

SERVOMECHANISM TRANSFER FUNCTIONS

1. INTRODUCTION

This Item presents the basic equations and transfer functions of a selection of servomechanisms that are commonly encountered. Each servomechanism is briefly introduced and associated tables illustrate schematically each type of component, the basic equations that represent each component and the transfer function of the output to input of each device. The formulation of a complete feedback control system can be accomplished by connecting together the required servomechanism elements defined by the transfer functions in this Item. The underlying principles of operation and construction of the servomechanisms are, in general, not included.

The Laplace transform is used extensively throughout this Item. For information on this method, Data Item No. 69025 (Reference 1) may be consulted.

2. TRANSFER FUNCTIONS

A control system component, or element, may be defined as one of a number of parts which, when connected together, forms a complete feedback control system. The transfer function of an element is the mathematical expression which indicates the dynamic characteristics of the element in terms of the ratio of its output to input. Such an element may only be described by a transfer function when its operation is linear, *i.e.* it can be represented by a set of simultaneously-linear differential equations with constant coefficients. Many elements which are nonlinear over their complete range of operation may be considered linear over a restricted range and so allow a transfer function approach for analysis or synthesis. Strictly, transfer functions are defined under the condition that the input is supplied from an ideal (zero impedance) source and that the output is unloaded. Thus the loading effects of succeeding control system elements, if appreciable, must be included in the expression for the transfer function that precedes it, or conversely, the regulation effect on an element should be included with the transfer function of the element that follows. It is important to be continually aware of the actual physical characteristics represented by the transfer function and the limitations under which a transfer function is valid.

The transfer function is normally expressed in the form of the Laplace transform for the output to input ratio in which all the initial values have been set to zero. It expresses the magnitude, the units and the dynamic characteristics of the element in terms of its parameters. Transfer functions have two general uses: for analysis in the time domain, where the time response can be found by the inverse Laplace transform, and in the frequency domain, where the steady-state sinusoidal response of the system is obtained by replacing the Laplace transform operator, s , by $i\omega$. The expression obtained in the frequency domain is a complex function from which the magnitude and phase angle may be determined.

Elements which are used as shaping networks or filters are usually classified according to the function they perform, for example, the terms “lag” and “lead” are used to describe certain of the electrical network circuits given in Tables 11.1. and 11.2. A lag element is one in which the phase angle of the output lags the input and the magnitude of the output decreases with increasing frequency. It is alternatively referred to as a low pass filter or integral controller. The compound lag element still provides low pass filtering but limits the high frequency attenuation of the input. A lead element is one in which the phase angle of the output leads the input and the magnitude of the output increases with increasing frequency. It is alternatively known as a high pass filter or derivative controller. The compound lead element limits the low frequency attenuation of the input. For a fuller discussion of lag and lead elements and combinations of such elements see Appendix C of Data Item No. 80018 (Reference 2).

3. NOTATION

In this notation units are given in the SI system only. However, any coherent system of units may be used.

A	area	m^2
B	damping coefficient – translational – rotational	N s/m N s/rad
C	capacitance	farad
F	force	N
I	current	amp
J	moment of inertia	kg m^2
K	stiffness coefficient – translational – rotational	N/m N/rad
k	machine constant	depends on servo
L	inductance	henry
M	mutual inductance	henry
M	mass	kg
N	number of turns of winding	–
n	amplification factor	–
p	pressure difference	N/m^2
Q	mass flow rate	kg/s
q	volumetric flow rate	m^3/s
R	resistance	ohm
R	gas constant	N m/(kg K)
s	Laplace operator	/s
T	torque	N m
T	temperature	K
t	time	s
V	voltage	volt
V	volume	m^3
x	input	depends on servo

y	output	depends on servo
Z	impedance	ohm
γ	ratio of specific heats	—
ζ	damping ratio	—
θ	shaft angular rotation	rad
κ	bulk modulus	N/m ²
τ	time constant	s
ϕ	magnetic flux	weber
Ω	rotational speed	rad/s
ω	frequency	rad/s

Subscripts

A	amplifier
a	armature
c	control or compressibility
d	direct
e	eddy
f	field
g	generator or grid
i	input
l	load or leakage
m	motor
n	natural
o	output
p	plate, pump or piston
q	quadrature
r	reservoir
s	stall or spin

t amplifier vacuum tubes

v valve

A bar ($\bar{}$) above a quantity indicates the Laplace transform of that quantity

One or two dots above a quantity indicates the first or second derivative respectively, of that variable with respect to time.

The symbol Δ preceding a quantity denotes the difference of that quantity between two states.

4. ELECTRICAL NETWORK CIRCUITS

Electrical networks provide inexpensive and effective means of improving the performance of servomechanisms. They can be used to improve stability and speed of response, and to suppress unwanted frequencies and resonances.

The assumptions that are made to derive the transfer functions for these circuits are strictly that the source feeding the network has zero impedance and that the load into which the network feeds has an infinite impedance, *i.e.* the network is unloaded. However, providing the element into which the output of a network is being fed has a much higher impedance than the network, only a small loss of accuracy will result. For accurate applications when the output is loaded, the network must be isolated by an amplifier of some sort or the transfer function derived with the output impedance taken into account.

4.1 Passive Circuits

Networks that which contain only resistors, capacitors and/or conductors are called passive circuits and these networks result only in d-c attenuation. However, because of the difficulty of securing good, general purpose, low cost constant inductive elements, most shaping networks are only designed with resistive and capacitive elements. Such circuits are known as RC circuits.

Table 11.1 presents a number of the most common RC networks used in servo systems together with a “voltage smoothing” circuit which additionally contains an inductance. A more comprehensive list of RC networks may be found in Reference 17.

4.2 Active Circuits

Electrical networks containing amplifiers in addition to resistors and capacitors are called active circuits. The advantages of these circuits compared with the passive circuits include:

- (a) a more generalised transfer function can be obtained with fewer components,
- (b) the use of inductors can be avoided,
- (c) amplification can be provided,
- (d) they are less sensitive to load impedance.

The type of amplifier used in active compensation circuits is known as an operational amplifier and has a very high negative gain, high input impedance and low output impedance. The output of circuits containing a single amplifier (or an odd number of amplifiers), are of opposite sign to the input due to the negative gain of the amplifier(s).

Table 11.2 presents a number of common active network circuits.

5. SERVOMOTORS

The basic requirements of a servomotor are that the torque be easily reversible and that it can be controlled at various speeds in either direction.

For very low power requirements of the order of a few watts, such as is used in instrument type servos, two-phase a-c motors are normally used. Large a-c motors can be economical for constant and multispeed applications but are difficult to control through continuous-torque or -speed ranges.

For larger power requirements, from approximately 0.05 horsepower to several hundred horsepower, the d-c motor is used because it is both more efficient than the a-c motor and exhibits superior control characteristics. At the smaller end of the power range, d-c motors are driven by vacuum-tube amplifiers or permanent magnets. Larger d-c motors are driven by single- or two-stage d-c generators (alternatively called rotary amplifiers) and such a generator/motor combination is known as a Ward-Leonard system. When power requirements exceed around 20 horsepower an electro-hydraulic servosystem may be more efficient than an electric motor.

5.1 D-C Motors

D-c motors used in servomechanism applications are available in three general types, series wound, separately excited shunt wound, including both the armature- and field-controlled motors, and the permanent magnet motor.

Permanent magnet motors are a particular form of the armature-controlled motor where the field is virtually constant and is supplied by the magnet. They are generally rugged and relatively inexpensive compared with the shunt motor equivalent, having the same or slightly better efficiency figures and a similar performance. They are used in both the fractional horsepower range and the multihorsepower range but are also used as torque motors suitable for heavy power applications (several horsepower) at slow speeds. These latter motors are ideal for direct drive applications where acceleration, deceleration and positioning of inertia loads can be accomplished with great precision and accuracy.

Separately excited shunt wound motors have the field and armature circuits connected in parallel and are suitable for driving pumps, fans and other types of general machinery that require normal starting torque. They can be field-weakened to provide a constant horsepower adjustable speed range. There are two types; the **field-controlled motor** where the armature current is held constant, and the **armature-controlled motor** where the field current is held constant. A schematic representation of these two motors together with their basic relationships and transfer functions are presented in **Table 11.3**.

Series wound motors have the field and armature circuits connected in series and are best suited for cranes, hoists and traction loads where a high starting torque is required and linearity is less important. These motors are operated at a minimum of about 15% of full load to prevent their running at excessive speeds. The transfer function for this motor together with a schematic representation and basic relationships is given in **Table 11.3**.

5.2 A-C Motors

For a-c servomechanisms, actuator requirements are most satisfactorily met by the **two-phase induction motor**. When one voltage is applied to the fixed phase and a second voltage, 90° out of phase with the first, is applied to the control or variable phase, a rotating field is set up in the motor stator. The most satisfactory general-purpose actuator is a motor having a linear characteristic where stall torque is proportional to control voltage and torque for any control voltage decreases at a uniform rate with speed. These motors are run at high speed, have high inertia and are used for low torque applications. These motors are designed

for a maximum ratio of stall torque to rotor inertia and have relatively small diameter rotors for their frame size. Two phase motors are available in sizes up to 1 horsepower. Above this size they are much too inefficient and the majority of units tend to be under 100 watts. A schematic representation together with the basic relationships and transfer function for a two-phase induction motor is given in **Table 11.4**

6. D-C GENERATORS

D-c generators are rotating electromagnetic devices driven by prime movers where the output power is controlled by small separate field power units. They are used either as **power amplifiers**, when they are known as **rotary amplifiers**, or velocity measurement devices, when they are known as **d-c tachometers** or **rate generators**. The no-load output voltage generated is proportional to the speed of rotation of the armature of the generator, Ω , multiplied by the magnetic-flux, ϕ .

When the flux is held constant and the load impedance is sufficiently high the voltage is directly, and accurately, proportional to the speed with no time lag. This is the principle used in the d-c tachometer. At very high speeds brush bounce has the effect of reducing this accuracy and also increasing both the noise and ripple component of the output. The best available commercial tachometer has linear characteristics with an accuracy of within 0.25% whilst "high precision" units have linear characteristics with accuracies within 2%.

When the unit is driven at constant speed it is used for power amplification. In the linear operating region the magnetic flux is directly proportional to the field current. These units differ from the lower-rating, better known, amplifier types in that they respond more slowly, typical uncompensated response times being of the order of 1 second. They also do not respond to input signals below a certain signal level and thus do not operate satisfactorily with much less than a few tenths of a watt of control power. The rotating amplifier can be split into three basic types; the **separately excited shunt generator**, the **tuned field generator** and the **armature reaction generator**. The basic relationships, schematic representation and the transfer functions of these three types of generator together with circuits accounting for eddy currents and a push-pull control field are presented in **Table 11.5**. When the generator is used to drive a motor it is known as a Ward-Leonard drive. In this arrangement the open circuit, or unloaded, generator is attached directly to, the controlling circuit, field or armature, of the motor to obtain the combined generator and motor transfer function. When higher powers are required than is possible from a single generator then the more elaborate two-stage generators are used.

7. MECHANICAL ELEMENTS

7.1 Gyroscopes

A mechanical gyroscope is normally classified according to either the number of degrees of freedom of angular motion with which the spinning wheel is provided or to the function it performs. A gyro suspended in a single gimbal is said to be a single-axis gyro and has a single degree-of-freedom, since no distinction is normally made, for classification purposes, between restrained and unrestrained motion and the gyro is insensitive to angular velocities about its own axis. Consequently a gyro with two gimbals is said to be a double-axis gyro and has two degrees of freedom. The wheel is spun about the spin axis, s , and the support axis of the gyro gimbal is called the output or precession axis, o . The third axis is the input or stabilizing axis, i . (See **Table 11.6**).

When a torque is applied about a particular axis, other than the spin axis, the gyro rotates in such a fashion that the spin axis precesses into the axis about which the torque is applied. The rotor of a practical gyroscope is usually not a disc but consists of the rotor of an induction motor and is designed to operate at speeds as

high as 20000 rev/min. Modern gyros are enclosed in a temperature-stabilised environment and may be sealed where the sealed case is supported or floated in a damping fluid. Aircraft applications usually employ single-axis gyros for each coordinate to be sensed. To minimise drift, caused by imbalance of the gimbals and bearing friction, gyros are constructed with extremely fine bearings and have also no inherent damping. For this reason some form of external damping is usually added.

Table 11.6 presents the general equations and associated transfer functions of a double-axis gyroscope (gimbals not shown) where T_o and T_i are the external torques applied (by torque motors) to the output and input axes respectively. It has been assumed in deriving these equations that the gimbal angles θ_y and θ_z are small, *i.e.* the gimbals of the gyro are at right angles to one another and to the frame. If these conditions are not met then the equations will be nonlinear, containing terms involving the product of the two angles in addition to trigonometrical terms. Assuming that the output torque, T_o , is zero, then a transfer function relating precession angle (output angle) to applied torque (input torque or precession torque since the input torque causes precession) can be obtained. This transfer function is called the direct transfer function of the gyro. The relation between torque and angular displacement about the same axis is referred to as the cross-coupling transfer function. A large cross-coupling is undesirable in a gyro and the transfer function shows that it can be minimised by reducing friction and making the spin velocity as large as possible.

Two basic types of gyro are used in practice; the **free gyro** and the **restrained gyro**. A free gyro is mounted in a set of gimbals so that angular motion of the supporting frame is not transmitted to it and therefore ideally the direction of the spin axis is fixed in space. These gyros are used to provide a fixed reference position, for example the **vertical** and **directional gyros** found in inertial navigation equipment and automatic pilots. In some critical applications the gyro is not completely free but can be slaved to a more accurate primary reference source by small torque motors. Thus a vertical gyro can be slaved to a pendulum; the position of the gyro is continuously compared to that of the pendulum and the difference is used to precess the gyro. The precessing torque applied to the gyro is kept as small as possible so that the gyro can follow rapid fluctuations of the pendulum position. Thus the gyro is considered as an integrating or smoothing device. In a similar way directional gyros may be slaved to a magnetic compass to provide a north-south reference. The **gyro compass** is a free gyro equipped with a weight that tends to keep the spin axis in a horizontal plane.

Double-axis gyro can be used for the measurement of angular velocity where use is made of the fact that the precession velocity is directly proportional to the precessing torque (see the expression for the direct transfer function ($s\bar{\theta}_o/\bar{T}_i$) in **Table 11.6**). The gyro is equipped with synchro devices which sense the relative displacement between the gimbals and the outer frame. Torque is applied in such a way that the resulting gyro precession returns the output of the synchro to zero, thus keeping the gyroscope aligned with the outer frame. Such a system is able to measure the velocity in two axes.

A simpler device is the restrained or **rate-gyro** (see **Table 11.6**) where the rotation of a single gimbal relative to the outer frame is checked by means of restraining springs. An input torque is applied to the gyro through the gimbal and the gyro begins to precess about the output axis. This precession is arrested by the output springs and this generated output torque precesses the gyro about the input axis so that the gyro follows the rotation of the outer frame. Thus the torque is a direct measure of the speed of the outer frame. Since the restraining springs are linear the displacement of the gimbal relative to the outer frame is proportional to the torque.

The **integrating rate-gyro** (see **Table 11.6**) is a single-axis gyro and is similar to the rate-gyro except that there is no spring restraint on the output axis. The inner gimbal is floated in a highly viscous fluid which provides viscous damping. This gyroscope is referred to as a HIG (hermetically sealed integrating gyro) and is very rugged and extremely accurate. The inner can or gimbal is filled with an inert gas which acts as a neutral atmosphere and the outer case is filled with a viscous fluid whose density is such that the inner can neither floats nor sinks. Thus the loads on the bearings are negligibly small and this minimises the

effects of drift. A rotation of the outer case about the input axis, i , causes a precession of the gyrowheel about the output axis, o . By placing an electrical pickup on the output shaft, a signal, θ_o , is obtained which is a measure of the angular position, θ_i . By making the damping, B_o , extremely large and the inertia J_o , small, the time constant, τ ($= J_o/B_o$), can be neglected. This makes θ_o directly proportional to θ_i and is why the device is called an integrating rate gyro.

7.2 Accelerometers

In its simplest form an accelerometer consists of a mass which is free to move in a single direction against a restraining spring together with some form of damping which may be either mechanical or electrical. The frame of the accelerometer is mounted rigidly to the vehicle whose acceleration is required and when the frame accelerates the spring deflects until it generates enough force to accelerate the mass at the same rate as the frame. The deflection of the spring is then a direct measure of the acceleration of the outer frame. The displacement of the system at steady acceleration is determined by the ratio of mass to spring stiffness and it is these same parameters that define the undamped natural frequency. Thus a basic problem associated with accelerometer design is the compromise between the choice of gain and natural frequency, since the resonant frequency is inversely proportional to the distance squared. The usual value of damping ratio, ζ , used for most accelerometers is 0.7 since this gives a good steady-state sinusoidal response and reasonable overshoot for transient inputs.

Table 11.7 presents the transfer function and schematic arrangement of both the linear and pendulum-type accelerometer.

7.3 Vibration Systems

Springs and dampers are often used to minimise the transmission of forces from a machine to its foundation, or conversely, from a moving foundation to a machine. The transfer functions of these two cases are presented in **Table 11.8**

In the former case the equation relating the ratio of the force transmitted to the foundation, F_o , to the driving force, F_i , is called the transmissibility force equation. It can be shown that for a constant speed machine where the amplitude of the exciting force is constant, the transmitted force is greater than the driving force (i.e. transmissibility ≥ 1) below a frequency ratio, ω/ω_N , of $\sqrt{2}$. Thus it is advantageous to operate the machine at a frequency, ω , greater than $\sqrt{2}\omega_N$. For the case when the foundation is excited, a transfer function relating the distance moved by the foundation, x_i , and the distance moved by the mass, x_o , is seen from **Table 11.8** to be exactly the same as the former case. This equation is called the transmissibility motion equation. This particular case is not restricted to a machine mounted on a foundation but may equally apply to other types of motion, for example, a vehicle moving over rough ground or a seismic instrument mounted on a base.

Another type of vibration system is encountered when, for example, a constant speed machine is operating near resonance conditions and the vibration of the machine itself is considered excessive. Often it is not feasible to alter the mass of the machine or the spring stiffness and damping between the machine and the foundation and in such a case a **dynamic absorber** is attached to the machine. This consists of a mass attached to the machine via a spring and/or damper. The transfer function for the dynamic absorber is given in **Table 11.8**. Conversely, to dampen foundation vibrations that are the result of the operation of a machine, a **vibration isolator** may be used. This consists of a mass, spring and/or damper interposed between the machine system and the foundation. An example of a vibration isolator is given in **Table 11.8**.

8. HYDRAULIC COMPONENTS

Hydraulics is normally considered, in the control system sense, to be the study of fluids that are incompressible or almost so. This is to differentiate hydraulics from pneumatics in which compressibility is the major consideration. Hydraulic systems have several advantages over equivalent electrical systems and some disadvantages.

Among the advantages are:-

- (a) they are smaller for the same power (a typical volume factor of 20 to 1 for a 5 hp motor),
- (b) high torque to inertia ratio,
- (c) rugged,
- (d) not usually susceptible to noise from shocks and vibration.

The disadvantages include:-

- (a) sensitive to certain other environmental effects, particularly temperature,
- (b) they are more complex and consequently can be more expensive, particularly for low power requirements,
- (c) damage of the fluid lines can mean complete loss of system supply oil,
- (d) not as flexible as electrical systems.

There are two types of hydraulic system, the **pump-controlled system** which consists of a single variable stroke pump and a fixed stroke motor, and the **valve controlled system**, where a single fixed stroke pump can supply several motors each of which are valve controlled. Servo-valves are accurate and fast acting (and consequently expensive) but they can result in a lighter, more flexible system. Basically two types of servo-valve are available; the **pressure control valve** where the valve is supplied with sufficient pressure to meet the flow demands of the system, and the **constant flow valve** where the flow is supplied by a fixed displacement hydraulic pump driven at constant speed.

Table 11.9 presents various components associated with hydraulic circuits.

9. PNEUMATIC COMPONENTS

Pneumatic control systems are usually valve controlled since a pneumatic pump or compressor is normally incapable of supplying the maximum load of a system by itself and variable displacement compressors are not commercially available. Where they are used an accumulator together with a pressure regulating valve are included in the pressure supply. Some systems that have a limited operational life omit the pump altogether and the system operates directly from an accumulator that has been charged before operation.

Many of the components used in high pressure pneumatic systems are identical, apart from performance, with their hydraulic counterparts, as is the case of several types of hydraulic valves and motors.

Comparison of hydraulic and pneumatic servomechanisms with valve controlled drives has shown that overall efficiencies of around 60% and 30%, respectively, can be expected. Much of the energy loss results from throttling because it is usually impossible within the time available for the process to achieve the isothermal expansion for which nozzles are designed. Additionally, the work required to compress a gas between given pressure limits is greater than that for a liquid. Hydraulic response times are approximately 50 times faster than those for a pneumatic system for the same supply pressure and load. The hydraulic system has also a smaller sensitivity to load than a pneumatic system and has a stiffness which is approximately 400 times greater. Pneumatic systems, however, do have some advantages which include: