

ELASTIC STRESS AND STRAIN DISTRIBUTIONS AROUND CIRCULAR HOLES IN INFINITE PLATES OF ORTHOTROPIC MATERIAL (Applicable to Fibre Reinforced Composites)

1. APPLICATION AND DERIVATION OF PROGRAM

1.1 Introduction

This Data Item together with ESDUpac A8501 presents a Fortran computer program for calculating the elastic stress and strain distribution around the perimeter of circular holes in homogeneous orthotropic flat plates subject to any combination of in-plane loads. The data may be applied to fibre reinforced laminated composites containing a circular hole*, a typical example being a composite bulkhead or web containing a circular cut-out for the passage of a fuel or hydraulic pipe.

The Data Item is provided in two parts. Section 1 describes the theoretical solution to the title problem and its application to practical plates. Section 2 describes the input and output of the Fortran program of ESDUpac A8501 together with example calculations. The compiled program is available on disk within ESDUview. The Fortran source code is also available on the same disk which is located within the Sub-series Software Volume.

1.2 Notation

The notation shows both SI and pounds-force inch units. Either system, or any other coherent system, may be used in the computer program in ESDUpac A8501.

1.2.1 Stresses (see Sketch 1.1)

$\sigma_{c\theta}$	circumferential stress at point θ on edge of hole	N/m ²	lbf/in ²
σ_x, σ_y	direct stress in plate, remote from hole, acting parallel to x- and y-axes respectively	N/m ²	lbf/in ²
$\sigma_{x\theta}, \sigma_{y\theta}$	direct stress in plate parallel to x- and y-axes respectively, acting at point θ on circumference of hole	N/m ²	lbf/in ²
τ_{xy}	shear stress in plate, remote from hole, acting parallel to x- and y-axes	N/m ²	lbf/in ²
$\tau_{xy\theta}$	shear stress in plate parallel to x- and y-axes, acting at point θ on circumference of hole	N/m ²	lbf/in ²

* Data Item No. 86003, "Example of the use of Data Item No. 85001. Choice of reinforcement for a circular hole in a fibre reinforced laminated plate (data relating to one particular set of laminate properties)", shows how the program of this Item may be used to generate curves for selection of lay-up arrangements around a hole.

1.2.2 Strains

$\epsilon_{c\theta}$	circumferential strain in plate acting at point θ on edge of hole
$\epsilon_{r\theta}$	radial strain in plate acting at point θ on circumference of hole
$\epsilon_{x\theta}, \epsilon_{y\theta}$	direct strain in plate parallel to x- and y-axes respectively, acting at point θ on circumference of hole
γ_{xy}	shear strain in plate, remote from hole, acting parallel to x- and y-axes
$\gamma_{rc\theta}$	radial shear strain in plate acting at point θ on circumference of hole
$\gamma_{xy\theta}$	shear strain in plate parallel to x- and y-axes, acting at point θ on circumference of hole

1.2.3 Plate properties

A	in-plane stiffness matrix for plate (see Data Item No. 94003*)	N/m	lbf/in
E_x, E_y	moduli of elasticity of plate in x- and y-directions respectively	N/m ²	lbf/in ²
G_{xy}	shear modulus of plate in xy-plane	N/m ²	lbf/in ²
t	total plate thickness	m	in
ν_{xy}	Poisson's ratio of plate resulting from load applied in x-direction		
ν_{yx}	Poisson's ratio of plate resulting from load applied in y-direction ($\nu_{yx} = \nu_{xy} E_y / E_x$)		

* Data Item No. 94003 "Stiffnesses of laminated flat plates".

1.2.4 Layer properties

C	layer reference number (see Section 2.3.1.1)		
E_α, E_β	moduli of elasticity of a layer in α - and β -directions respectively	N/m ²	lbf/in ²

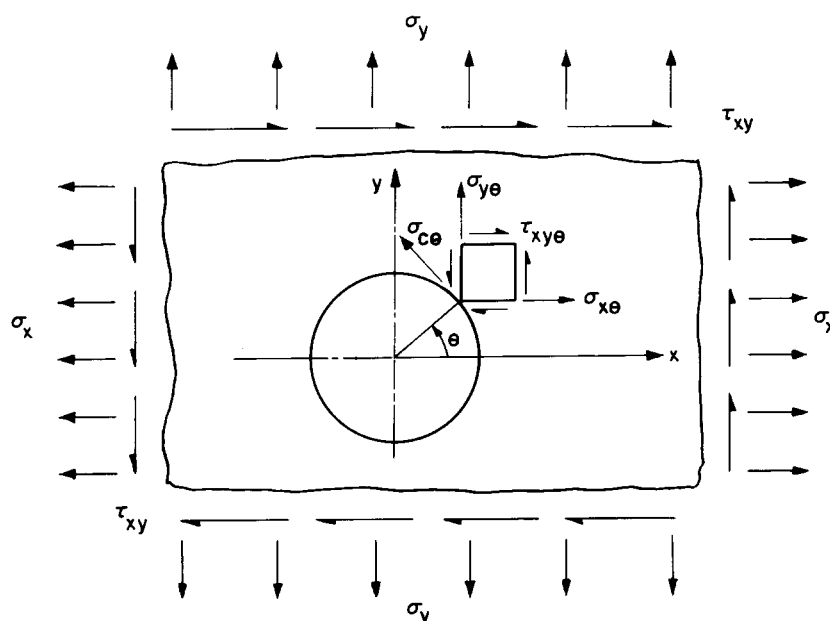
$G_{\alpha\beta}$	shear stiffness of a layer in $\alpha\beta$ -plane	N/m ²	lbf/in ²
$\nu_{\alpha\beta}$	Poisson's ratio for a layer resulting from load applied in α -direction		
ψ	angle of principal axis of orthotropy, α , of a layer with respect to x -axis	degree	degree

1.2.5 Terms used in governing equations

α, δ	see Equations (1.5)
β_1, β_2	see Equations (1.4)
ξ, λ, μ	see Equations (1.3)

1.2.6 Coordinates

x, y	plate coordinates, chosen such that x -axis lies along a principal axis of orthotropy		
α, β	layer coordinates, chosen such that α -axis lies along principal axis of orthotropy of layer		
θ	angle measured from x -axis around circumference of hole (see Sketch 1.1)	degree	degree



Sketch 1.1

1.3 Application to Laminated Fibre Reinforced Composites

The computer program of ESDUpac A8501 provides a solution to the elastic stress and strain distributions around the perimeters of circular holes in thin homogeneous orthotropic plates under any combination of in-plane loadings. The application of the results of the program to laminated fibre reinforced composites is only approximate since these materials are not homogeneous. Further, such materials experience inter-laminar stresses near the hole perimeter and load shedding near the ends of fibres that terminate at the hole and neither of these effects is modelled. Nevertheless, the stresses and strains obtained from the program may be used to assess the relative strengths of various lay-ups with regard to in-plane failure of the fibres and/or matrix.

At a very small hole the high stresses are localised to a region where the constant through-the-thickness strain assumption is invalid. Thus, the maximum stresses obtained using the data presented in this Item are more appropriately applied to larger holes in thin laminates, that is, **diameter greater than about 5t**. Since orthotropic plate theory assumes the existence of an axis system about which there is no coupling between in-plane direct and shear stiffnesses, the program can only be applied to arrangements of laminates for which this is the case. This covers the vast majority of practical laminates. The theory used is strictly only valid for an infinite plate. Nevertheless, the results of the computer program can be used with sufficient accuracy for plates that have a boundary, or other feature, no closer than about $1\frac{1}{2}$ hole diameters away from the hole perimeter.

In order to ensure the integrity of a plate containing a hole it is necessary to ensure that everywhere the material strength exceeds the local loading. In a fibre reinforced composite this is complicated by the fact that the stress distribution is determined not only by the loading but also by the material properties, which are a function of the lay-up arrangement. Failure does not necessarily occur at the point of maximum stress but at some other point where the local stress exceeds the material strength in the direction of the local loading. In order to facilitate the process of checking the strength against the local loading the computer program provides the complete stress and strain distribution around the hole both in the xy coordinates and circumferentially and tangentially to the hole perimeter. The results of the program can be employed in the computer program of Data Item No. 84018* to carry out a layer-by-layer failure criterion check.

* Data Item No. 84018 "Failure analysis of fibre reinforced composite laminates".

1.4 Equations Used Within ESDUpac A8501

1.4.1 Stiffnesses

Values of the in-plane stiffness matrix, **A**, may be calculated using the method set out in Data Item No. 94003. The values of E_x , E_y , G_{xy} and ν_{xy} may be calculated from the values of **A** using

$$\left. \begin{aligned} E_x &= A_{11}(1 - \nu_{xy}\nu_{yx})/t \\ E_y &= A_{22}(1 - \nu_{xy}\nu_{yx})/t \\ G_{xy} &= A_{33}/t \\ \nu_{yx} &= A_{12}/A_{11} \\ \nu_{xy} &= A_{12}/A_{22} \end{aligned} \right\} \quad (1.1)$$

1.4.2 Equations for stresses around a hole perimeter

The figures are derived from the following equation which calculates the circumferential stress around a hole for an orthotropic plate subjected to biaxial and shear loading.

$$\sigma_{c\theta} = \frac{(\sigma_x + \sigma_y)\mu - \sigma_y\xi + \tau_{xy}\lambda}{(1 + \beta_1^2 - 2\beta_1 \cos 2\theta)(1 + \beta_2^2 - 2\beta_2 \cos 2\theta)}, \quad (1.2)$$

where

$$\left. \begin{aligned} \mu &= (1 + \beta_1)(1 + \beta_2)(1 + \beta_1 + \beta_2 - \beta_1\beta_2 - 2 \cos 2\theta) \\ \xi &= 4\{\beta_1 + \beta_2 - (1 + \beta_1\beta_2) \cos 2\theta\} \\ \lambda &= 4(\beta_1\beta_2 - 1) \sin 2\theta, \end{aligned} \right\} \quad (1.3)$$

where

$$\left. \begin{aligned} \beta_1 &= (\alpha - 1)/(\alpha + 1) \\ \beta_2 &= (\delta - 1)/(\delta + 1) \end{aligned} \right\} \quad (1.4)$$

for which,

$$\left. \begin{aligned} \alpha &= \left[\left\{ \frac{E_y}{2G_{xy}} - \nu_{yx} \right\} + \left[\left\{ \frac{E_y}{2G_{xy}} - \nu_{yx} \right\}^2 - \frac{E_y}{E_x} \right]^{\frac{1}{2}} \right]^{\frac{1}{2}} \\ \delta &= \left[\left\{ \frac{E_y}{2G_{xy}} - \nu_{yx} \right\} - \left[\left\{ \frac{E_y}{2G_{xy}} - \nu_{yx} \right\}^2 - \frac{E_y}{E_x} \right]^{\frac{1}{2}} \right]^{\frac{1}{2}} \end{aligned} \right\} \quad (1.5)$$

1.4.3 Stress and strain relationships

The direct stresses and shear stress acting in the x - and y -directions equivalent to $\sigma_{c\theta}$ can be obtained using the transform matrix given in Section 4.1 of Data Item No. 94003. Since, at the hole boundary, both the radial stress and the shear stress are zero it follows that

$$\left. \begin{aligned} \sigma_{x\theta} &= \sigma_{c\theta} \sin^2 \theta, \\ \sigma_{y\theta} &= \sigma_{c\theta} \cos^2 \theta \\ \tau_{xy\theta} &= -\sigma_{c\theta} \sin \theta \cos \theta. \end{aligned} \right\} \quad (1.6)$$

Hence, the x - y strains around the edge of the hole are

$$\left. \begin{aligned} \epsilon_{x\theta} &= \frac{1}{E_x} (\sigma_{x\theta} - \nu_{xy} \sigma_{y\theta}), \\ \epsilon_{y\theta} &= \frac{1}{E_y} (\sigma_{y\theta} - \nu_{yx} \sigma_{x\theta}) \\ \gamma_{xy\theta} &= \frac{\tau_{xy\theta}}{G_{xy}}. \end{aligned} \right\} \quad (1.7)$$

These strains can be transferred back into the circumferential and radial direction using the same transform matrix noted above. Thus,

$$\begin{bmatrix} \epsilon_{r\theta} \\ \epsilon_{c\theta} \\ \gamma_{rc\theta} \end{bmatrix} = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & \frac{1}{2} \sin 2\theta \\ \sin^2 \theta & \cos^2 \theta & -\frac{1}{2} \sin 2\theta \\ -\sin 2\theta & \sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} \epsilon_{x\theta} \\ \epsilon_{y\theta} \\ \gamma_{xy\theta} \end{bmatrix} \quad (1.8)$$

1.5 Derivation

This section lists selected sources that have assisted in the preparation of this Item.

1. GRESZCZUK, L.B. Stress concentration and failure criteria for orthotropic and anisotropic plates with circular openings. A.S.T.M. STP 497, April 1971
2. GREEN, A.E. ZERNA, W. *Theoretical elasticity*. Chapter 9. Oxford Clarendon Press, 1954.
3. de JONG, T. Stresses around rectangular holes in orthotropic plates. *Journal of Composite Materials*, Vol. 15, July 1981

2. ESDUpac A8501

2.1 Introduction

ESDUpac A8501 contains a Fortran computer program that accepts the properties of an orthotropic plate and its applied in-plane loading. The properties may be provided as details of the properties of the individual layers of a laminated plate or as stiffnesses or moduli of the complete plate. Using the equations of Section 1.4 the program calculates the stresses and strains around the edge of a circular hole in an infinite orthotropic plate of the given properties and under the given loading. Section 2.2 lists program limitations, Section 2.3 details the input requirements and Section 2.4 gives an explanation of the output, including all messages. Section 2.5 provides a worked example that illustrates use of the program and may be used as a test case.

The program is provided on a diskette which is located in the Sub-series Software Volume. Information on how to install and run the program is contained in the introduction to the Software Volume. The ESDUpac also contains example inputs and the resulting outputs. In addition the disk contains the compiled program and files ready for use with ESDUview, see "Using ESDUview" in the Sub-series software volume. If the program is to be used within ESDUview it is only necessary to consult Section 2.4 and, if required, Section 2.5 on the program output.

The basis of the FORTRAN computer Program of ESDUpac A8501 is described in Part 1 of this Data Item. Every reasonable effort has been made to ensure that the program performs the intended calculations satisfactorily. No representation as to the suitability or fitness of the program for any particular purpose is made and no liability for any loss occasioned by any persons as a direct or indirect result of use of the program whether arising from negligence or otherwise can be accepted. In no event shall ESDU or any individuals associated with the development of the program be liable for any damages, including loss of profit or consequential loss, arising out of or in connection with the program.

2.2 Program Limitations

This section gives a synopsis of the limitations of the program. The plate properties may be input as stiffnesses, moduli, or, for laminated plates, as the individual layer properties and their orientations. In the latter case the plate stiffnesses are calculated by the program.

Input of individual layer properties.

- (a) The number of layer types is limited to 9.
- (b) The number of layers in a plate is limited to 500.
- (c) Positive finite values must be provided for E_{α} , E_{β} , $G_{\alpha\beta}$, $\nu_{\alpha\beta}$, t and ψ for each layer type.

Plate stiffnesses input directly.

- (i) Positive finite values of the elements of **A** must be provided.
- (ii) A positive finite value for the plate thickness, t , must be provided.

Plate moduli input directly.

- (1) Positive finite values must be provided for E_x , E_y , ν_{xy} and G_{xy} .

All inputs.

- (α) A value must be provided for each of the applied stresses σ_x , σ_y and τ_{xy} (two of which may be zero).
- (β) The number of different loading cases that can be examined for one plate in a single run is limited to 10.

Sections 1.3 and 1.4 contain a full description of the limitations of the theory and Section 2.4 gives full details of the program output including details of all messages.

2.3 Data Input

Prior to data input 3 lines are provided for run identification information. These lines may be empty or can carry up to 75 alpha numeric characters each. The contents of these lines appear in the output.

Also prior to the input of materials values, 2 lines are provided for the declaration of the units in which all the input data are to be supplied. These lines may be left empty or can carry up to 75 alpha numeric characters each. These lines also appear in the output. (Note that this is NOT an option to select units but simply the invitation to state the units used in the input data and to echo these statements in the output.)

Following the statement on units two distinct sets of input data are required. They are the plate properties and the applied loading. Three input options are available for the properties and these, together with the loading data, are explained in Tables 2.1 to 2.3. The following sections, referred to on the tables, give further details. A coherent system of units must be used throughout the input.

2.3.1 Input of plate properties

The program offers three options for input for the plate properties. They are

- (i) for laminated plates, details of the individual layer moduli, thickness, Poisson's ratio, orientation and the layer stacking sequence,
- (ii) the plate in-plane stiffnesses and total thickness or
- (iii) the plate in-plane moduli and a Poisson's ratio.

The required input is selected via an input code number as detailed in the following three sections.

2.3.1.1 Input of laminated plate layer properties (input code = 1, Table 2.1)

This type of input is signalled by the integer 1 which is followed by the layer properties and their stacking sequence. Values for layers having the same properties, thickness and orientation are listed only once with a prefixing layer reference number, C . Note that for orthotropic layers the angle ψ must be set at or between 0° and 90° . Preceding the list of layer properties is a number giving the total number of layers for which the properties are listed. The example in Table 2.7 shows two layers. However, it may be convenient to declare other layers at this stage, which can be held on file for future use, and provision is made for up to 9.