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Natural frequencies of rectangular singly-curved plates

Associated software: ESDUpac A8702
ESDUpac B8702

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THE PREPARATION OF THIS DATA ITEM

The work on this particular Item, which supersedes ESDU 72004, was monitored and guided by the following Acoustic Fatigue Panel on behalf of the Vibration and Structural Dynamics Steering Group:

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NATURAL FREQUENCIES OF RECTANGULAR SINGLY-CURVED PLATES

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NATURAL FREQUENCIES OF RECTANGULAR SINGLY-CURVED PLATES

1. NOTATION

a	length of plate in x-direction (see Sketch 1.1)	m	in
b	length of plate in y-direction, or arc length of curved plate (see Sketch 1.1)	m	in
E	Young's modulus of plate material	N/m^2	lbf/in^2
f	natural frequency of vibration of plate	Hz	Hz
K	natural frequency parameter for fixed-edge plates	m/s	in/s
$K_{m,n}$	natural frequency parameter for simply-supported plate in (m,n)th mode	m/s	in/s
M	number of terms in x in displacement function [see Equations (B.1) to (B.3)]		
m	mode number in x-direction		
N	number of terms in y in displacement function [see Equations (B.1) to (B.3)].		
n	mode number in y-direction		
p,q	integers in Equations (B.1) to (B.3)		
R	radius of curvature of plate	m	in
t	thickness of plate	m	in
U,V,W	displacement parameters [see Equations (A.1) to (A.3)]	m	in
U_{pq}, V_{pq}, W_{pq}	displacement parameters [see Equations (B.1) to (B.3)]	m	in
u,v,w	displacements (see Sketch 1.1)	m	in
V_s	velocity parameter for plate material†		
x,y	plate coordinate system (see Sketch 1.1)		
θ	subtended angle (see Sketch 1.1)	rad	rad

For footnote see end of Notation

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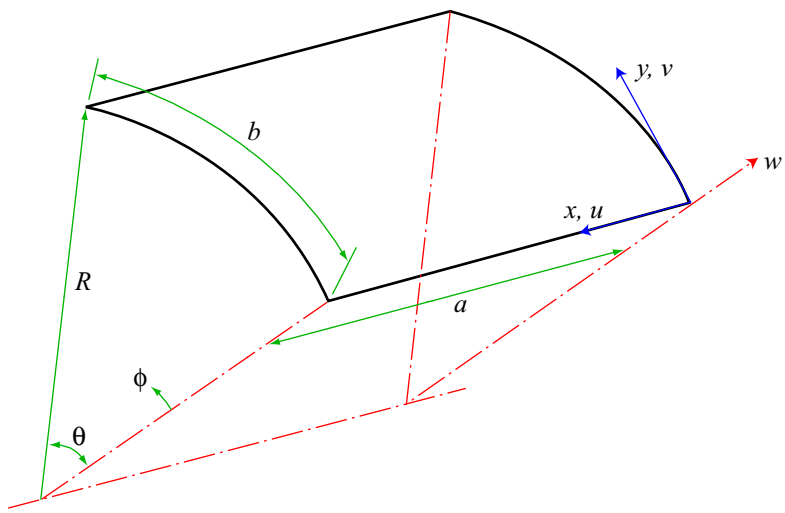
ν	Poisson's ratio of plate material		
ρ	density of plate material	kg/m^3	‡
ϕ	angular location (see Sketch 1.1)	rad	rad

† The velocity parameter is defined in SI units by $V_s = (E/\rho)^{1/2}/5080$ or in British units by $V_s = (E/\rho)^{1/2}/200\,000$. The value of V_s is approximately unity for most common structural metallic materials.

‡ A density value expressed in British units as pounds per cubic inch has to be divided by 386.4 before it can be used in the formula for V_s given here. (A force of 1 lbf acting on a mass of 1 lb produces an acceleration of 386.4 in/s^2.) For example if the material density is 0.097 lb/in^3, the value input should be 2.510×10^{-4} lb s^2/in^4.



Sketch 1.1 Singly-curved rectangular plate geometry

2. NOTES

2.1 General

This Item provides a means of estimating the lower natural frequencies of initially unstressed, singly-curved rectangular plates. The data are applicable to thin, isotropic plates of shallow curvature having either all edges simply-supported or all edges fixed in both translation and rotation (*i.e.* the clamped edge condition). Natural frequency data are provided in graphical form. In Appendices A and B, computer programs, referred to as ESDUpacs A8702 and B8702*, are described for the estimation of modal data for simply-supported and fixed-edge plates, respectively.

* The full ESDUpac number has an appended V followed by two digits. The digits indicate the ESDUpac version number. For example, the full ESDUpac number for Version 1.2 of ESDUpac A8702 is A8702V12. The version appendage will not be used when referring to ESDUpacs in the text of this Item.

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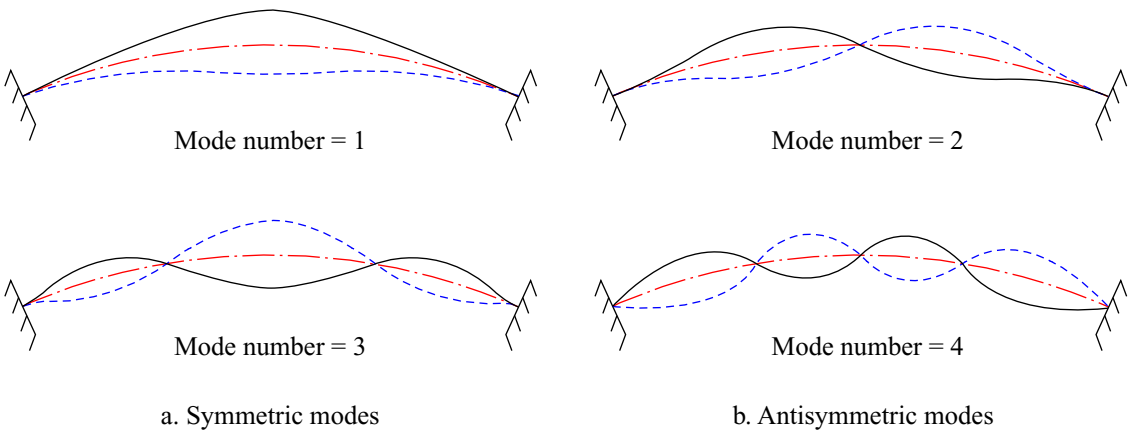
2.2 Mode Numbers and Mode Definition

For any mode of vibration, the mode numbers m and n may be related to the nodal lines provided that these lines are parallel, or approximately parallel, to the plate sides. For these cases m and n are one plus the number of nodal lines crossing any line parallel to the x-axis and y-axis, respectively. In this definition plate edges, clamped or simply-supported, are not counted as nodal lines.

For plates having all edges simply-supported the modes are exactly defined by the number of half waves across the plate in the x- and y-directions. For simply-supported plate vibration modes, nodal lines are parallel to the plate edges and the condition at a nodal line is the same as that at the boundaries parallel to the nodal line.

For plates having all edges fully fixed, the modes of vibration are not necessarily uniquely defined by the number of half waves across the plate in the x- and y-directions. In this Item the fixed-edge plate modes are defined by the symmetry or antisymmetry in two directions, stating first the symmetry condition parallel to the x-direction followed by the symmetry condition parallel to the y-direction. For example, for the first symmetric-antisymmetric mode shown in Figure 11, for all geometric conditions illustrated, there is one half wave across the plate in the x-direction (the straight side of the curved plate is parallel to the x-direction) so that the shape is symmetric in that direction. Similarly there are two half waves across the plate in the y-direction (the curved plate side is parallel to the y-direction) so that the shape is antisymmetric in that direction.

For fixed-edge plate modes having nodal lines parallel, or approximately parallel, to the plate boundaries, a symmetric condition in a particular direction corresponds to an odd mode number in that direction. Similarly, an antisymmetric condition in a particular direction corresponds to an even mode number in that direction. Sketch 2.1 shows typical curved plate displacement forms for fixed edge plates. The Sketch illustrates the relationship between mode number and symmetry/antisymmetry.



Sketch 2.1 Curved plate displacement forms

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2.3 Natural Frequencies

2.3.1 Plates with simply-supported edges

The natural frequencies are given by

$$f = V_s K_{m,n} \frac{n^2 t}{b^2}. \tag{2.1}$$

In Figure 1 values of $K_{m,n}$ are plotted against $an/(bm)$ for a range of $b^2/(n^2 Rt)$ for initially unstressed, cylindrically curved, rectangular plates of uniform thickness having all edges simply-supported. Data plotted in Figure 1 are for the bending mode in which the predominant displacement is w.

2.3.2 Plates with fixed edges

The natural frequencies are given by

$$f = V_s K \frac{t}{b^2}. \tag{2.2}$$

In Figures 2 to 8 values of K are plotted against a/b for a range of values of $b^2/(Rt)$ for a number of natural modes of initially unstressed, cylindrically curved, rectangular plates of uniform thickness having all edges fully fixed. The order of modes in any mode type is given in ascending order of frequency.

In Figures 9 to 15 typical mode shapes are presented.

The Figure numbers for natural frequency parameters and mode shapes for the fixed-edge plate modes considered in this Item are tabulated below.

Mode		Figure number for frequency parameter	Figure number for mode shape
First	(S, S)	2	9
Second	(S, S)	3	10
First	(S, A)	4	11
Second	(S, A)	5	12
First	(A, S)	6	13
Second	(A, S)	7	14
First	(A, A)	8	15

S = symmetric A = antisymmetric

The expression used to calculate the fixed-edge plate frequencies contains parameters $a/b, ba/(Rt)$ and b/R. The curves in this Item are for a value of b/R of 0.1, but the curves may be used for shallow plates generally (see Section 2.4).

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2.3.3 Lowest natural frequency

For flat plates with simply-supported edges the lowest natural frequency of vibration occurs in the mode having a single half wave across the length and breadth of the plate, *i.e.* the ($m = 1$, $n = 1$) mode. For flat plates with fixed edges the lowest natural frequency occurs in the first symmetric-symmetric mode where the mode shape is similar to the simply-supported (1, 1) mode. As the curvature of the plate increases (radius decreases) the lowest frequency, for both simply-supported and fixed-edge plates, tends to occur in a mode where the number of half waves across the curved side is greater than one.

2.4 Accuracy of and Limitations on Data

For the simply-supported plates, the natural frequency parameters are obtained from an exact solution to the equations of motion using displacement functions that completely satisfy the boundary conditions. For plates with fully fixed edges, no exact solution to the equations of motion and boundary condition is known. For this latter case, the analysis is based on an extended Rayleigh-Ritz method (Derivations 6 and 8). Therefore, for the fixed-edge plate, the natural frequencies are upper bounds to the flexural modes of the plates.

In the theory on which this Item is based it is assumed that plates are thin and of shallow curvature. These conditions are satisfied when a/t, b/t and R/t are greater than 20, and $b/R < 1.5$. Also in the method of analysis for both simply-supported and fixed-edge plate natural frequencies the effects of shear deflection and rotary inertia have been neglected. This is a generally accepted simplification if the ratio of half wave length to plate thickness is greater than 10.

In deriving the graphical data presented in this Item the value of ν was assumed to be 0.3. This value gives sufficiently accurate frequencies for all common structural metallic materials. Graphical data presented for the fixed-edge plates are evaluated for $b/R = 0.1$. At higher values of b/R, the natural frequencies are slightly less than those obtained using Figures 2 to 8. For the modes of vibration and range of parameters presented in those figures, the difference between frequencies estimated using $b/R = 0.1$ and $b/R = 1.0$ is usually less than 5 per cent, the greatest difference occurring in the higher range of a/b ratios. The effects of varying ν and the ratio b/R may be readily investigated using the computer programs described in the Appendices to this Item.

It is difficult in practice to achieve the theoretical simply-supported or fixed-edge boundary condition. Comparisons with measured natural frequencies over a wide range of plate geometries have shown that, with few exceptions, the measured natural frequencies lie within the band of frequencies bounded by the estimated simply-supported and fixed-edge frequencies. For one set of test data (reported in Derivation 8) the fixed-edge boundary condition was very closely approximated. For those tests, a comparison between the estimated and measured frequencies of the first ten modes is presented in Figure 16.

3. APPLICATION TO STIFFENED STRUCTURAL PANELS

Structural panels, such as those used in aerospace structures, are typically constructed using stiffening members such as ribs, frames and stringers that divide the panel into plates. Owing to the mechanical coupling between plates, modes result in which arrays of plates vibrate together. Theoretical analysis shows that an array of plates has groups of resonant frequencies. Each group is bounded by a lower frequency mode where stringers twist and a higher frequency mode where stringers bend. These lower and higher frequency modes correspond to the single plate modes with simply-supported and fixed edges, respectively. In between these two bounding frequencies there are a number of intermediate modes in which the stringers both twist and bend. The second and subsequent frequency groups correspond to higher order modes of the fundamental group.

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The number of plates showing predominant vibration at any one natural frequency of the group depends on the relative dimensions of the plates and the phase relationships of the exciting force over the surface of the panel.

The lowest natural frequency in a frequency group is generally associated with a mode of vibration in which the response of plates is such that half waves across the plates on the two sides of a stringer are out of phase. If the stringers are of low torsional stiffness the lowest frequency for the panel will be close to the value given by this Item for a single plate with simply-supported edges. The highest natural frequency in a frequency group is associated with a mode of vibration in which the half waves across the plates on the two sides of a stringer are in phase. This latter behaviour corresponds to that resulting from fixed-edge conditions. This Item therefore provides upper and lower boundary values for the individual frequency groups. In practice, the upper and lower frequencies of a frequency group are dependent on the torsional stiffness of the stringers. With increasing torsional stiffness the lowest resonant frequency of the group rises to approach the value given by this Item for a single plate with fixed edges.

4. DERIVATION

This Section lists selected sources that have assisted in the preparation of this Data Item.

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WARBURTON, G.B. — Discussion of “Approximate methods for the determination of the natural frequencies of stiffened and curved plates”. *J. Sound Vib.*, Vol. 18, No. 1, pp. 139-141, September 1971.

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5. EXAMPLE

It is required to estimate the lowest natural frequency of an aluminium alloy plate in both the simply-supported and fixed-edge conditions. The plate has the following dimensions and physical properties.

$a = 240$ mm, $b = 300$ mm, $R = 1250$ mm,

$t = 1.2$ mm, $E = 70\,000$ MN/m^2, $\rho = 2770$ kg/m^3.

Firstly
$$\frac{a}{b} = \frac{240}{300} = 0.8,$$
$$\frac{b^2}{Rt} = \frac{300^2}{1250 \times 1.2} = 60,$$
and
$$V_S = \sqrt{\frac{70\,000 \times 10^6}{2770}} \times \frac{1}{5080} = 0.99.$$

5.1 All Edges Simply-Supported

Natural frequencies are given by

$$f = V_s K_{m,n} \frac{n^2 t}{b^2},$$

where values of $K_{m,n}$ are obtained from Figure 1. The calculation steps are tabulated below.

Mode (m, n)	$\frac{an}{bm}$	$\frac{b^2}{n^2 Rt}$	$K_{m,n} \times 10^{-3}$ (m/s)	$\frac{n^2 t}{b^2}$ (m^{-1})	$f = V_s K_{m,n} \frac{n^2 t}{b^2}$ (Hz)
1, 1	0.8	60	30.2	0.0133	398
1, 2	1.6	15	4.78	0.0533	252
2, 1	0.4	60	45.3	0.0133	596
2, 2	0.8	15	9.64	0.0533	509
1, 3	2.4	6.67	2.94	0.1200	349
3, 1	0.267	60	58.1	0.0133	765

Hence the lowest natural frequency with all edges simply-supported occurs in the (1, 2) mode and is 252 Hz.

5.2 All Edges Fixed

The natural frequencies are given by

$$f = V_s K \frac{t}{b^2},$$

where K is obtained from Figures 2 to 9.

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The lowest frequency is the first frequency of one of the four symmetry conditions.

For lowest symmetric-symmetric mode, from Figure 2,

$$K \times 10^{-3} = 33.5 \text{ m/s.}$$

For lowest symmetric-antisymmetric mode, from Figure 4,

$$K \times 10^{-3} = 30.8 \text{ m/s.}$$

For lowest antisymmetric-symmetric mode, from Figure 6,

$$K \times 10^{-3} = 49.3 \text{ m/s.}$$

For lowest antisymmetric-antisymmetric mode, from Figure 8,

$$K \times 10^{-3} = 48.5 \text{ m/s.}$$

Hence the lowest natural frequency with all edges fixed occurs in the first symmetric-antisymmetric mode and is

$$0.99 \times 30.8 \times 10^{3} \times \frac{1.2 \times 10^{-3}}{0.3^{2}} = 407 \text{ Hz.}$$

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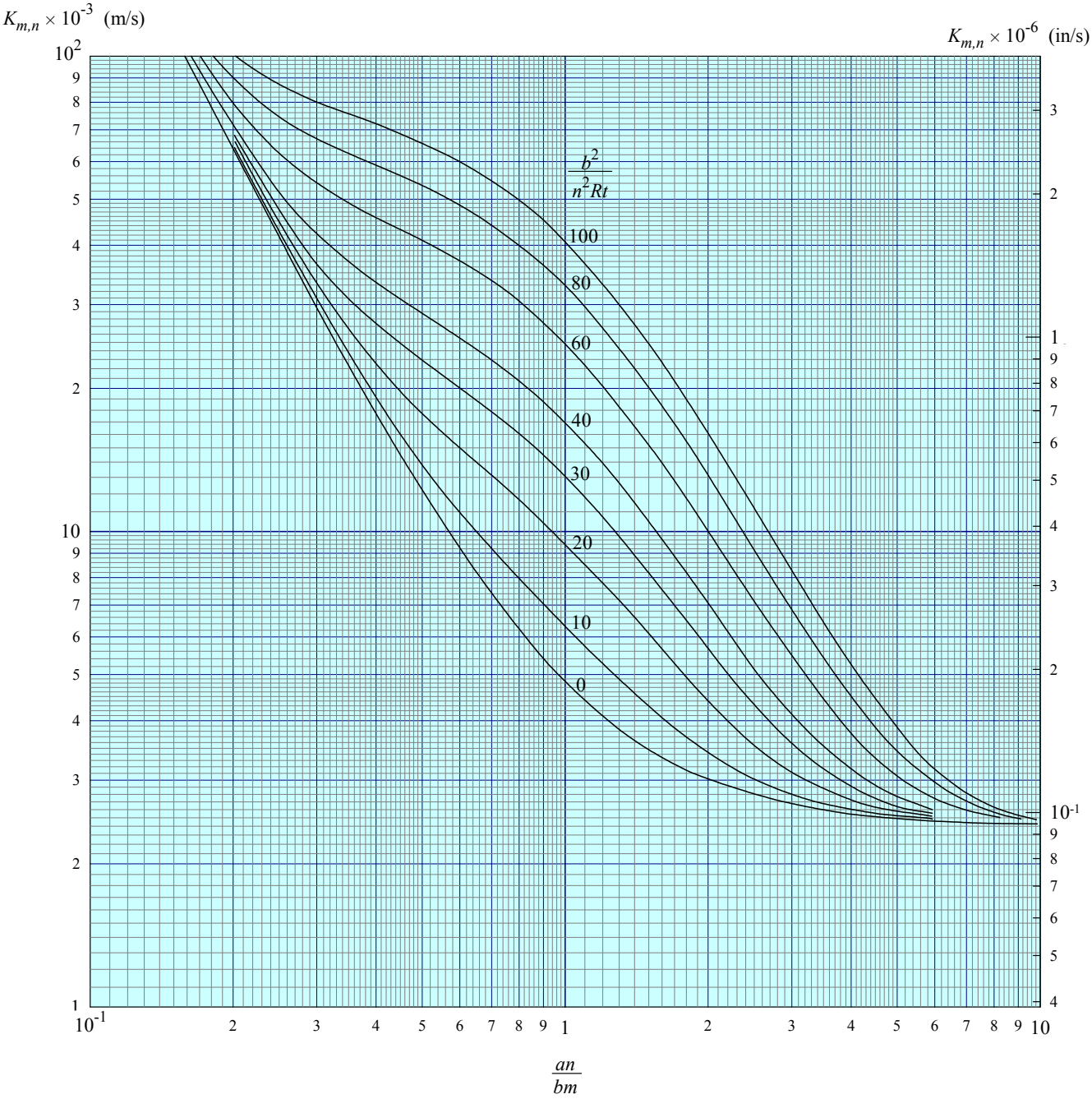

FIGURE 1 FREQUENCY PARAMETERS FOR PLATES WITH ALL EDGES SIMPLY-SUPPORTED

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