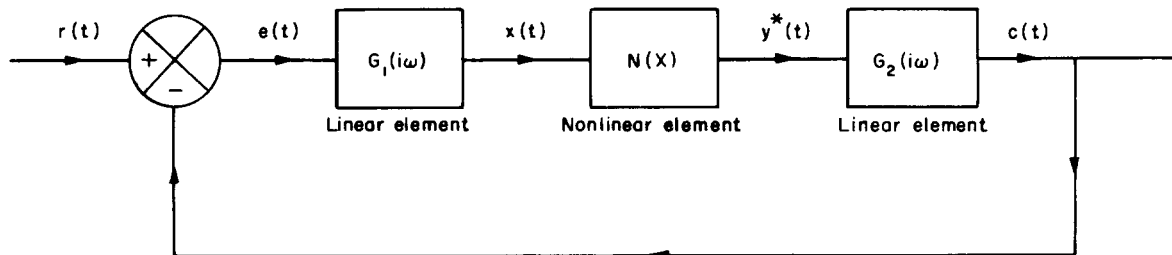
When the period of input and the period of the system self-oscillation are different, the “almost periodic” motion of Category (iii) may occur, for example, in a system on the verge of self-oscillation, although there may be other circumstances when this can occur. The output wave in such a case does not have a distinct period although on casual inspection the waveform seems to be periodic. Chaotic motion, as the name implies, is a random-like output and has been observed in some very simple nonlinear differential equations (see References 6 and 7). However its occurrence is rare and at present there is no established means of checking whether or not a given general nonlinear system will give rise to chaotic behaviour.

There are essentially two reasons for a closed-loop system consisting of a linear element and a nonlinear element to be on the verge of oscillation. A system which is third-order or higher can exhibit oscillatory tendencies because of the nature of the nonlinear element in the loop, or because the linear element has a resonant peak at a particular frequency. By guarding against these two possibilities, the behaviour of Category (iv) is normally assured.

The one further type of anomalous system behaviour that can occur in some lightly damped systems is the phenomenon of jump resonance where the closed-loop frequency response is found to be multi-valued. This is illustrated in the examples of Section 6 of this Item.

4. SYSTEM RESPONSE

Consider the general unity-feedback nonlinear system illustrated in Sketch 4.1 which contains three elements in its forward path, one representing a nonlinear element and the other two representing linear elements.



Sketch 4.1 Nonlinear closed-loop unity feedback system

The behaviour of the nonlinear element is a function only of the amplitude of its input, X , and is represented by the Describing Function, $N(X)$. The behaviour of the linear elements is a function of frequency only and the transfer functions of the two elements are denoted by $G_1(i\omega)$ and $G_2(i\omega)$.

Taking the input to the nonlinear element, $x(t)$, as the reference, the following relationships for sinusoidal input signals may be denoted by,

$$x(t) = X \sin(\omega t), \quad (4.1)$$

$$y^*(t) = Y \sin(\omega t + \phi_N), \quad (4.2)$$

$$c(t) = C \sin(\omega t + \phi_N + \phi_2), \quad (4.3)$$

$$e(t) = E \sin(\omega t - \phi_1), \quad (4.4)$$

$$r(t) = R \sin(\omega t + \phi_C + \phi_N + \phi_2), \quad (4.5)$$

where ϕ_N , ϕ_1 and ϕ_2 are the phase angles of the nonlinear element and the linear elements, $G_1(i\omega)$ and $G_2(i\omega)$, respectively. The phase angle of the closed-loop response is denoted by ϕ_C and the total phase

shift around the loop, ϕ_L , is given by

$$\phi_L = \phi_1 + \phi_N + \phi_2. \quad (4.6)$$

To determine the steady-state response of the system subject to an input of known magnitude, R , and frequency, ω , it is necessary to determine the input to the nonlinear element, $x(t)$. Once the input to the nonlinear element is known the Describing Function (see Item No. 84009) can be used to evaluate the output, $y^*(t)$, and consequently the closed-loop response may be found.

5. ANALYSIS METHODS

This Item will describe two methods for obtaining the closed-loop response of nonlinear systems. The first may be described as an Amplitude Response Method since it relies directly on finding the amplitude, X , of the sine wave input to the nonlinear element for a given R and ω (Derivation 8 and 9). The second method is essentially a Graphical Method and uses a Universal Chart, constructed from the phasor diagram of the complete system, to determine the system response (Derivation 10).

5.1 Amplitude Response Method

The output-to-input transfer function for the complete system of Sketch 4.1 is given by,

$$\frac{c(t)}{r(t)} = \frac{G_1(i\omega) G_2(i\omega) N(X)}{1 + G_1(i\omega) G_2(i\omega) N(X)}. \quad (5.1)$$

The error signal is related to the input signal of the nonlinear element by

$$e(t) = x(t)/G_1(i\omega), \quad (5.2)$$

and to the output signal by

$$e(t) = r(t) - c(t). \quad (5.3)$$

Thus the transfer function relating the input of the nonlinear element to the input of the closed-loop system is

$$\frac{x(t)}{r(t)} = \frac{G_1(i\omega)}{1 + G_1(i\omega) G_2(i\omega) N(X)}, \quad (5.4)$$

which may be rearranged to give

$$\frac{x(t)}{r(t)} = \frac{1/G_2(i\omega)}{1/[G_1(i\omega)G_2(i\omega)] + N(X)}. \quad (5.5)$$

Substituting the terms on the right-hand side of Equation (5.5) by the following expressions

$$\begin{aligned} 1/G_2(i\omega) &= a + ib, \\ 1/[G_1(i\omega)G_2(i\omega)] &= c + id, \end{aligned} \quad (5.6)$$

and
$$N(X) = n_1 + in_2,$$

Equation (5.5) becomes,

$$\frac{x(t)}{r(t)} = \frac{a + ib}{(c + n_1) + i(d + n_2)}. \quad (5.7)$$

The amplitude and phase responses of this equation are given by,

$$\left| \frac{X}{R} \right|^2 = \frac{a^2 + b^2}{(c + n_1)^2 + (d + n_2)^2} \quad (5.8)$$

$$\text{and} \quad \phi_C + \phi_N + \phi_2 = \tan^{-1}(b/a) - \tan^{-1}[(d + n_2)/(c + n_1)]. \quad (5.9)$$

Thus, once the input to the nonlinear element, X , is known for a given R and ω , the quantities n_1 and n_2 (which are functions only of X) may be determined and hence the amplitude ratio, X/R , and phase angles, ϕ_N and ϕ_2 , found from Equations (5.8) and (5.9).

The output-to-input transfer function of the closed-loop system may then be found from,

$$\begin{aligned} \frac{c(t)}{r(t)} = 1 - \frac{1}{|G_1(i\omega)|} \left| \frac{X}{R} \right| \cos(\phi_C + \phi_N + \phi_2 - \phi_1) \\ - \frac{1}{|G_1(i\omega)|} \left| \frac{X}{R} \right| \sin(\phi_C + \phi_N + \phi_2 - \phi_1). \end{aligned} \quad (5.10)$$

The solutions of Equation (5.8) cannot be obtained explicitly, since n_1 and n_2 are functions of X , and so this equation must be solved by a trial-and-error iterative technique or by graphical means. One suitable graphical approach is to plot the square-root of the right-hand side of Equation (5.8) against $|X|$ as a family of curves for different values of ω , and on the same diagram to plot the square-root of the left-hand side of this equation for different values of $|R|$. An illustration of such a plot, for a nonlinear element depicting saturation, is given in Sketch 5.1 (see Item No. 84009 for the Describing Function for saturation).

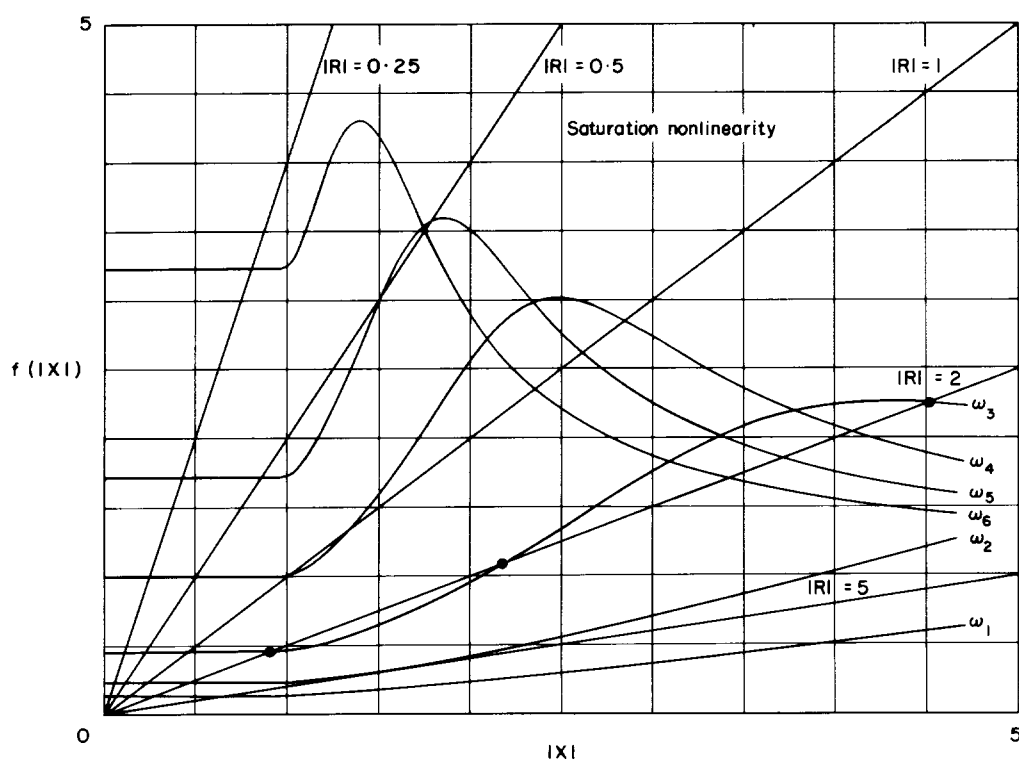
The plots of the left-hand side of Equation (5.8) are straight lines which pass through the origin and have a slope of $1/|R|$. The intersection of a particular $|R|$ curve with an ω curve yields as the abscissa the magnitude, $|X|$, for the particular $|R|$ and ω values chosen. It might have been expected to be more natural to plot lines of constant values of $|X|$ taking the frequency, ω , as the abscissa. However the presentation was found to be inferior to that of Sketch 5.1. The curves of Sketch 5.1 in themselves have no significance other than the intersection points, but they can indicate the existence of jump resonance. This is where more than one $|X|$ value can be obtained from a single ω curve and is illustrated in Sketch 5.1 by the curve for ω_3 and $|R| = 2$.

Significant simplification occurs in the analysis of Equation (5.8) if the nonlinear element of the system is a single valued function, in which case the phase shift of the Describing Function for the nonlinear element is zero, (i.e. $n_2 = 0$) and so

$$N(X) = n_1.$$

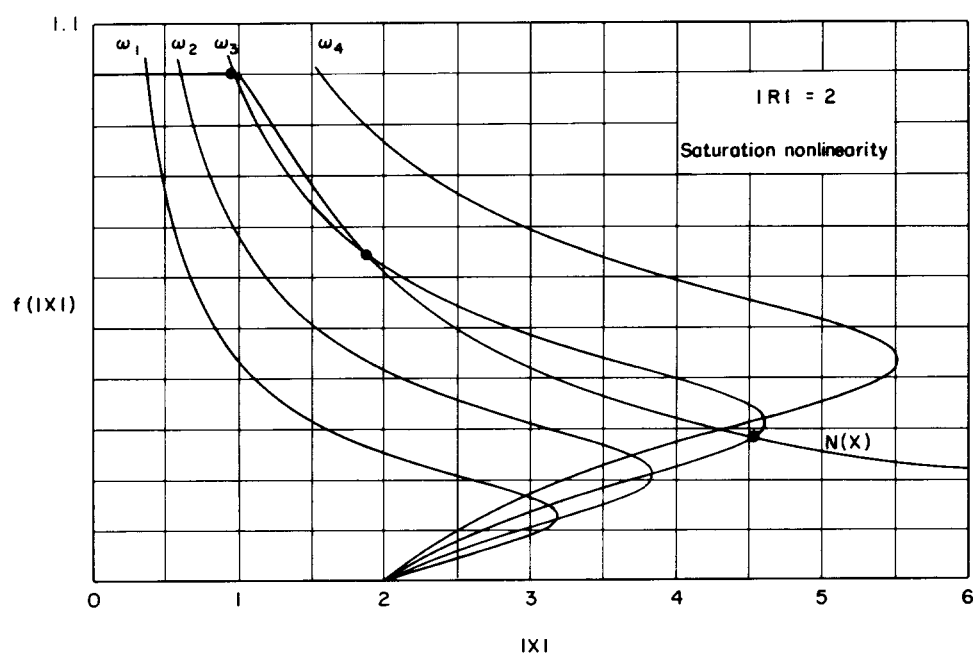
Equation (5.8) for this case can be rearranged to give

$$n_1 = -c \pm \left((a^2 + b^2) |R/X|^2 - d^2 \right)^{1/2}. \quad (5.11)$$



Sketch 5.1 Graphical solution of Equation (5.8)

The solutions of this equation may once again be obtained by graphical means. The left-hand side of this equation is the Describing Function of the nonlinear element and may be plotted directly against $|X|$. The right-hand side is plotted on the same diagram as curves for different values of ω for a chosen value of $|R|$. Such a plot, for a nonlinear element depicting saturation is given in Sketch 5.2.



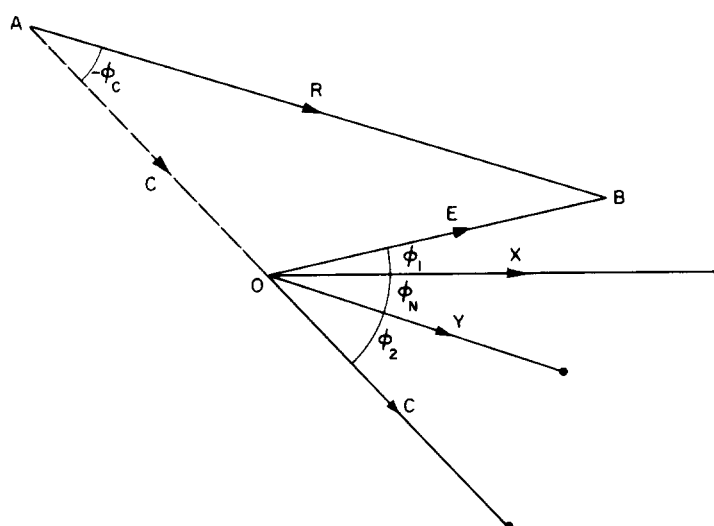
Sketch 5.2 Graphical solution of Equation (5.11)

The ω curves of Sketch 5.2 are seen to have two branches, the lower branch having a common point for all ω at $|X| = |R|$. Jump resonance is again illustrated by the curve for ω_3 where it crosses the DF curve at more than one point. Although the analysis required to produce Sketch 5.2 had been reduced from that required to produce Sketch 5.1, each sketch contains information relating only to a single value of $|R|$.

5.2 Graphical Method

5.2.1 Underlying concepts

The graphical method to be presented here uses a phasor diagram of the complete system to construct a Universal Chart for determining the frequency response of the system. Consider the system given in Sketch 4.1 in which, as before, the input to the nonlinear element, $x(t)$, is taken as the reference signal. The phase angle changes of each element of the system are given by Equations (4.1) to (4.5) which can be used to sketch a phasor diagram of the system. Assuming ϕ_N and ϕ_2 represent phase lags of the nonlinear element and the linear element, $G_2(i\omega)$, respectively, and ϕ_1 represents a phase lead of the linear element, $G_1(i\omega)$, relative to the reference, then the phasor diagram of Sketch 5.3 may be drawn.



Sketch 5.3 Phasor diagram of the system of Sketch 4.1

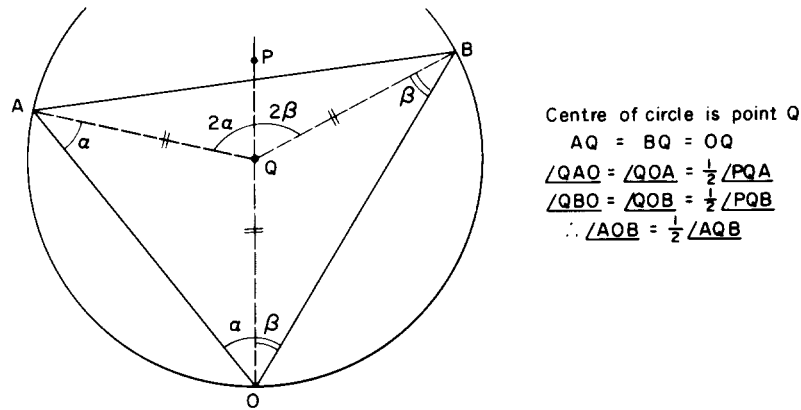
The input to the closed-loop system, R , is found from the relationship

$$e(t) = r(t) - c(t), \quad (5.12)$$

and is seen in the sketch to be the side AB which phase lags the output of the system, $c(t)$, by an angle ϕ_C . (Note, the angle $\angle(\text{BAO})$ in Sketch 5.3 is $-\phi_C$ which is consistent with the Analytical Method.)

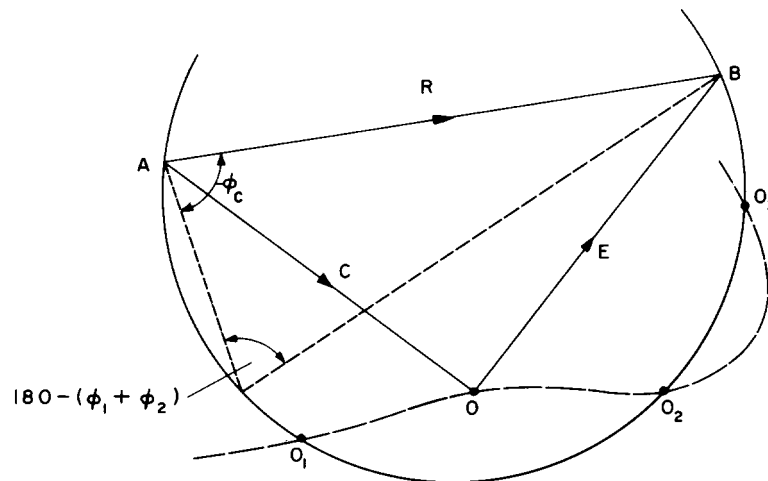
Consider the simpler case initially, that of a single-valued nonlinear element. For this case $\phi_N = 0$. Thus for a given system at a particular frequency, ω , the angles ϕ_1 and ϕ_2 are fixed, and further, for a given value of R the angle AOB is fixed. So for a given R and ω , the magnitude of the output, C , is given by the side AO provided that the following two conditions are satisfied.

- (i) The point O must lie on a circle with segment AB, subtending an angle $[180 - (\phi_1 + \phi_2)]$ at O. (This can be seen from Sketch 5.4).
- (ii) $C = Y|G_2(i\omega)|$ and $E = X/|G_1(i\omega)|$, where the relation between X and Y is the Describing Function for the nonlinear element.



Sketch 5.4 Proof that point O lies on a circle if AB and $\angle(AOB)$ are fixed

The triangle AOB must satisfy Conditions (i) and (ii) and so Sketch 5.5 may be constructed where the points O_1 , O_2 and O_3 satisfy Condition (i) since they lie on a circle with segment AB, and the points on the dotted line satisfy Condition (ii) since $AO = C$ and $BO = E$. The sketch recognises the possibility of more than one solution for the closed-loop response of the system, *i.e.* jump resonance.



Sketch 5.5 Nonlinear system satisfying Conditions (i) and (ii)

5.2.2 Universal Chart construction

For the triangle AOB, satisfying Condition (i), the following relationships may be obtained,

$$\frac{E}{\sin(-\phi_C)} = \frac{R}{\sin(\pi - \phi_L)}$$

where $\phi_L = \phi_1 + \phi_2$

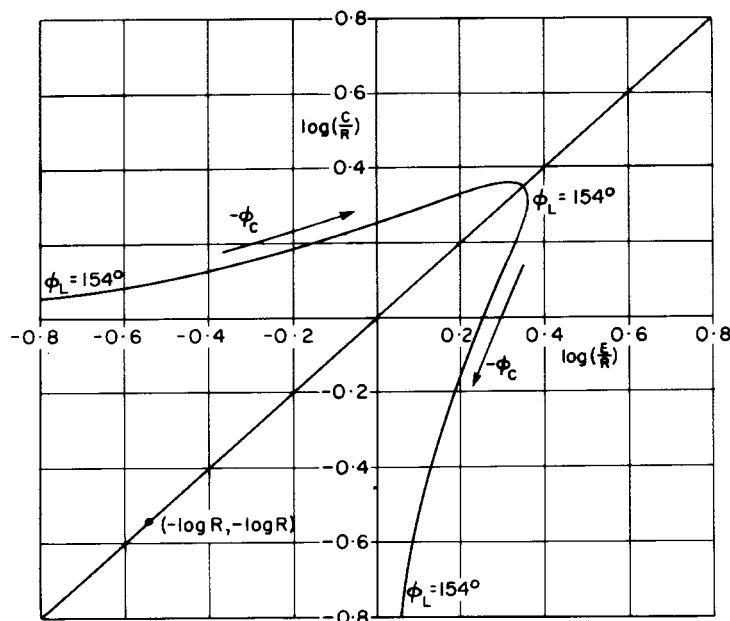
$$\text{or, } \frac{E}{R} = \frac{\sin(-\phi_C)}{\sin \phi_L} \quad (5.13)$$

$$\text{Also } \frac{C}{\sin(\phi_1 + \phi_2 + \phi_C)} = \frac{R}{\sin \phi_L},$$

$$\text{or } \frac{C}{R} = \frac{\sin(\phi_L + \phi_C)}{\sin \phi_L} \quad (5.14)$$

$$\left. \begin{aligned} \text{Hence, } \log(E/R) &= \log[\sin(-\phi_C)] - \log(\sin \phi_L) \\ \text{and } \log(C/R) &= \log[\sin(\phi_L + \phi_C)] - \log(\sin \phi_L) \end{aligned} \right\} \quad (5.15)$$

Choosing $-\phi_C$ as the variable, the graph of $\log(C/R)$ versus $\log(E/R)$ may be drawn for a constant value of ϕ_L . This is shown in Sketch 5.6.



Sketch 5.6 Curve of $\log\left(\frac{C}{R}\right)$ versus $\log\left(\frac{E}{R}\right)$ for a fixed value of ϕ_L

The curve is symmetrical about a line at 45° to the axes and a value on this line is the point $[-\log(R), -\log(R)]$. If the origin of this figure is shifted to this point, then the curve represents the variation of $\log(C)$ with $\log(E)$ for the given loop phase shift, ϕ_L . This curve is unique for the given value of ϕ_L . If a set of these curves is drawn for various values of ϕ_L , then a Universal Chart is obtained which may be used for any system. Such a chart has been drawn and is given as Figure 1.

5.2.3 Curves for nonlinear and linear elements

From Condition (ii) the following relationships may be found,

$$\left. \begin{aligned} \log(E) &= \log(X) - \log[G_1(i\omega)] \\ \log(C) &= \log(Y) + \log[G_2(i\omega)] \end{aligned} \right\} \quad (5.16)$$

From the Describing Function for the nonlinear element and the characteristics of the linear elements of the system, $\log(Y)$ versus $\log(X)$ and $\log|G_1(i\omega)|$ versus $\log|G_2(i\omega)|$ may be plotted, as illustrated in Sketch 5.7.

The curve of the linear elements is a function of frequency ω , for which there is a corresponding phase angle, ϕ_L . Consider a chosen point on the linear elements curve, say P_1 , of frequency ω_1 and phase angle