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Introduction to aerodynamic derivatives, equations of motion and stability (including the classical criteria of longitudinal stability and control, and description of the lateral modes of motion)

Associated software: ESDUpac A8621



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THE PREPARATION OF THIS DATA ITEM

The work on this particular Data Item which supersedes ESDU Aero A.00.00.02, ESDU Aero A.00.00.03, ESDU Aero A.00.00.04 and ESDU 67039, was monitored and guided by the Aerodynamics Committee. This Committee first met in 1942 and at the time of the first issue had the following membership:

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PREFACE

Many Items throughout the ESDU Aerodynamics Series give methods for the prediction of aerodynamic derivatives. To illustrate the use of such derivatives in the study of the dynamic motion of an aircraft, this Item describes how they appear in the rigid-body equations of motion to describe the response of an aircraft to a small disturbance from steady trimmed flight. The dependence of the stability of the aircraft on the values of the aerodynamic derivatives is displayed.

INTRODUCTION TO AERODYNAMIC DERIVATIVES, EQUATIONS OF MOTION AND STABILITY (INCLUDING THE CLASSICAL CRITERIA OF LONGITUDINAL STABILITY AND CONTROL, AND DESCRIPTION OF THE LATERAL MODES OF MOTION)

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INTRODUCTION TO AERODYNAMIC DERIVATIVES, EQUATIONS OF MOTION AND STABILITY (INCLUDING THE CLASSICAL CRITERIA OF LONGITUDINAL STABILITY AND CONTROL, AND DESCRIPTION OF THE LATERAL MODES OF MOTION)

1. AXIS SYSTEMS AND NOTATION

1.1 Axis Systems

A number of different axis systems are used to describe the disturbed motion of an aircraft and the forces and moments acting on it. The necessary definitions for the systems used in this Item are given below. Each is right-handed and orthogonal. It is assumed that the aircraft has a longitudinal plane of symmetry.

1.1.1 Body-axis systems, $Oxyz$

These systems are fixed in the body and move with it. For simplicity, in this Item the origin O is taken to be at the centre of gravity position of the aircraft in a steady (datum) flight condition. The x -axis and z -axis lie in the longitudinal plane of symmetry, x positive forwards and z positive downwards, with the y -axis positive to starboard. The direction of the x -axis is fixed to define particular systems. Two are considered.

- (a) **Aerodynamic-body axes*** constitute a system where the undisturbed direction of the x -axis is along the projection onto the longitudinal plane of symmetry of the tangent to the flight path in the datum condition. In a datum condition of straight and symmetrical flight the direction of the x -axis is in the direction of motion of the aircraft.
- (b) **Geometric-body axes** constitute a system where the fixed x -axis is defined parallel to some convenient geometric longitudinal datum.

1.1.2 Earth-axis systems, $O_0x_0y_0z_0$, $Ox_0y_0z_0$

These systems have the direction of their axes fixed in space. The z_0 -axis is directed vertically downwards, the x_0 -axis is directed forwards in the vertical plane containing the aircraft datum direction of motion and the y_0 -axis completes the right-handed system. An Earth-fixed system has an origin O_0 that is fixed in space. An aircraft-carried system has an origin O that is fixed in the aircraft and moves with it, and in this Item O is taken to be coincident with the centre of gravity position, as in the body-axis systems.

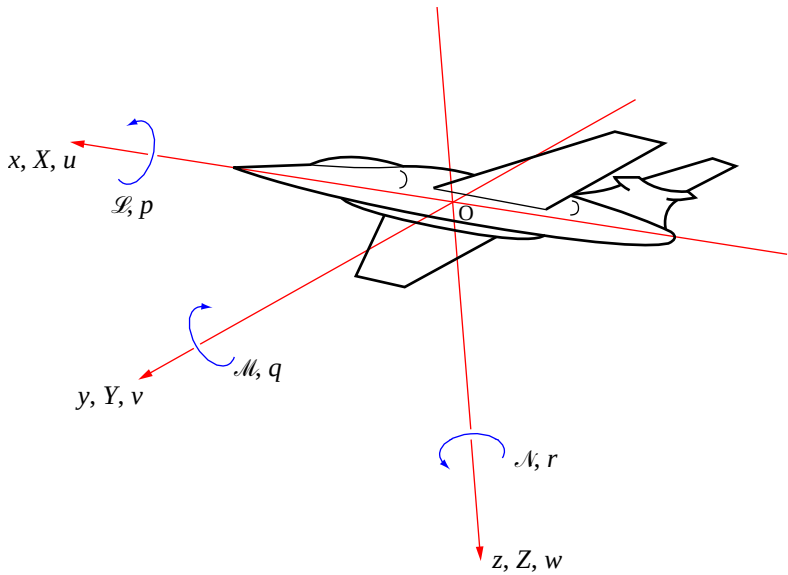
1.2 Notation and Units

Alphabetical trios are used to denote components of force, moment, linear displacement, velocity and angular velocity. They are listed in the table in Sketch 1.1. The diagram in Sketch 1.1 shows a body-axis system $Oxyz$ and indicates positive directions. Forces and linear velocities are positive in the direction of the axes. Angular velocities are positive if, when looking along the appropriate axis in the positive sense, the aircraft is seen to be rotating clockwise. Moments are positive in the same sense.

The same symbols are used for dimensional quantities and the corresponding dimensionless ones represented in normalised systems (see Sections 3.4 and 4.5).

* At one time referred to in the UK as wind axes or wind-body axes. In the USA usually referred to as stability axes.

Quantity	Components		
Force	X	Y	Z
Moment	$\mathcal{L}$	$\mathcal{M}$	$\mathcal{N}$
Linear displacement	x	y	z
Linear velocity	u	v	w
Angular velocity	p	q	r



Sketch 1.1 System of body axes showing positive directions

		SI	British
a_i	constant in typical solution of equations of motion, see Equation (5.3)	m/s	ft/s
$\underline{\mathbf{A}}$	matrix in Equation (5.1)		
$\mathbf{A}_{ij}$	partition matrix from $\underline{\mathbf{A}}$		
A, B, C, D, E	constants in characteristic equations; subscripts 1, 2 denote longitudinal, lateral equations; see Section 5.3		
$\underline{\mathbf{B}}$	matrix in Equation (5.1)		
$\mathbf{B}_{ij}$	partition matrix from $\underline{\mathbf{B}}$		
B	hinge moment, positive when tending to increase positive control deflection	N m	ft lbf
ΔB	increment in hinge moment	N m	ft lbf
b	wing span	m	ft
b_x, b_y, b_z	inertia ratios $(I_z - I_y)/I_x, (I_x - I_z)/I_y, (I_y - I_x)/I_z$		

$\underline{\mathbf{C}}$	matrix in Equation (5.1)		
$\mathbf{C}_{ij}$	partition matrix from $\underline{\mathbf{C}}$		
C_B	hinge moment coefficient (B divided by $\frac{1}{2}\rho V^2$ and representative length and area of control)		
C_D	drag coefficient, $D/\frac{1}{2}\rho V^2 S$		
C_L	lift coefficient, $L/\frac{1}{2}\rho V^2 S$		
C_{LT}	lift coefficient of tailplane (tailplane lift $/\frac{1}{2}\rho V^2 S_T$)		
C_R	force coefficient, $R/\frac{1}{2}\rho V^2 S$		
C_X, C_Y, C_Z	force coefficients, $X/\frac{1}{2}\rho V^2 S, Y/\frac{1}{2}\rho V^2 S, Z/\frac{1}{2}\rho V^2 S$		
C_l, C_m, C_n	moment coefficients, $\mathcal{L}/\frac{1}{2}\rho V^2 S l, \mathcal{M}/\frac{1}{2}\rho V^2 S l, \mathcal{N}/\frac{1}{2}\rho V^2 S l$; see Section 3.1		
$\bar{c}$	standard (or geometric) mean chord (or first mean chord)	m	ft
$\bar{\bar{c}}$	aerodynamic mean chord (or second mean chord)	m	ft
D	drag	N	lbf
D	differential operator d/dt	s^{-1}	s^{-1}
d_T	perpendicular distance from centre of gravity position to thrust vector, positive when thrust gives positive pitching moment; see Section 3.5.1	m	ft
d_x, d_y, d_z	inertia ratios, $-I_{yz}/I_x, -I_{yz}/I_y, -I_{yz}/I_z$		
e_x, e_y, e_z	inertia ratios, $-I_{zx}/I_x, -I_{zx}/I_y, -I_{zx}/I_z$		
F	free-control factor, see Section 6.5.1		
f_x, f_y, f_z	inertia ratios, $-I_{xy}/I_x, -I_{xy}/I_y, -I_{xy}/I_z$		
$\mathbf{g}$	acceleration due to gravity vector	m/s^2	ft/s^2
g	magnitude of acceleration due to gravity	m/s^2	ft/s^2
g_x, g_y, g_z	components of acceleration due to gravity in x -, y -, z -direction	m/s^2	ft/s^2
g_1	$g \cos \theta_e$	m/s^2	ft/s^2
g_2	$g \sin \theta_e$	m/s^2	ft/s^2
$\mathbf{H}$	angular momentum vector	$kg\ m^2/s$	$slug\ ft^2/s$
h_{cg}	dimensionless distance of centre of gravity position aft of reference point, see Section 6		

h_m, h_{mf}	dimensionless distance of pitch manoeuvre point aft of reference point controls-fixed, controls-free; see Section 6		
h_n, h_{nf}	dimensionless distance of neutral point aft of reference point controls-fixed, controls-free; see Section 6		
$\mathbf{I}_i$	unitary ($i \times i$) matrix		
$\mathbf{I}_n$	inertia matrix	kg m ²	slug ft ²
I_x, I_y, I_z	moments of inertia of aircraft about x -, y -, z -axis	kg m ²	slug ft ²
I_{xy}, I_{yz}, I_{zx}	products of inertia of aircraft about x - and y -axes, y - and z -axes, z - and x -axes	kg m ²	slug ft ²
i_x, i_y, i_z	inertia parameters $I_x / m_e l_0^2, I_y / m_e l_0^2, I_z / m_e l_0^2$		
$\mathcal{J}(\lambda)$	denotes imaginary part of λ		
$\mathbf{J}$	angular momentum vector of rotating parts of aircraft or engines	kg m ² /s	slug ft ² /s
J_x, J_y, J_z	components of $\mathbf{J}$ about x -, y -, z -axis	kg m ² /s	slug ft ² /s
k_{cg}, k_{cgf}	dimensionless centre of gravity margin (c.g. margin) controls-fixed, controls-free; see Section 6		
k_m, k_{mf}	dimensionless manoeuvre margin controls-fixed, controls-free; see Section 6		
k_s, k_{sf}	dimensionless static margin controls-fixed, controls-free; see Section 6		
L	lift	N	lbf
$\mathcal{L}, \mathcal{M}, \mathcal{N}$	moments about x -, y -, z -axis (rolling moment, pitching moment, yawing moment), with associated aerodynamic derivatives L_u, M_u, M_w, N_v etc.; see Section 3.2	N m	lbf ft
l, m, n	concise moments, $-\mathcal{L}/I_x, -\mathcal{M}/I_y, -\mathcal{N}/I_z$	N/kg m	lbf/slug ft
l	a characteristic length for the aircraft, usually $\bar{c}$ or $\bar{c}$ in longitudinal equations and b or $b/2$ in lateral equations	m	ft
l_0	unit of length in aero-normalised system, see Section 3.4	m	ft
l_T	distance of aerodynamic centre of tailplane aft of centre of gravity position	m	ft
$\mathbf{M}$	moment vector	N m	lbf ft
M	Mach number		
m_e	datum value of aircraft mass, unit of mass in dynamic-normalised system; see Section 4.5	kg	slug
n	normal acceleration in units of g , see Section 6		

P	ambient pressure of air	N/m ²	lbf/ft ²
p, q, r	angular velocity about x -, y -, z -axis, (roll rate, pitch rate, yaw rate)	rad/s	rad/s
$\mathbf{R}$	aerodynamic force vector	N	lbf
R	magnitude of aerodynamic force vector	N	lbf
R	Reynolds number, $\rho V l / \mu$		
R	Routh's discriminant		
$\mathcal{R}(\lambda)$	denotes real part of λ		
S	characteristic (reference) area	m ²	ft ²
S_T	tailplane planform area	m ²	ft ²
S_Φ, S_ϕ	axis transformation matrices, see Section 4.1		
T	period of oscillation	s	s
T	thrust	N	lbf
T_x, T_z	component of thrust in x -, z -direction	N	lbf
t	time	s	s
$t_{1/2}, t_{2/1}$	time to half amplitude, time to double amplitude	s	s
$\mathbf{u}$	state vector of perturbation variables		
$\mathbf{u}_1, \mathbf{u}_2$	partitions of $\mathbf{u}$		
u, v, w	components of aircraft velocity relative to air in x -, y -, z -direction	m/s	ft/s
$\mathbf{V}$	aircraft velocity vector relative to air	m/s	ft/s
V	magnitude of velocity vector	m/s	ft/s
V_e	datum value of V , unit of speed in aero-normalised and dynamic-normalised systems	m/s	ft/s
W	aircraft weight	N	lbf
X, Y, Z	components of force in x -, y -, z -direction, with associated aerodynamic derivatives X_u, X_w, Y_v, Z_u, Z_w etc.; see Section 3.2	N	lbf
x, y, z	concise forces, $-X/m_e, -Y/m_e, -Z/m_e$	N/kg	lbf/slug
x, y, z	body axis co-ordinates in x -direction (positive forwards), y -direction (positive to starboard), z -direction (positive downwards)	m	ft
x_0, y_0, z_0	co-ordinates in Earth-axis system	m	ft

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x_1, y_1, z_1 x_2, y_2, z_2	co-ordinates in intermediate axis systems used in defining attitude angles, Ψ, Θ, Φ ; see Sketch 4.1	m	ft
α	angle of attack, $\tan^{-1}(w/u)$	rad	rad
β	angle of sideslip, $\sin^{-1}(v/V)$	rad	rad
γ	angle of climb	rad	rad
γ	ratio of specific heat capacities of air		
$\partial\epsilon/\partial\alpha$	downwash gradient at tailplane		
ζ, η, ξ	rudder, elevator, aileron deflection angle	rad	rad
$\bar{\eta}$	elevator angle required to trim	rad	rad
$\Delta\bar{\eta}$	increment in $\bar{\eta}$	rad	rad
$\underline{\eta}$	vector of control perturbations	rad	rad
η_1, η_2	partitions of $\underline{\eta}$	rad	rad
Φ, Θ, Ψ	bank, inclination, azimuth attitude angles	rad	rad
ϕ, θ, ψ	roll, pitch, yaw attitude-deviation angles	rad	rad
λ	variable of characteristic stability equation		
λ_i	root of characteristic equation		
μ	relative density parameter, $m_e / \frac{1}{2}\rho_e S l_0$		
μ	dynamic viscosity of air	kg/m s	slug/ft s
ρ	density of air	kg/m ³	slug/ft ³
τ	unit of time in dynamic-normalised system, $m_e / \frac{1}{2}\rho_e V_e S$ see Section 4.5	s	s
τ_η	elevator trim-tab deflection angle	rad	rad
$\bar{\tau}_\eta$	value of τ_η for trim	rad	rad
$\underline{\Omega}$	angular velocity vector	rad/s	rad/s

Dressings

Dot (·)	as in $\dot{w}$, denotes differentiation with respect to time
Dip (✓)	as in $\check{X}_u$, denotes an aero-normalised quantity
Cap (^)	as in $\hat{x}_u$, denotes a dynamic-normalised quantity
Ord (°)	as in $\overset{\circ}{X}_u$, denotes a quantity expressed in ordinary dimensional units

Prime (')	as in $u' = u - u_e$, denotes a perturbation
Subscripts	
e	as in m_e , denotes a datum quantity
i, j	denote integers
$\left. \begin{matrix} u, v, w, \dot{w}, \dot{v} \\ p, q, r \\ \eta, \xi, \zeta \end{matrix} \right\}$	as in X_u etc., denote partial derivatives with respect to variable
T	denotes tail
WB	denotes wing-body

2. INTRODUCTION

2.1 General Remarks

To determine whether an aircraft is stable in a given trimmed steady flight condition it is necessary to analyse the motion of the aircraft after it has suffered a small disturbance (or perturbation) from that datum flight condition. If the disturbance dies out the aircraft is stable and if it does not then the aircraft is unstable. It is often convenient to analyse the motion of an aircraft by deriving representative equations. This requires knowledge of the aerodynamic and inertial forces acting on the aircraft.

The motion after a small disturbance can be described by linearised quasi-steady perturbation equations developed from the general rigid-body equations of motion. Although an aircraft is a flexible structure it is acceptable to study its dynamic behaviour through the equations of motion for a rigid body by modifying the aerodynamic derivatives to account for quasi-steady structural flexure under low frequency aerodynamic and inertial loads. The perturbation equations can be analysed in terms of the roots of a so-called 'characteristic' polynomial equation. The values of the roots determine whether or not the aircraft is stable. In the perturbation equations the influence of aerodynamic forces and moments is expressed through aerodynamic derivatives. The aim of the present Item is to show the connection between these aerodynamic derivatives and the roots of the characteristic equation.

The natural dynamic behaviour of the basic aircraft can be artificially changed by the inclusion in the overall aircraft system of certain automatic flight-control features of varying degrees of complexity. The aircraft is then said to be augmented. Simple 'stability augmentation' may, for instance, be provided by yaw or pitch dampers where controls operate to oppose a sensed disturbance. ESDU 83024 (Reference 26) of the Dynamics Series illustrates the operation of a yaw damper. In more sophisticated cases the design features incorporated into the control system can encompass 'manoeuvre-command' systems that improve handling qualities in response to pilot commands, relaxed requirements on natural stability, protective systems to guard against mishandling of the aircraft, and the many aspects of active control technology with its incorporation of direct-force controls. All of these are discussed in Reference 28. Any such systems need to be modelled within the equations of motion. Nevertheless the dynamic behaviour of the unaugmented aircraft remains important as a sub-system even in advanced designs. For the introductory purposes of this Item it is sufficient to restrict attention to the unaugmented aircraft.

The symbols adopted for describing the dynamic motion of an aircraft have varied over the years and between different authors. For many years the system of Reference 5 (Bryant and Gates) was in widespread use in the UK. However, this and other systems had the disadvantage that there was often a loss of recognisable connection between the symbols themselves and the physical quantities they represented. To remedy this, a general system that can be recommended for universal adoption has been published by Hopkin in Reference 16. That reference contains a complete description of a scheme of nomenclature for aircraft dynamics and the associated aerodynamics. The versatility of the complete form allows it to be unambiguous in the most exacting circumstances. In less demanding circumstances considerable simplification is usually possible. The system has provided much of the background material for the notation adopted by the International Organisation for Standardisation (ISO) in Reference 18, and is used in the ESDU Dynamics Series where it is described in some detail in ESDU 67001 to 67003 (References 12 to 14).

The system of Reference 16 has also become standard in the Aerodynamics Series. In Reference 16 there are tables of factors for conversion to or from the system of Reference 5.

2.2 Organisation of Item

Section 3 introduces the concepts of aerodynamic forces and moments and their derivatives. A systematic method for expressing these quantities in dimensionless form is given. Equations are provided for calculating the derivatives associated with longitudinal motion variables and reference is made to existing Items that contain methods for predicting the derivatives associated with lateral motion variables. A table is given that shows the conversion factors between derivatives expressed in the notation of this Item and in the notation most widely used in the USA.

Section 4 develops the general equations of motion for an aircraft treated as a rigid body, and shows how linearised quasi-steady perturbation equations can be formed. The equations apply rigorously only to infinitesimal disturbances but they can be used with good accuracy for quite large amplitudes provided there are no great departures from linearity.

Section 5 describes the principles underlying the analysis of the perturbation equations and shows that under certain simplifying assumptions they separate into two independent sets involving the longitudinal and the lateral motions of the aircraft. A detailed study is made of an aircraft slightly disturbed from straight and symmetrical steady flight, with controls fixed. Reference is made to the connections with the classical concepts of static margin and manoeuvre margin. Consideration is mainly confined to controls-fixed stability.

Section 6 contains a classical treatment of the problem of controls-fixed and controls-free longitudinal stability, starting from a consideration of static stability, which is concerned only with the equilibrium of the static forces and moments that develop on an aircraft in straight flight when it is disturbed slightly from a trimmed state. It then introduces the concepts of static margin and manoeuvre margin and discusses the relation between static stability and general stability in a dynamic sense. It is to be noted that the general stability of an aircraft can be studied without explicit reference to the concepts of static stability. Conversely, it is a requirement that an aircraft should have positive static stability unless some form of stability augmentation is provided. Determination as to whether an aircraft is statically stable can be made without reference to the dynamic stability equations.

Section 7 provides a description in simple physical terms of the commonly occurring lateral modes of motion of an aircraft that may follow a small disturbance.

Section 8 contains a list of references. Section 9 sets out a worked example for the investigation of the controls-fixed longitudinal and lateral stability of a civil jet-transport aircraft.

Appendix A gives tables summarising conversion factors for aerodynamic derivatives, for dynamic-normalised concise derivatives, and for obtaining dynamic-normalised quantities. Also tabulated is a comparison of American and aero-normalised derivatives. Details of a computer program to obtain the roots of the characteristic equations for the controls-fixed longitudinal and lateral stability of an aircraft, ESDUpac A8621, are given in Appendix B.

3. AERODYNAMIC FORCES AND MOMENTS AND THEIR EXPANSIONS

3.1 Form of Forces and Moments

The solution of problems in flight dynamics requires the aerodynamic forces and moments to be expressed in a form that adequately reflects the nature of the motion being considered and is convenient for the solution of the equations of motion. In order to set up a suitable form it is necessary to relate the forces and moments to the motion variables and other quantities that specify the aircraft flight condition (or state). For conditions of steady flight the forces and moments are fully determined by a reduced set of variables.

A general description of unsteady motion requires the whole set of time-varying state variables, and forces and moments that depend not only on the present values of the variables but also on the whole time history of the motion. However, the mathematical arguments developed by Tobak (see References 6, 20 and 29 and the discussion in Reference 27), together with practical experience, show that it is permissible in most problems of aircraft stability to adopt an approach whereby the forces and moments are assumed to depend on the values of the state variables and the first derivatives with respect to time of the linear velocity components.

The aerodynamic forces depend on

- (a) the overall shape of the aircraft and its skin roughness,
- (b) the size of the aircraft typified by some length, l ,
- (c) the properties of the air, specified by the ambient pressure, P , density, ρ , and dynamic viscosity, μ ,
- (d) the motion of the aircraft relative to the air, defined by the components of linear velocity, u , v , w , and of angular velocity, p , q , r ,
- (e) the control settings, where the elevator, aileron and rudder angles η , ξ , ζ are used as examples but other controls such as thrust vectors or direct force generators are assumed to be included,
- (f) the nearness and orientation of the aircraft relative to another large object, usually the Earth (ground effect). With an Earth-fixed axis system this may be defined in terms of the co-ordinates x_0 , y_0 , z_0 and three attitude angles, Φ (bank), Θ (inclination) and Ψ (azimuth).

For an aircraft of given shape and surface condition a typical force component Z can be expressed

$$Z = Z(l, P, \rho, \mu, x_0, y_0, z_0, \Phi, \Theta, \Psi, u, v, w, p, q, r, \eta, \xi, \zeta, \dot{u}, \dot{v}, \dot{w}), \quad (3.1)$$

where although they are not independent variables $\dot{u}$, $\dot{v}$ and $\dot{w}$ have been included to emphasise their part in the representation of knowledge of the history of the aircraft motion.*

A particular force or moment will usually be mainly determined by a sub-set of the variables in Equation (3.1). For the force Z and symmetrical flight a sensibly abbreviated form is

$$Z \approx Z(l, P, \rho, \mu, x_0, z_0, \Theta, u, w, q, \eta, \dot{u}, \dot{w}). \quad (3.2)$$

At this stage it is usually convenient to restrict attention to motion well away from the Earth or other large object so that x_0 , z_0 and Θ may be omitted, so that

$$Z \approx Z(l, P, \rho, \mu, u, w, q, \eta, \dot{u}, \dot{w}). \quad (3.3)$$

This can be written in a dimensionless form such as

$$\frac{Z}{\rho V^2 l^2} \approx Z\left(\frac{P}{\rho V^2}, \frac{\mu}{\rho V l}, \frac{w}{V}, \frac{q l}{V}, \eta, \frac{\dot{u} l}{V^2}, \frac{\dot{w} l}{V^2}\right) \quad (3.4)$$

where V is the magnitude of the aircraft velocity.

* See Reference 29 for a mathematically rigorous formulation.

There is the alternative and more familiar form involving Mach number, $M = V(\rho/\gamma P)^{1/2}$, Reynolds number, $R = \rho V l / \mu$, and a reference area, S ,

$$\frac{Z}{\frac{1}{2}\rho V^2 S} \approx Z\left(M, R, \frac{w}{V}, \frac{ql}{V}, \eta, \frac{\dot{u}l}{V^2}, \frac{\dot{w}l}{V^2}\right). \quad (3.5)$$

For symmetrical flight at low angles of attack $\alpha \approx w/V$ and $\dot{\alpha} \approx \dot{w}/V$ and there is the form

$$\frac{Z}{\frac{1}{2}\rho V^2 S} \approx Z\left(M, R, \alpha, \frac{ql}{V}, \eta, \frac{\dot{u}l}{V^2}, \frac{\dot{\alpha}l}{V}\right). \quad (3.6)$$

The dimensionless quantity $Z/\frac{1}{2}\rho V^2 S$ is known as the aerodynamic force coefficient C_Z and similar relationships exist for C_X and C_Y . The aerodynamic moment coefficients C_l , C_m and C_n are similar except that the length l appears also in the divisor. Different choices of l will normally be made for the longitudinal and lateral coefficients; $\bar{c}$ or $\bar{c}$ is usual for C_m and b or $b/2$ for C_l and C_n .

3.2 Expansion of Forces and Moments

When considering the motion of an aircraft after a small perturbation from a datum flight condition a return is made to a form such as Equation (3.1), since it is reasonable to express the aerodynamic forces and moments in linear expansions about their datum values. Before developing such expansions it is noted that as P , ρ and μ are all functions of the height above sea level, for air in equilibrium they may be replaced by this single variable. Therefore if, as is customary, it is assumed that the atmosphere is essentially uniform for small changes in height the variation of forces and moments with height can be ignored.

For an aircraft with a longitudinal plane of symmetry the usual expansion of the forces and moments with respect to a system of body-axes $Oxyz$ (defined in Section 1.1), where the x -axis and z -axis lie in the plane symmetry, is

$$X' = X - X_e = X_u u' + X_w w' + X_{\dot{w}} \dot{w}' + X_q q' + X_{\eta} \eta' + \dots, \quad (3.7)$$

$$Y' = Y - Y_e = Y_v v' + Y_p p' + Y_r r' + Y_{\xi} \xi' + Y_{\zeta} \zeta' + \dots, \quad (3.8)$$

$$Z' = Z - Z_e = Z_u u' + Z_w w' + Z_{\dot{w}} \dot{w}' + Z_q q' + Z_{\eta} \eta' + \dots, \quad (3.9)$$

$$\mathcal{L}' = \mathcal{L} - \mathcal{L}_e = L_v v' + L_p p' + L_r r' + L_{\xi} \xi' + L_{\zeta} \zeta' + \dots, \quad (3.10)$$

$$\mathcal{M}' = \mathcal{M} - \mathcal{M}_e = M_u u' + M_w w' + M_{\dot{w}} \dot{w}' + M_q q' + M_{\eta} \eta' + \dots, \quad (3.11)$$

$$\mathcal{N}' = \mathcal{N} - \mathcal{N}_e = N_v v' + N_p p' + N_r r' + N_{\xi} \xi' + N_{\zeta} \zeta' + \dots, \quad (3.12)$$

where the subscript e denotes datum conditions and primes denote perturbation from the datum, for example $u' = u - u_e$. The quantities X_u etc. in the expansions are commonly referred to as aerodynamic derivatives. They are evaluated at datum conditions and hence at a specified Mach number and Reynolds number. They correspond to the partial differentials $\partial X / \partial u$ etc. when it is valid to assume that the influence of the history of the motion on the aerodynamic forces and moments is adequately represented through present values of $\dot{u}$, $\dot{v}$ and $\dot{w}$.

The selected terms shown in Equations (3.7) to (3.12) are those known to be significant for a wide range of flight conditions. In certain other cases additional terms and control variables may be needed. For example, for aircraft with highly swept wings at high angles of attack the terms $L_v \dot{v}$ and $N_v \dot{v}$ may be important (see References 11 and 22). No cross-coupling terms such as $X_v v$, $Y_u u$ etc. have been included since for a symmetrical aircraft they may normally be treated as zero, although the pitching moment due to sideslip, $M_v v$, can assume significant values for some aircraft (see Reference 23). In the case of vertical-lift aircraft some further cross terms may be important.

Aerodynamic coefficients may themselves be expanded in analogous forms to those developed for the dimensional forces and moments. In such an expansion the dependence on speed and density is largely embodied within the factor $\frac{1}{2}\rho V^2$. As indicated in Reference 16 such expansions are appropriate when there is a substantially varying speed or density.

3.3 Concise Forms

To provide a compact way of writing the equations of motion, concise forms of forces and moments are introduced whereby the forces are divided by the aircraft mass and the moments by the moment of inertia about the appropriate axis. In both cases a change of sign is introduced. The concise forms are denoted by a lower case letter corresponding to the original force or moment. Concise forms of aerodynamic derivatives are produced in an identical manner. For example, the concise forms of the force Z and the moment $\mathcal{L}$ are

$$z = -Z/m_e \text{ and } l = -\mathcal{L}/I_x, \quad (3.13)$$

and concise forms of typical derivatives are

$$z_u = -Z_u/m_e \text{ and } l_v = -L_v/I_x. \quad (3.14)$$

3.4 Aero-normalised System

Thus far it has been tacitly assumed that the forces and moments and aerodynamic derivatives are quantities measured in a system of ordinary dimensional units, *e.g.* the SI system.

Just as the conversion of aerodynamic forces and moments into dimensionless coefficients is a convenient means of handling such data in that it isolates the dependence on velocity, density and size, a parallel 'normalising' procedure is used to express aerodynamic derivatives and motion variables in dimensionless form. The standard system adopted for aerodynamic derivatives in the Aerodynamics Series is the aero-normalised system recommended in Reference 16.

The aero-normalised system uses divisors based on the appropriate combinations of three reference units of force, speed and length. The unit of force is $\frac{1}{2}\rho_e V_e^2 S$, where ρ_e and V_e are datum values of air density and aircraft speed, and can be sensibly taken as the conditions existing immediately before a disturbance. The unit of speed is V_e and the unit of length is l_0 , a characteristic length for the aircraft. The divisor required for any other physical quantity can be constructed by dimensional analysis, for example the divisor for a moment is $\frac{1}{2}\rho_e V_e^2 S l_0$ and that for an angular velocity is V_e/l_0 . If the equations of motion are expressed in the aero-normalised system a single value of l_0 must be used throughout.

It is a feature of the aero-normalised system that the same symbols are used to represent both dimensional and dimensionless quantities as there will seldom be confusion as to which system is being used. If it is necessary, a dressing (°) as in $\dot{X}_u^\circ$ may be used to distinguish the dimensional form or a dressing (✓) as in $\check{X}_u$ may be used to denote the aero-normalised form. The first three columns of Table A1.1 demonstrate the normalising process for aerodynamic derivatives.

3.5 Estimation of Aerodynamic Derivatives

Numerical values of stability derivatives can be obtained from wind-tunnel tests of a particular model, from theoretical or semi-empirical prediction methods, or from an analysis of flight-test results. The values depend on the particular body-axis system chosen. Although conversion from one system to another can be made, it is convenient to adopt a system in which the derivatives are available directly.

In cases where the angle of attack is small or moderate, an aerodynamic-body axis system, defined in Section 1.1, is convenient. In the UK aerodynamic-body axes have in the past been referred to as wind-axes or wind-body axes. In the USA they are usually referred to as stability axes. They are used in many semi-empirical prediction methods.

The chief alternative to the use of aerodynamic-body axes are geometric-body axes, defined in Section 1.1, and these are well suited to the study of motion at high angles of attack. In some cases it is possible to choose the geometric-body axes to coincide with the principal axes of inertia. This has the benefit that all the cross products of inertia are eliminated from the equations of motion.

3.5.1 Longitudinal derivatives

For studies of small disturbances from straight and symmetrical steady flight the controls-fixed longitudinal derivatives about aerodynamic-body axes are often approximated by simple formulae that involve the lift, drag and thrust characteristics of the aircraft, see Reference 25. The relevant equations for the aero-normalised derivatives at subcritical speeds are

$$X_u = - \left(2C_D + V \frac{\partial C_D}{\partial V} \right)_e + \left(\frac{1}{\frac{1}{2}\rho V S} \frac{dT}{dV} \right)_e, \quad (3.15)$$

$$X_w = \left(C_L - \frac{\partial C_D}{\partial \alpha} \right)_e = \left(C_L - \frac{\partial C_D}{\partial C_L} \frac{\partial C_L}{\partial \alpha} \right)_e, \quad (3.16)$$

$$X_{\dot{w}} = 0, \quad (3.17)$$

$$X_q = 0, \quad (3.18)$$

$$Z_u = - \left(2C_L + V \frac{\partial C_L}{\partial V} \right)_e, \quad (3.19)$$

$$Z_w = - \left(C_D + \frac{\partial C_L}{\partial \alpha} \right)_e, \quad (3.20)$$

$$Z_{\dot{w}} = 0, \quad (3.21)$$

$$Z_q = (Z_q)_{WB} - \left(\frac{\partial C_{LT}}{\partial \alpha} \right)_e \frac{S_T l_T}{S l_0}, \quad (3.22)$$

$$M_u = \left(V \frac{\partial C_m}{\partial V} \right)_e, \quad (3.23)$$

$$M_w = \left(\frac{\partial C_m}{\partial \alpha} \right)_e, \quad (3.24)$$

$$\dot{M}_w = (\dot{M}_w)_{WB} - \left(\frac{\partial C_{LT}}{\partial \alpha} \right)_e \frac{S_T l_T^2}{S l_0^2} \left(\frac{\partial \varepsilon}{\partial \alpha} \right)_e, \quad (3.25)$$

$$M_q = (M_q)_{WB} - \left(\frac{\partial C_{LT}}{\partial \alpha} \right)_e \frac{S_T l_T^2}{S l_0^2}. \quad (3.26)$$

The derivatives X_q , X_w and Z_w which are of negligible importance, have been set to zero. The derivatives Z_q , M_w and M_q are represented as the sum of a wing-body contribution and a tail contribution, with the two wing-body moment derivatives being referred to the aircraft centre of gravity position. For aircraft with long tail arms the tail contributions dominate and the wing-body contributions $(Z_q)_{WB}$, $(M_w)_{WB}$ and $(M_q)_{WB}$ may be omitted. The derivative $\partial C_{LT}/\partial \alpha$ is the tailplane lift-curve slope based on tailplane planform area, S_T , and l_T is the distance of the aerodynamic centre of the tailplane aft of the centre of gravity position. The factor $\partial \varepsilon/\partial \alpha$ represents the downwash at the tailplane. The lift and drag coefficients and their derivatives are all evaluated at the steady datum conditions.

It is assumed in the equations that the thrust acts along the x -axis. If the thrust axis is inclined to the x -axis and its line of action passes at a perpendicular distance d_T below the centre of gravity position then the component of thrust T_x acting parallel to the x -axis replaces T in Equation (3.15). There are also small additional terms, $(dT_z/dV)_e / \frac{1}{2} \rho_e V_e S$ in Equation (3.19) associated with the thrust component parallel to the z -axis and $(dT/dV)_e d_T / \frac{1}{2} \rho_e V_e S l_0$ in Equation (3.23) associated with the thrust moment about the centre of gravity position. These two terms are usually small enough to be ignored (see Reference 22).

Equation (3.15) involves the variation of the engine thrust with speed, which will depend on the operating conditions and type of engine. For example, in the case of a piston-engined aircraft with variable-pitch propellers a reasonable approximation is that the power and efficiency of the engine are constant, *i.e.* $TV = \text{constant}$ and $(dT/dV)_e = -T_e/V_e$. There is much less dependence of thrust on speed for jet engines.

The derivatives of the aerodynamic coefficients C_D , C_L and C_m with respect to V reflect changes in slipstream (see Reference 9), compressibility and Reynolds number effects, although the last of these is insignificant for full-scale flight. The general classical treatment (Reference 2), that is not confined to the assumption of a rigid body, also allows for the effect of a change in aircraft shape due to low frequency structural flexure that is in phase with the local aerodynamic loading. If such derivatives are negligible then some of the equations simplify. In particular, and for steady horizontal flight, Equation (3.24) can be replaced by

$$M_w = \left(\frac{dC_m}{dC_L} \frac{\partial C_L}{\partial \alpha} \right)_e = -k_s \left(\frac{\partial C_L}{\partial \alpha} \right)_e, \quad (3.27)$$

where k_s is the static margin (see Equation (6.1) of Section 6).

Note that Chapter 15 of Reference 9 considers the effects of compressibility on the aerodynamic derivatives.

3.5.2 Lateral derivatives

Items in the Aerodynamics Series may be used for predicting controls-fixed lateral stability derivatives about aerodynamic-body axes. For subcritical speeds there is a complete coverage of the derivatives due to sideslip, roll rate and yaw rate. Items for predicting the rolling moment derivatives are introduced by ESDU Aero.06.01.00 (Reference 8) and those for predicting yawing moment and sideforce derivatives are introduced in ESDU Aero.07.01.00 (Reference 3).

3.5.3 Comparison of derivative definitions

In Reference 16 it is explained that various systems of stability notation are used in the USA. This Section contains information on the differences between aerodynamic derivatives expressed in a widely used American system and the aero-normalised system of Reference 16.

In this American system the basic symbols such as X , Y , Z , u , v , w , α , β etc. remain standard but derivatives of dimensionless coefficients based on constant values of ρ and V are used to denote aerodynamic derivatives, for example the symbol C_{xu} is used instead of X_u . Also, unlike the aero-normalised system, the American system uses different representative lengths for the normalised motion variables and the moment coefficients, respectively $\bar{c}/2$ and $\bar{c}$ for the longitudinal derivatives and $b/2$ and b for the lateral derivatives. When derivatives with respect to α and β are formed there is usually the implication that $\alpha \approx w/V$ and $\beta \approx v/V$ so that $C_{l\beta} = C_{lv}$ etc.

If it is assumed that in the American system all coefficients are evaluated at datum conditions and that the reference dimensions for the longitudinal coefficients are S and $\bar{c}$ and for the lateral coefficients are S and b , then Table A1.2 allows for all of these differences and relates the American derivatives to the corresponding aero-normalised derivatives. In practice, symbols other than η , ξ and ζ will probably be used to denote control deflections.

All of the individual Items in the Aerodynamics Series which give methods for the prediction of stability derivatives have before their Notation a note that gives the American style of the derivative, with a warning to check on reference dimensions.

4. EQUATIONS OF MOTION

This Section sets out the general equations of motion for an aircraft treated as a rigid body. It goes on to develop the linearised perturbation equations that are widely used to describe the motion following a small disturbance.

It is assumed throughout that the variation of atmospheric properties with small changes in height is insignificant, that the effects of the Earth's curvature and rotation may be neglected, and that the velocity of the surrounding air relative to the Earth is zero. The aircraft is taken to be well clear of the Earth so that there is no ground effect on the aerodynamic forces and moments.

4.1 Attitude Angles

The inertial quantities and instantaneous motion of the aircraft can be defined with respect to any set of body-axes $Oxyz$, as defined in Section 1.1. The choice of particular systems can introduce simplifications into the equations of motion and those brought about by aerodynamic-body axes are demonstrated in later sections.

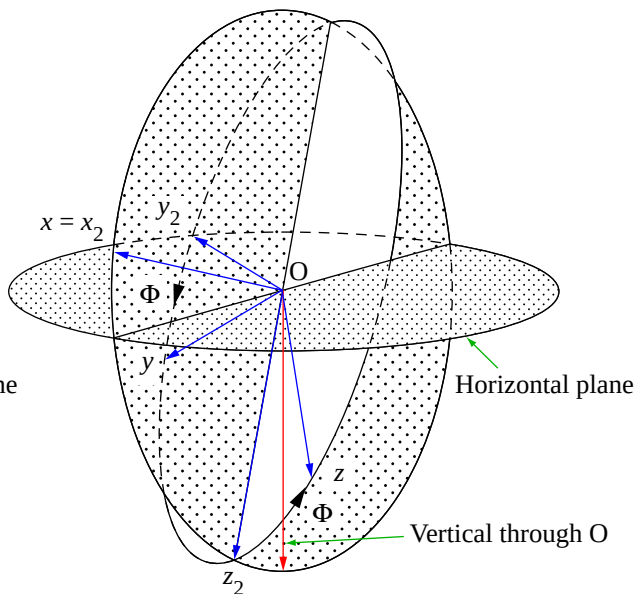
Because the body-axes move in space a complete description of the aircraft motion requires the definition of a reference axis system whose axes are in fixed directions in space. In this Item an aircraft-carried Earth-axis system $Ox_0y_0z_0$ is chosen, as defined in Section 1.1.2.

The position occupied by an aircraft in space may then be specified by the position of the axis origin O relative to the Earth and by the aircraft's attitude relative to the axis system $Ox_0y_0z_0$. The latter may be defined by specifying three independent angular rotations, Φ (bank), Θ (inclination) and Ψ (azimuth), which if applied in a standard sequence to an axis system initially coincident with the Earth-axis system would bring it into alignment with the body-axis system.

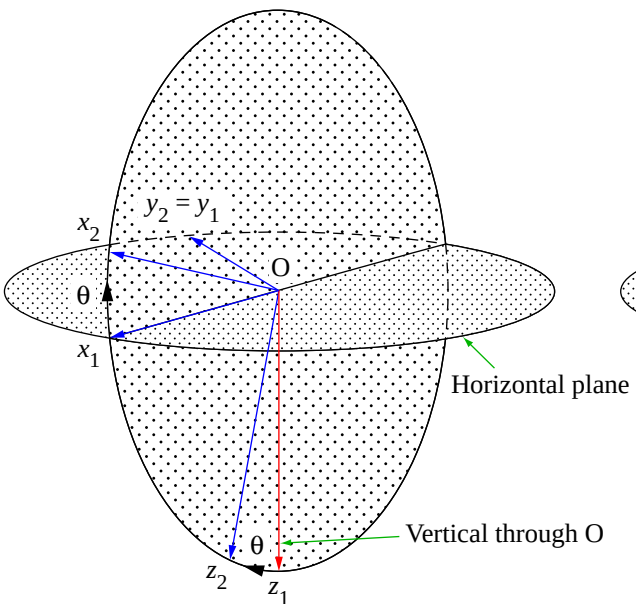
The standard sequence of rotations, which is illustrated in Sketch 4.1, is

- (i) an initial rotation Ψ in the positive sense about the z_0 -axis,
- (ii) a second rotation Θ in the positive sense about the resulting (intermediate) position of the y -axis,
- (iii) a third rotation Φ in the positive sense about the resulting final position of the x -axis.

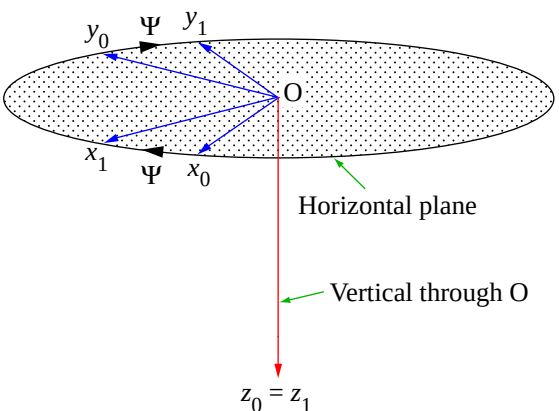
For small perturbation studies it is convenient to express the disturbed orientation of the aircraft relative to a datum attitude defined by Φ_e , Θ_e , Ψ_e . This is done by defining attitude-deviation angles ϕ , θ , ψ . Application of these rotations to the body-axes system in its datum condition, in the same way that the rotations Φ , Θ , Ψ are applied to the Earth-axis system, changes the orientation of the body-axis system from the datum position to the disturbed position.



(iii) THIRD ROTATION:
THROUGH Φ ABOUT Ox_2 ($\equiv Ox$)



(ii) SECOND ROTATION:
THROUGH Θ ABOUT Oy_1 ($\equiv Oy_2$)



(i) FIRST ROTATION:
THROUGH Ψ ABOUT Oz_0 ($\equiv Oz_1$)

- $Ox_0y_0z_0$ Initial position of axis system, coincident with Earth-axes system
- $Ox_1y_1z_1$ Position of axis system after first rotation ($Oz_1 \equiv Oz_0$)
- $Ox_2y_2z_2$ Position of axis system after second rotation ($Oy_2 \equiv Oy_1$)
- $Oxyz$ Final position of axis system ($Ox \equiv Ox_2$), coincident with body-axis system

Sketch 4.1 SEQUENCE OF ROTATIONS DEFINING ATTITUDE ANGLES Ψ , Θ , Φ