

CONTACT PHENOMENA III: CALCULATION OF INDIVIDUAL STRESS COMPONENTS IN CONCENTRATED ELASTIC CONTACTS UNDER COMBINED NORMAL AND TANGENTIAL LOADING

1. NOTATION

Most of the quantities are expressed in non-dimensional form, stresses being normalised by dividing them by the maximum compressive stress p_0 , and distances by dividing them either by the major semi-axis, a (subscript 1), or by the minor semi-axis, b (subscript 2) of the ellipse of contact. They are therefore valid in any self-consistent system of units.

a	major semi-axis of contact ellipse, or radius of circular contact
a_0, a_1, a_2	coefficients defined in Section 16.3
b	minor semi-axis of contact ellipse, or half-width of line contact
b_0, b_1, b_2	coefficients defined in Section 6.1
C, D, E, F	auxiliary quantities defined in Section 5
c, d, e, f	recursion variables used in Section 8
$c_0, \dots, c_4$	constants defined in Section 8.6
$d_0, \dots, d_4$	constants defined in Section 8.6
$e_0, \dots, e_4$	constants defined in Section 8.6
$f_0, \dots, f_4$	constants defined in Section 8.6
$E(\phi, n)$	incomplete elliptic integral of the second kind, $= \int_0^\phi \left(1 - n^2 \sin^2 \theta\right)^{1/2} d\theta$
$F(\phi, n)$	incomplete elliptic integral of the first kind, $= \int_0^\phi \left(1 - n^2 \sin^2 \theta\right)^{-1/2} d\theta$
f_x, f_y, f_z	direct stresses within solid body
G, H	auxiliary quantities defined in Sections 5.4 and 5.5, respectively
I_1, I_2, I_3	integrals defined in Sections 5.8, 5.9 and 5.10, respectively
J, K	auxiliary quantities defined in Sections 5.18 and 5.21 respectively
L, N	auxiliary quantities defined in Sections 5.6 and 5.11 respectively
N_i	number of sub-divisions into which an integration interval is divided
n	$= \left(1 - \beta^2\right)^{1/2}$

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P	total normal load
p_0	maximum compressive stress in contact region
Q	total force of friction
q, r	auxiliary quantities defined in Section 6.2
q_e	effective or equivalent shear stress
q_{xy}, q_{yz}, q_{zx}	shear stresses within solid body
r_1	auxiliary quantity defined in Section 5.3
s_1, s_2	auxiliary quantities defined in Sections 5.2 and 13.2, respectively
T	tensile stress
t	recursion variable used in Section 8
V_1, V_2	auxiliary quantities defined in Sections 9.4 and 13.3, respectively
v, w	recursion variables used in Section 8
W_1, W_2	auxiliary quantities defined in Sections 9.5 and 13.4, respectively
w_1	dummy integration variable used in definitions of I_1, I_2 and I_3 , Sections 5.8, 5.9 and 5.10, respectively
x	distance measured from centre of ellipse, parallel to major axis
x_1	$= x/a$
x_2	$= x/b$
y	distance measured from centre of ellipse, parallel to minor axis
y_1	$= y/a$
y_2	$= y/b$
z	distance measured from centre of ellipse, normal to surface and directed towards the interior of the body
z_1	$= z/a$
z_2	$= z/b$
α	modular angle $= \cos^{-1}\beta = \sin^{-1}n$
β	axis ratio of ellipse $= b/a \leq 1$

γ	angle between force of friction and minor axis of ellipse
δ	auxiliary quantity defined in Section 7.3
Δ	range, <i>i.e.</i> difference between maximum and minimum values
ε_1	auxiliary quantity defined in Section 12.1
θ_1, θ_2	auxiliary quantities defined in Sections 5.16 and 5.17 respectively
μ	coefficient of friction for gross sliding
ρ_1	$= \left(x_1^2 + y_1^2 \right)^{1/2}$
σ	Poisson's ratio
ϕ	$= \tan^{-1}(1/s_1), \text{ except in Section 10 where } \phi = \tan^{-1}(1/z_1) \text{ and in Section 12 where } \phi = \sin^{-1}(1/\rho_1)$
ψ	auxiliary quantity defined in Section 6.2

Subscripts

max	maximum value
0	pertaining to contact centre at surface
$orth$	orthogonal, applied to shear stress
x, y, z	single subscripts applied to direct stresses, denoting action in directions parallel to, and over planes perpendicular to, the x, y and z axes, respectively
xy, yz, zx	double subscripts applied to shear stresses, in which the first subscript denotes the direction normal to the plane in which the stresses act and the second gives the direction of the stress
$0, 1, 2, \dots$	serial number of recursion in Section 8
$()_1, ()_2, ()_3$	subscripts applied outside brackets to stress components, denoting origin in one of the three loading systems defined in Section 4, Equations (4.1), (4.2) and (4.3)

Superscript

A bar superscript over a stress symbol denotes that the stress has been rendered dimensionless by dividing it by the maximum compressive stress p_0 .

2. INTRODUCTION

When two bodies, one or both of whose surfaces are curved, are brought into contact and subjected to a normal load, the nominal point of contact becomes an elliptical area of contact as a result of elastic

deflections within the bodies. Two limiting cases may be identified. In the first, the contact area is circular, for example, sphere on sphere, sphere on plane, identical cylinders crossed at right-angles. In the second limiting case, a nominal line contact becomes a rectangle, for example, cylinder on cylinder with parallel axes, cylinder on plane, cylinder in groove with parallel axes.

The elastic deflections imply the presence of stress fields within and on the surface of the bodies. These stress fields are characterised by the values at each point of the six components of the stress tensor, defined in Section 3.

A previously published Data Item, No. 78035 (Reference 1), dealt with the calculation of stresses and contact dimensions for normally loaded contacts. The maximum values of all the stress components then occur along the line of symmetry, normal to the surface and passing through the centre of the ellipse, and much of the information of interest could be presented by a plot of the values of the stress components, and of quantities derived from them, against position along this line.

A second data Item in this series, No. 84017 (Reference 2), considered the effect on the stress fields of the addition of tangential shear stresses over the surface of the ellipse, the ratio of the shear stress to the normal stress having everywhere a constant value, equal to the coefficient of friction for gross slip. The inclusion of these shear stresses destroyed the symmetry of field, and the maximum values no longer necessarily occurred on the axis of symmetry. This complicated the presentation of the results of calculations of the stress field, and the coverage of the effect of different combinations of the various factors, such as coefficient of friction, Poisson's ratio, axis ratio of ellipse of contact, *etc.* was less comprehensive than in the previous Item, No. 78035. Some further results will be given in an ESDU Memorandum No. 50 (Reference 3), which is available to all holders of Reference 2. Results for cases not covered adequately in References 2 and 3 may be calculated by the procedures explained in the present Item.

Analytic expressions for the stress fields in the two limiting cases of circular and line contact under combined normal and tangential loading have been available in the literature for some time and are well known (References 4 to 7 inclusive and Derivations 9 and 10), but solutions for the general elliptical contact are more complicated, involve elliptic integrals, and have been published only recently (Reference 7, Derivations 10 to 12 inclusive). Some degree of mathematical sophistication is required in the application of the results of these Derivations, and there is a large quantity of data to be handled.

In Reference 2 the presentation of the results was limited to the consideration of the maximum values of three criteria for failure, attention being concentrated on three simplified models of failure:

Plastic yielding of ductile materials,

Cracking of brittle materials,

Fatigue failure under repeated stressing.

The treatment was over-simplified, in that very few materials behave exactly in the manner assumed, but it was considered to be adequate for many purposes.

For accurate and reliable prediction of failure, a detailed knowledge of the behaviour of the material, and a detailed knowledge of the variation of the individual stress components with position and with time, are required.

The present Item explains the procedure for the calculation of these individual stress components. Although it may appear excessively long and formidably complicated, no great mathematical expertise is required to apply the procedures. All the operations are broken down into simple steps, and in many cases the treatments

given in the Derivations have been simplified considerably. The expressions given for the general case may become indeterminate in some special cases and these special cases are covered in separate sections. In other special cases the general treatment remains valid but the expressions can be simplified, and again, full coverage is given for these cases.

In the application of the information in the present Data Item to the solution of practical problems, it will be assumed that the essential quantities, the maximum compressive stress and the lengths of the axes of the ellipse of contact, have been calculated from the information given in Data Item No. 78035. It will also be assumed that the values of these quantities are unaffected by the imposition of the tangential stresses. This latter assumption is not strictly valid if the elastic properties of the two bodies are different, but in general the effects of the differences are small. Further details are given in Reference 2, Section 8.3.

Shear stresses of the form assumed arise in the case of gross sliding between dry (unlubricated) bodies. The distribution of shear stresses is more complicated if the friction is insufficient to cause gross sliding, but the assumption of gross sliding will result in higher values of the stresses than will occur in other circumstances, that is, the situation that has been assumed is the worst that could arise. The stress distribution may differ from that assumed if the surfaces are lubricated, but the results given in this Data Item are valid for boundary lubrication, and are approximately correct for the limiting case of very thin films in elastohydrodynamic lubrication, that is, with low surface velocities and low lubricant viscosities. A more detailed discussion of the limitations to the application of the results is given in Reference 2.

3. REFERENCE AXES AND STRESS COMPONENTS

Sketch 3.1 represents the area of contact between two solid elastic bodies as the ellipse,

$$(x/a)^2 + (y/b)^2 = 1, \quad a \geq b.$$

This area supports the total load P imposed on the contact by a distribution of compressive stress p over the ellipsoid:

$$p = p_0 \left(1 - (x/a)^2 - (y/b)^2 \right)^{1/2}, \quad (x/a)^2 + (y/b)^2 \leq 1.$$

A force of friction $Q = \mu P$ acts at a general angle γ to the y -axis and it is assumed that the resultant shear stress, q , on the surface is everywhere equal to μp .

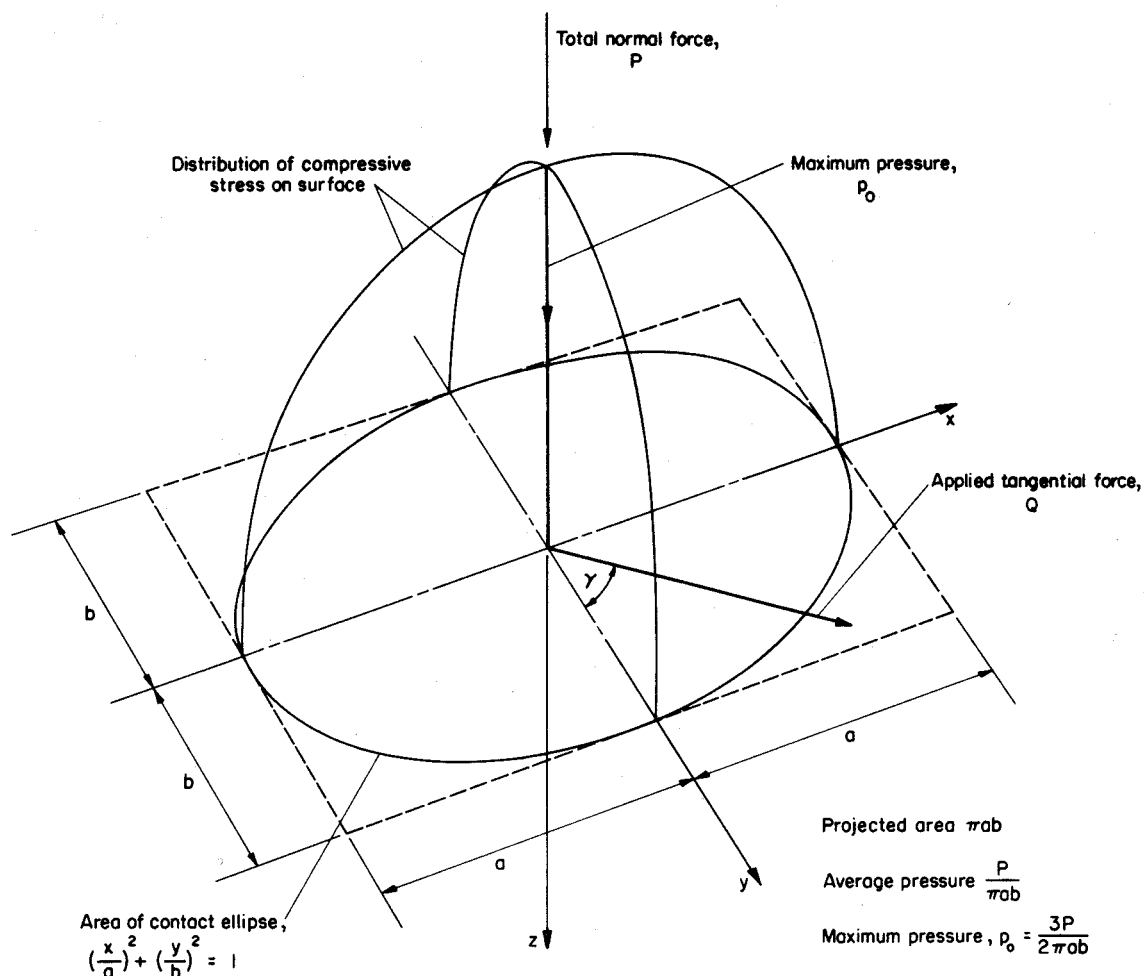
The stress field within and on the surface of one of the bodies is characterised by the values of the six components of the stress tensor. These are the three direct stresses;

$$f_x, f_y, f_z$$

acting in directions parallel to, and over plane perpendicular to, the x , y and z axes respectively, and the three shear stresses;

$$q_{xy} = q_{yx}, \quad q_{yz} = q_{zy}, \quad q_{zx} = q_{xz}.$$

The z -axis is normal to the tangent plane at the centre of the ellipse and directed toward the interior of the body. The usual convention that positive direct stresses are tensile will be adopted here. In the symbol for the general shear stress q_{xy} , the first subscript gives the direction of the normal to the plane in which the stress acts and the second subscript gives the direction of the stress.



Sketch 3.1 Area of contact, forces and stresses

4. GENERAL DESCRIPTION OF THE CALCULATIONS

These calculations were obtained from Derivations 10 to 13 inclusive, but some modifications and simplifications have been made.

For the purposes of the calculations, spatial coordinates are rendered non-dimensional by dividing them by the major semi-axis of the ellipse of contact:

$$x_1 = x/a; \quad y_1 = y/a; \quad z_1 = z/a.$$

For the special case of line contacts, when $a \rightarrow \infty$ the dimensional coordinates are divided by the half-width b :

$$x_2 = x/b = x_1/\beta; \quad y_2 = y/b = y_1/\beta; \quad z_2 = z/b = z_1/\beta;$$

where $\beta = b/a$ is the axis ratio of the ellipse, and $\beta \rightarrow 0$ for a line contact. It will normally be convenient to express the results for the general case in terms of (x_2, y_2, z_2) rather than of (x_1, y_1, z_1) .

The stress components are rendered non-dimensional by dividing them by the maximum compressive stress p_0 ,

$$\bar{f}_i = f_i/p_0; \quad \bar{q}_{ik} = q_{ik}/p_0; \quad \text{etc}, \quad i, k = x, y, z \quad .$$

Three sets of components are given, corresponding with the effects of the three loading cases assumed:

1. Normal loading:
$$f_z = -p_0 \left(1 - (x/a)^2 - (y/b)^2 \right)^{1/2} \quad (4.1)$$

2. Tangential loading parallel to short dimension:
$$q_{zy} = -\mu p_0 \left(1 - (x/a)^2 - (y/b)^2 \right)^{1/2} \cos \gamma \quad (4.2)$$

3. Tangential loading parallel to long dimension:
$$q_{zx} = -\mu p_0 \left(1 - (x/a)^2 - (y/b)^2 \right)^{1/2} \sin \gamma \quad (4.3)$$

The forces of friction are in the positive directions of the x - and y -axes. The three sets are distinguished by numerical subscripts, *e.g.* $(f_x)_1$, $(q_{xy})_2$, *etc.* Sets 1 and 3 have been obtained directly from the Derivations, with μ replaced by $\mu \sin \gamma$, while set 2 has been obtained from set 3 by interchanging x and y , a and b , β and $1/\beta$, $\sin \gamma$ and $\cos \gamma$ and by reversing the direction of x , *i.e.* reversing the sign of $(q_{xy})_2$ and $(q_{zx})_2$. The justification for this procedure is shown in Sketch 4.1.

The general elliptical case is considered first, and is followed by the simpler cases of circular and line contacts. The general case requires the evaluation of three numerical integrals, either directly or by conversion to the standard forms of the elliptic integrals, for which a recursive method of evaluation is provided.

Simplified forms are given for use with points which are either on the surface ($z = 0$) or on the axis ($x = 0$, $y = 0$). Unless these special forms are used, it is recommended that any zeros in the dimensionless space coordinates be replaced by small numbers, *e.g.* 10^{-6} . These should be small compared with β , but their squares should be large compared with the smallest digit retained by the calculator or computer. In this way, trouble with infinities and indeterminate numbers may be avoided. The signs of square roots may be taken as positive unless otherwise indicated.

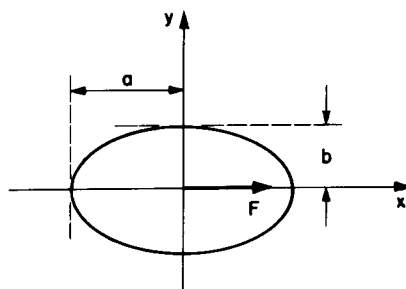
Many of the quantities are defined in different ways in the different sections, *e.g.* s_1^2 is defined in Section 5.2 as the root of the equation

$$\frac{x_1^2}{1 + s_1^2} + \frac{y_1^2}{\beta^2 + s_1^2} + \frac{z_1^2}{s_1^2} = 1,$$

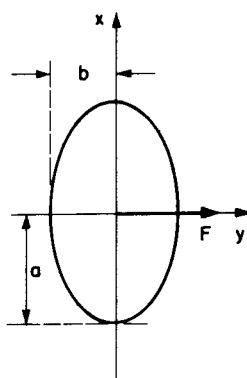
but in Section 9.2 as the root of

$$\frac{\rho_1^2}{1 + s_1^2} + \frac{z_1^2}{s_1^2} = 1.$$

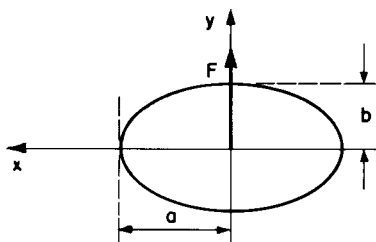
The second version is derived from the first as the special case $\beta = 1$, but because of the large number of symbols needed, the same symbol s_1 has been employed in both cases.



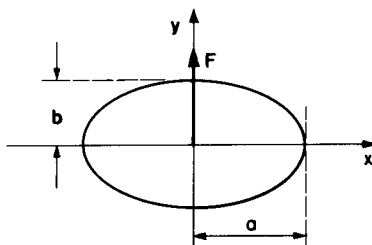
(a) Loading Case 3. Configuration used in Derivation 12



(b) Interchange x and y , a and b



(c) Rotate 90° anticlockwise



(d) Reverse direction of x . This is now the configuration required for loading Case 2

Sketch 4.1 Transformation from loading Case 3 to loading Case 2

This example is an obvious one, but in the case of the function J and its partial derivatives, the definitions for the circular case in Sections 9.3, 9.6 and 9.7 are apparently unrelated to those for the general elliptical case in Sections 5.18 to 5.20. However, the special case may be shown to be derived from the general case in the limit $B \rightarrow 1$, $n \rightarrow 0$.

In many cases, some checks on the working have been provided. These involve the sums of the three direct stresses for each of the three loading stresses given by Equations (4.1) to (4.3).

5. THE GENERAL ELLIPTICAL CONTACT: POINTS WITHIN THE CONTACT OR ON THE SURFACE AND OUTSIDE THE CONTACT ELLIPSE

5.1 Establish dimensionless coordinates based on the major semi-axis of the ellipse of contact:

$$x_1 = x/a; \quad y_1 = y/a; \quad z_1 = z/a.$$

5.2 Find s_1^2 as the highest root of the equation:

$$\left(\frac{x_1^2}{1 + s_1^2} \right) + \left(\frac{y_1^2}{\beta^2 + s_1^2} \right) + \left(\frac{z_1^2}{s_1^2} \right) = 1$$

where $\beta = b/a$, the axis ratio of the ellipse of contact ($\beta \leq 1$).

Notes on the procedure for solving this equation are given in Section 6. If the point (x, y, z) is both on the surface and on or within the ellipse of contact, $s_1 = 0$ and the special forms provided for this case should be used (Section 7). If $\beta = 1$ the forms provided for the special case of circular contact should be used (Sections 9 to 12).

$$5.3 \quad r_1 = (s_1^2 + 1)^{1/2} (s_1^2 + \beta^2)^{-1/2}$$

$$5.4 \quad G = (s_1^2 + 1)^{1/2} (s_1^2 + \beta^2)^{1/2}$$

$$5.5 \quad H = \left(\frac{x_1}{1 + s_1^2} \right)^2 + \left(\frac{y_1}{\beta^2 + s_1^2} \right)^2 + \left(\frac{z_1}{s_1^2} \right)^2$$

$$5.6 \quad L = z_1 / (s_1^3 H G)$$

$$5.7 \quad n = (1 - \beta^2)^{1/2}$$

$$5.8 \quad I_1 = \int_{s_1}^{\infty} \left(\frac{dw_1}{(1 + w_1^2)^{3/2} (\beta^2 + w_1^2)^{1/2}} \right)$$

$$5.9 \quad I_2 = \int_{s_1}^{\infty} \left(\frac{dw_1}{(1 + w_1^2)^{1/2} (\beta^2 + w_1^2)^{3/2}} \right)$$

$$5.10 \quad I_3 = \int_{s_1}^{\infty} \left(\frac{dw_1}{w_1^2 (1 + w_1^2)^{1/2} (\beta^2 + w_1^2)^{1/2}} \right), \quad s_1 \neq 0.$$

A note on the evaluation of the integrals I_1 , I_2 and I_3 is given in Section 8.

$$5.11 \quad N = Gz_1/s_1$$

$$5.12 \quad C = s_1^2 + 1 + N$$

$$5.13 \quad D = s_1^2 + \beta^2 + N$$

$$5.14 \quad E = \left(2Ns_1^4 + z_1^2(s_1^4 - \beta^2) \right) / \left(NHs_1^4 G^2 \right), \text{ for the case of } z_1 = 0, E = 2/(HG^2)$$

$$5.15 \quad F = C + D - x_1^2 - y_1^2$$

$$5.16 \quad \theta_1 = \tan^{-1}(ny_1/D), \quad -\pi/2 \leq \theta_1 \leq \pi/2$$

$$5.17 \quad \theta_2 = \tanh^{-1}(nx_1/C)$$

- Notes: (a) These definitions of θ_1 and θ_2 may be shown to be equivalent to those given in the Derivations, with a change of sign.
- (b) If the inverse tangent is expressed in degrees, it must be multiplied by $(\pi/180)$ to convert it to radians.
- (c) $\tanh^{-1}Z = \frac{1}{2} \ln \left(\frac{1+Z}{1-Z} \right).$

$$5.18 \quad J = \frac{1}{n^3} (x_1 \theta_1 - y_1 \theta_2)$$

$$5.19 \quad \frac{\partial J}{\partial x_1} = \frac{1}{n^3} \left[\theta_1 - \frac{ny_1 C}{(s_1^2 + 1)F} - \frac{n^3 x_1^2 y_1 E}{(s_1^2 + 1)F} \right]$$

$$5.20 \quad \frac{\partial J}{\partial y_1} = \frac{1}{n^3} \left[-\theta_2 + \frac{nx_1 D}{(s_1^2 + \beta^2)F} - \frac{n^3 x_1 y_1^2 E}{(s_1^2 + \beta^2)F} \right]$$

$$5.21 \quad K = \frac{1}{n^3} (y_1 \theta_1 + x_1 \theta_2 - n)$$

$$5.22 \quad (\bar{f}_x)_1 = \beta \left[2(1 - \sigma) z_1 I_1 - 2\sigma z_1 I_3 - \left(\frac{x_1 s_1}{G r_1} \right)^2 L + (1 - 2\sigma) [K + z_1 / (n^2 r_1 s_1)] \right]$$

$$5.23 \quad (\bar{f}_y)_1 = \beta \left[2(1 - \sigma) z_1 I_2 - 2\sigma z_1 I_3 - \left(\frac{y_1 s_1 r_1}{G} \right)^2 L - (1 - 2\sigma) [K + z_1 r_1 / (n^2 s_1)] \right]$$

$$5.24 \quad (\bar{f}_z)_1 = -\beta z_1^2 L / s_1^2$$

$$5.25 \quad (\bar{q}_{xy})_1 = (1 - 2\sigma) \beta J - (\beta x_1 y_1 s_1^2 L / G^2)$$

$$5.26 \quad (\bar{q}_{yz})_1 = -\beta y_1 z_1 L r_1 / G$$

$$5.27 \quad (\bar{q}_{zx})_1 = -\beta z_1 x_1 L / (G r_1)$$

$$5.28 \quad (\bar{f}_x)_2 = \mu \beta \cos \gamma \left[\left(\frac{1}{n^2} (I_1 - I_2) - \frac{x_1^2 s_1}{r_1 H G^4} \right) 2\sigma y_1 + (1 - 2\sigma) z_1 \frac{\partial J}{\partial x_1} \right]$$

$$5.29 \quad (\bar{f}_y)_2 = \mu \beta \cos \gamma \left[- \left((n^2 - \sigma \beta^2) I_2 + \sigma I_1 \right) \frac{2y_1}{n^2} - (1 - 2\sigma) z_1 \frac{\partial J}{\partial x_1} + \frac{y_1 z_1^2 r_1}{H G^2 s_1^3} + \frac{2\sigma x_1^2 y_1 s_1}{r_1 H G^4} \right]$$

$$5.30 \quad (\bar{f}_z)_2 = -\mu \beta \cos \gamma \left(\frac{y_1 z_1^2 r_1}{s_1^3 G^2 H} \right)$$

$$5.31 \quad (\bar{q}_{xy})_2 = -\mu \beta \cos \gamma \left[\left(2\sigma (I_1 - \beta^2 I_2) - n^2 I_1 \right) \frac{x_1}{n^2} + (1 - 2\sigma) z_1 \frac{\partial J}{\partial y_1} - \frac{2\sigma x_1 y_1^2 r_1 s_1}{H G^4} \right]$$

$$5.32 \quad (\bar{q}_{yz})_2 = \mu \beta \cos \gamma \left(z_1 (I_2 - I_3) - \frac{y_1^2 z_1 r_1}{s_1 H G^3} \right)$$

$$5.33 \quad (\bar{q}_{zx})_2 = \mu \beta \cos \gamma \left(\frac{x_1 y_1 z_1}{s_1 H G^3} \right)$$

$$5.34 \quad (\bar{f}_x)_3 = \mu\beta \sin \gamma \left[- \left((n^2 + \sigma)I_1 - \sigma\beta^2 I_2 \right) \frac{2x_1}{n^2} - (1 - 2\sigma)z_1 \frac{\partial J}{\partial y_1} + \frac{x_1 z_1^2}{HG^2 r_1 s_1^3} + \frac{2\sigma x_1 y_1^2 s_1 r_1}{HG^4} \right]$$

$$5.35 \quad (\bar{f}_y)_3 = \mu\beta \sin \gamma \left[\left(\frac{\beta^2}{n^2} (I_1 - I_2) - \frac{y_1^2 s_1 r_1}{HG^4} \right) 2\sigma x_1 + (1 - 2\sigma)z_1 \frac{\partial J}{\partial y_1} \right]$$

$$5.36 \quad (\bar{f}_z)_3 = -\mu\beta \sin \gamma \left(\frac{x_1 z_1^2}{s_1^3 G^2 H r_1} \right)$$

$$5.37 \quad (\bar{q}_{xy})_3 = \mu\beta \sin \gamma \left[\left(2\sigma(I_1 - \beta^2 I_2) - n^2 I_2 \right) \frac{y_1}{n^2} + (1 - 2\sigma)z_1 \frac{\partial J}{\partial x_1} - \frac{2\sigma x_1^2 y_1 s_1}{HG^4 r_1} \right]$$

$$5.38 \quad (\bar{q}_{yz})_3 = -\mu\beta \sin \gamma \left(\frac{x_1 y_1 z_1}{s_1 H G^3} \right)$$

$$5.39 \quad (\bar{q}_{zx})_3 = \mu\beta \sin \gamma \left(z_1 (I_1 - I_3) - \frac{x_1^2 z_1}{s_1 H G^3 r_1^2} \right)$$

$$5.40 \quad \begin{aligned} \bar{f}_x &= (\bar{f}_x)_1 + (\bar{f}_x)_2 + (\bar{f}_x)_3 \\ \bar{f}_y &= (\bar{f}_y)_1 + (\bar{f}_y)_2 + (\bar{f}_y)_3 \\ \bar{f}_z &= (\bar{f}_z)_1 + (\bar{f}_z)_2 + (\bar{f}_z)_3 \\ \bar{q}_{xy} &= (\bar{q}_{xy})_1 + (\bar{q}_{xy})_2 + (\bar{q}_{xy})_3 \\ \bar{q}_{yz} &= (\bar{q}_{yz})_1 + (\bar{q}_{yz})_2 + (\bar{q}_{yz})_3 \\ \bar{q}_{zx} &= (\bar{q}_{zx})_1 + (\bar{q}_{zx})_2 + (\bar{q}_{zx})_3 \end{aligned}$$

Note that a useful check on the working is obtained by taking the sums of the three direct stresses for each of the three loading stresses:

$$\begin{aligned} (\bar{f}_x)_1 + (\bar{f}_y)_1 + (\bar{f}_z)_1 &= -2\beta(1 + \sigma)z_1 I_3 \\ (\bar{f}_x)_2 + (\bar{f}_y)_2 + (\bar{f}_z)_2 &= -2\mu\beta \cos \gamma (1 + \sigma)y_1 I_2 \\ (\bar{f}_x)_3 + (\bar{f}_y)_3 + (\bar{f}_z)_3 &= -2\mu\beta \sin \gamma (1 + \sigma)x_1 I_1. \end{aligned}$$

6. SOLVING THE EQUATION FOR s_1^2

6.1 The equation is

$$f(s_1) = \frac{x_1^2}{1 + s_1^2} + \frac{y_1^2}{\beta^2 + s_1^2} + \frac{z_1^2}{s_1^2} - 1 = 0.$$

In general, this last equation will have three real roots, of which only one, the required largest root, is positive. The equation may be re-written as

$$s_1^6 + 3b_2s_1^4 + 3b_1s_1^2 - 3b_0 = 0,$$

where

$$b_2 = \frac{1}{3}(1 + \beta^2 - x_1^2 - y_1^2 - z_1^2)$$

$$b_1 = \frac{1}{3}(\beta^2 - \beta^2x_1^2 - y_1^2 - (1 + \beta^2)z_1^2)$$

$$b_0 = \frac{1}{3}\beta^2z_1^2.$$

6.2 The following expressions are now calculated

$$q = b_2^2 - b_1$$

$$r = \frac{3}{2}(b_1b_2 + b_0) - b_2^3$$

$$\psi = \frac{1}{3}\cos^{-1}(rq^{-3/2})$$

In general q will be positive and its square root may also be taken as positive, while r may be positive, zero or negative. The inverse cosine should be placed in the first quadrant ($0 \leq \psi \leq \pi/2$) if r is positive and in the second quadrant ($\pi/2 \leq \psi \leq \pi$) if r is negative.

6.3 The required largest root is

$$s_1^2 = 2q^{1/2}\cos\psi - b_2$$

The other roots may be obtained by increasing ψ by $2\pi/3$ (120 deg) and $4\pi/3$ (240 deg) respectively.

6.4 If $\beta < 10^{-2}$ the above procedure may give inaccurate results. The value of the function $f(s_1)$ should always be checked, and if it differs significantly from zero, corrections may be made by Newton's method. The value of s_1 found by the method of Section 6.3 may be used as a starting value of $(s_1)_0$, subsequent corrections being made by

$$(s_1)_{i+1} = (s_1)_i + \frac{f[(s_1)_i]}{2(s_1)_i H[(s_1)_i]},$$

where
$$H[(s_1)_i] = \left(\frac{x_1}{1 + (s_1)_i^2} \right)^2 + \left(\frac{y_1}{\beta^2 + (s_1)_i^2} \right)^2 + \left(\frac{z_1}{(s_1)_i^2} \right)^2,$$

and $i = 0, 1, 2, \dots$

An example of the solution is given in Example 18.1.

7. THE GENERAL ELLIPTICAL CONTACT: POINTS BOTH ON THE SURFACE AND ON OR WITHIN THE ELLIPSE OF CONTACT

7.1 In general, the calculation procedure follows that given in Section 5, with $z_1 \rightarrow 0$, $s_1 \rightarrow 0$ but,

$$z_1/s_1 \rightarrow \left(1 - x_1^2 - (y_1/\beta)^2 \right)^{1/2}.$$

The functions I_1 , I_2 , J and K remain as defined in Section 5, but for ease of reference all the quantities required in the calculations are redefined in the following exposition of the procedure.

7.2
$$x_1 = x/a; \quad y_1 = y/a$$

7.3
$$\delta = \left(1 - x_1^2 - (y_1/\beta)^2 \right)^{1/2}$$

7.4
$$I_1 = \int_0^\infty \left(\frac{dw_1}{\left((1 + w_1^2)^{3/2} (\beta^2 + w_1^2)^{1/2} \right)} \right)$$

7.5
$$I_2 = \int_0^\infty \left(\frac{dw_1}{\left((1 + w_1^2)^{1/2} (\beta^2 + w_1^2)^{3/2} \right)} \right)$$

A note on the evaluation of these integrals is given in Section 8.

7.6
$$n = \left(1 - \beta^2 \right)^{1/2}$$

7.7
$$\theta_1 = \tan^{-1} \left(\frac{ny_1}{\beta(\beta + \delta)} \right), \quad -\pi/2 \leq \theta_1 \leq \pi/2$$

7.8
$$\theta_2 = \tanh^{-1} \left(\frac{nx_1}{1 + \beta\delta} \right)$$

Note:
$$\tanh^{-1} Z = \frac{1}{2} \ln \left(\frac{1 + Z}{1 - Z} \right)$$

- 7.9 $J = \frac{1}{n} (x_1 \theta_1 - y_1 \theta_2)$
- 7.10 $K = \frac{1}{n} (y_1 \theta_1 + x_1 \theta_2 - n)$
- 7.11 $(\bar{f}_x)_1 = \beta(1 - 2\sigma)K - (\delta/n^2)(2\sigma - \beta^2)$
- 7.12 $(\bar{f}_y)_1 = -\beta(1 - 2\sigma)K - (\delta/n^2)(1 - 2\sigma\beta^2)$
- 7.13 $(\bar{f}_z)_1 = -\delta$
- 7.14 $(\bar{q}_{xy})_1 = (1 - 2\sigma)\beta J$
- 7.15 $(\bar{q}_{yz})_1 = 0$
- 7.16 $(\bar{q}_{zx})_1 = 0$
- 7.17 $(\bar{f}_x)_2 = \mu \cos \gamma \, 2\sigma y_1 \beta (I_1 - I_2)/n^2$
- 7.18 $(\bar{f}_y)_2 = -\mu \cos \gamma \left([2\sigma y_1 \beta (I_1 - I_2)/n^2] + 2(1 + \sigma)y_1 \beta I_2 \right)$
- 7.19 $(\bar{f}_z)_2 = 0$
- 7.20 $(\bar{q}_{xy})_2 = -\mu \cos \gamma \left(\beta x_1/n^2 \right) \left(2\sigma(I_1 - \beta^2 I_2) - n^2 I_1 \right)$
- 7.21 $(\bar{q}_{yz})_2 = -\mu \delta \cos \gamma$
- 7.22 $(\bar{q}_{zx})_2 = 0$
- 7.23 $(\bar{f}_x)_3 = -\mu \beta \sin \gamma \left((2\sigma x_1 \beta^2/n^2)(I_1 - I_2) + 2(1 + \sigma)x_1 I_1 \right)$
- 7.24 $(\bar{f}_y)_3 = \mu \sin \gamma \, 2\sigma x_1 \beta^3 (I_1 - I_2)/n^2$
- 7.25 $(\bar{f}_z)_3 = 0$
- 7.26 $(\bar{q}_{xy})_3 = \mu \beta \sin \gamma \left(y_1/n^2 \right) \left(2\sigma(I_1 - \beta^2 I_2) - n^2 I_2 \right)$
- 7.27 $(\bar{q}_{yz})_3 = 0$
- 7.28 $(\bar{q}_{zx})_3 = -\mu \delta \sin \gamma$