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Transverse (through-the-thickness) shear stiffnesses of fibre reinforced composite laminated plates

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TRANSVERSE (THROUGH-THE-THICKNESS) SHEAR STIFFNESSES OF FIBRE REINFORCED COMPOSITE LAMINATED PLATES

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TRANSVERSE (THROUGH-THE-THICKNESS) SHEAR STIFFNESSES OF FIBRE REINFORCED COMPOSITE LAMINATED PLATES

1. ASSUMPTIONS, SOLUTION OUTLINE AND STIFFNESS TRENDS

1.1 Introduction

The transverse (through-the-thickness) elastic properties of continuous fibre reinforced composite laminated plates, of the type shown in Sketch 1.1, are, to a large extent, dependent upon the properties of the matrix material used to bond the fibres together. For many matrix materials, these properties can be very low compared with those in the direction of the reinforcing fibres, with the result that significant deflections can be caused by through-the-thickness shear stresses which arise in plate bending. In order to take this behaviour into account, it is necessary to obtain values for the shear stiffness terms that relate the transverse shear force resultants to the transverse shear strains. This Item together with the accompanying ESDUpac A8913 presents a Fortran computer program for the determination of these stiffnesses for laminated plates, without restriction on the type of lay-up. The plates are considered to be built up from thin orthotropic homogeneous layers with perfect bonding between them. The layers may have different material elastic properties, thicknesses and orientations of their principal axes of orthotropy. The analysis is also applicable to thick sandwich panels.

This Data Item consists of three parts. Section 1 describes the theoretical solution to the title problem and its limitations in application to practical panels. Section 2 describes the input and output for the program of ESDUpac A8913 and gives example calculations.

1.2 Notation

Both SI and British units are quoted but any coherent system of units may be used.

1.2.1 Plate overall stiffnesses

The numerical values given for A_{ij} , B_{ij} and D_{ij} correspond with the sign convention of ESDU 94003*.

A	in-plane stiffness matrix	N/m	lbf/in
A_{ij}	element of A	N/m	lbf/in
B	coupled in-plane and flexural stiffness matrix	N	lbf
B_{ij}	element of B	N	lbf
D	flexural stiffness matrix	N m	lbf in
D_{ij}	element of D	N m	lbf in
F	transverse shear flexibility matrix	m/N	in/lbf
F_{ij}	element of F	m/N	in/lbf

* ESDU 94003 "Stiffnesses of laminated flat plates".

$\mathbf{H}$	transverse shear flexibility matrix	N/m	lbf/in
H_{ij}	element of $\mathbf{H}$	N/m	lbf/in
H_{11}	transverse shear stiffness with respect to xz -plane	N/m	lbf/in
H_{22}	transverse shear stiffness with respect to yz -plane	N/m	lbf/in
H_{12}	coupling transverse shear stiffness with respect to xz - and yz -planes	N/m	lbf/in
α	see Equations (1.8)		

1.2.2 Layer, plate properties and reference axes

C	reference number for layer type		
$E_{\alpha k}, E_{\beta k}, E_{zk}$	moduli of elasticity of k th layer in α_k -, β_k - and z -directions respectively	N/m ²	lbf/in ²
$G_{\alpha\beta k}, G_{\alpha z k}, G_{\beta z k}$	shear moduli of k th layer in $(\alpha, \beta)_k$, $(\alpha, z)_k$ and $(\beta, z)_k$ directions respectively	N/m ²	lbf/in ²
t_k	thickness of k th layer	m	in
h	total plate thickness	m	in
ψ_k	angle of principal axes of orthotropy, α_k , of k th layer measured from plate reference axis, $0x$, as shown in Sketch 1.1	degree	degree
$\nu_{\alpha\beta k}$	Poisson's ratio for k th layer resulting from load applied in α_k - direction with induced strain in β_k - direction		
$\nu_{\alpha z k}, \nu_{\beta z k}$	Poisson's ratio for k th layer resulting from load applied in α_k and β_k -directions respectively with induced strains in z -direction		
$0(\alpha, \beta, z)$	principal axes of orthotropy of a layer (see Sketch 1.1)		
$0(x, y, z)$	plate main reference axes (see Sketch 1.1)		

1.2.3 Forces, moments and energy

M_x, M_y, M_{xy}	bending moments per unit length acting in xz - and yz -planes and twisting moment per unit length acting about x - and y -axes respectively	(N m)/m	(lbf in)/in
N_x, N_y, N_{xy}	direct and shear loads per unit length referred to (x,y) coordinates	N/m	lbf/in
Q_x, Q_y	transverse (through-the-thickness) shear forces per unit length acting on yz - and xz -planes respectively (see Sketch 1.1)	N/m	lbf/in
U_s	transverse shear strain energy per unit area of mid-plane	(N m)/m ²	(lbf in)/in ²

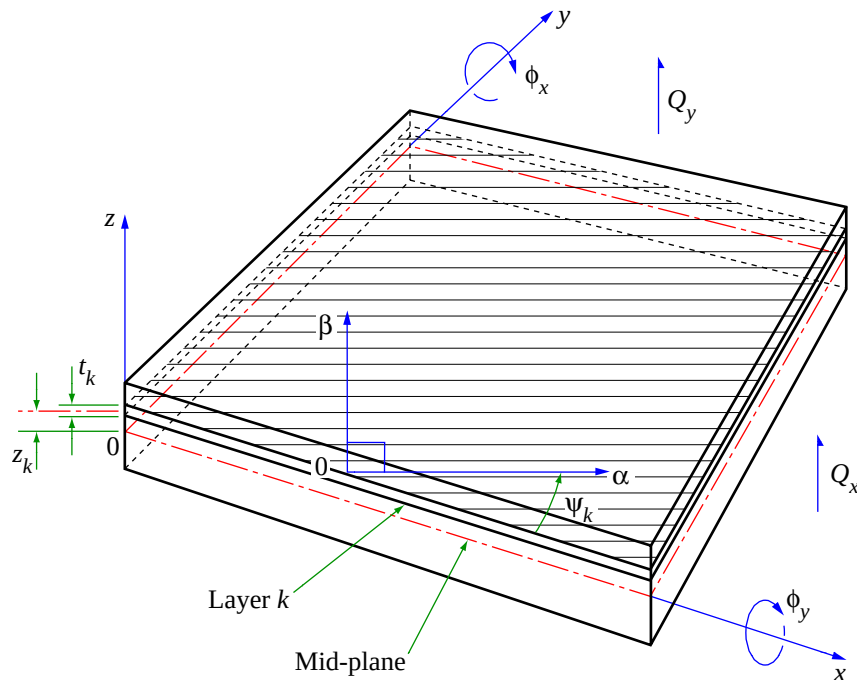
1.2.4 Displacements, strains and rotations

The terms enclosed by brackets do not appear in this Data Item, but are included here for completeness. They are used in the development of the theory in Derivation 1.

$(u_o), v_o$	displacements of mid-plane of plate in x - and y -directions respectively	m	in
$(\Gamma_x), \Gamma_y, (\Gamma_{xy})$	first derivatives of rotations	rad/m	rad/in
γ_{xz}, γ_{yz}	transverse shear strains in xz - and yz -planes respectively		
$(\epsilon_{xo}), \epsilon_{yo}$	direct strain components of plate mid-plane in x - and y -directions respectively		
ϕ_x, ϕ_y	rotations of normals to mid-plane about Oy - and Ox -axes respectively (see Sketch 1.1)	rad	rad

1.2.5 Suffixes

i, j	numbers signifying position in row and column respectively of element in matrix
k	layer identification number
x, y, z	relates to x -, y - and z -directions respectively
α, β	relates to α - and β -directions respectively



Sketch 1.1 Laminated plate geometry, shear loading and rotations

1.3 Statement of Basic Assumptions and Solution Outline

The main underlying assumption in the theory is that points on a normal to the middle plane of the plate before bending remain on a straight line, though not normal to the middle surface after bending. This allows for a rotation due to shear to be added to the usual rotation due to classical plate bending theory. It is known as Mindlin's plate theory (see Reference 2) which was developed for isotropic plates. The assumption implies that the transverse strains γ_{xz} and γ_{yz} are independent of z , with the result that the corresponding shear stresses will be constant in each layer of the laminated plate if they are calculated from the simple shear stress-strain equations. This constant strain-stress approach leads to transverse shear stiffnesses that are usually too high. In the present analysis, which is based on Derivation 1, an improvement is obtained by using the stress equilibrium equations in each layer to evaluate the transverse shear stress distribution (not taken to be constant in each layer). These stresses are then related to the transverse shear forces per unit length, Q_x and Q_y , using the overall plate equations for cylindrical bending. The planes on which Q_x and Q_y act are shown in Sketch 1.1. Having obtained expressions for the transverse shear stresses it is possible to evaluate the shear strain energy density, U_s , using the transverse shear stress-strain equations for each layer. The expression takes the form

$$U_s = \frac{1}{2} [Q_x, Q_y] [\mathbf{F}]_{2 \times 2} \begin{bmatrix} Q_x \\ Q_y \end{bmatrix}, \quad (1.1)$$

where $[\mathbf{F}]_{2 \times 2}$ is a flexibility matrix.

The required shear stiffness matrix is then given by

$$\begin{bmatrix} H_{11} & H_{12} \\ (sym) & H_{22} \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ (sym) & F_{22} \end{bmatrix}^{-1} \quad (1.2)$$

and the relation between the transverse shear forces per unit length and the transverse shear strains becomes

$$\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ (sym) & H_{22} \end{bmatrix} \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}. \quad (1.3)$$

It should be noted that the shear stiffness matrix contains the coupling term H_{12} . This will be non-zero for certain lay-up geometries, for example in a plate comprising two layers of which one has $\psi = 0^\circ$ and the other $\psi = 45^\circ$.

In the above theory, the full three-dimensional set of stress-strain equations is used for each orthotropic layer, with the assumption that the transverse direct stress is zero. Thus, a knowledge of the nine elastic constants $E_{\alpha k}$, $E_{\beta k}$, $E_{z k}$, $G_{\alpha \beta k}$, $G_{\alpha z k}$, $G_{\beta z k}$, $\nu_{\alpha \beta k}$, $\nu_{\alpha z k}$ and $\nu_{\beta z k}$ is required, as well as the lay-up geometry, before an analysis can be carried out.

1.3.1 Equations for Cylindrical Bending*

In the analysis outlined above it is found that, for the general loading case, the transverse shear stresses depend on the derivatives of the membrane forces and bending moments. The particular loading states appropriate to cylindrical bending about the x - and y -axes are used to relate the shear stresses to the transverse shear forces. In Derivation 1 these relations for bending about the y -axis are given by the following matrix equation in which $N_{x,x} = \partial N_x / \partial x$ etc.

$$\begin{bmatrix} N_{x,x} \\ N_{y,x} \\ N_{xy,x} \\ M_{x,x} \\ M_{y,x} \\ M_{xy,x} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Q_x \\ Q_y \end{bmatrix}. \quad (1.4)$$

Whilst it may be shown that Equation (1.4) satisfies the equilibrium equations, it also gives the two additional conditions

$$\frac{\partial N_y}{\partial x} = 0 \text{ and } \frac{\partial M_y}{\partial x} = 0. \quad (1.5)$$

* For ease of cross-reference the sign convention used for bending moments is the same as that used in Derivation 1.

In this present Item for the general laminated plate, these two conditions are not assumed; rather the variations of the mid-plane displacements, v_o , and the rotation of the normal to the mid-plane, ϕ_y , are assumed not to vary with y . This allows the following two equations to be written:

$$\varepsilon_{yo} \left(= \frac{\partial v_o}{\partial y} \right) = 0 \text{ and } \Gamma_y \left(= \frac{\partial \phi_y}{\partial y} \right) = 0. \quad (1.6)$$

Using these two conditions in the constitutive equations for the whole plate gives

$$\frac{\partial N_y}{\partial x} = a_1 Q_x + a_2 Q_y \text{ and } \frac{\partial M_y}{\partial x} = a_3 Q_x + a_4 Q_y, \quad (1.7)$$

where

$$a_1 = \frac{D_{12}^* B_{22}^* - D_{22}^* B_{12}^*}{D_{22}^* A_{22}^* - B_{22}^{*2}}; a_2 = \frac{D_{26}^* B_{22}^* - D_{22}^* B_{26}^*}{D_{22}^* A_{22}^* - B_{22}^{*2}}; \quad (1.8)$$

$$a_3 = \frac{B_{12}^* B_{22}^* - A_{22}^* D_{12}^*}{D_{22}^* A_{22}^* - B_{22}^{*2}}; a_4 = \frac{B_{26}^* B_{22}^* - A_{22}^* D_{26}^*}{D_{22}^* A_{22}^* - B_{22}^{*2}}.$$

The A^* , B^* and D^* terms in Equation (1.8) are obtained by inverting the standard 6×6 matrix that relates the in-plane forces and bending moments to the overall plate strains and curvatures. A similar analysis has been carried out for cylindrical bending about the x -axis.

Thus, the present analysis differs from that given in Derivation 1 and is considered to be more representative of actual plate behaviour.

1.4 Some Numerical Results and Trends

Figure 1 gives values for H_{11} and H_{22} for a two layer plate where the top layer is at $+\psi$ and the bottom layer $-\psi$. Values for the mechanical constants and layer thicknesses are given in the figure. The stiffness matrix is uncoupled with $H_{12} = 0$. As expected $H_{11} = H_{22}$ at $\psi_k = 45^\circ$. Also, when $\psi_k = 0^\circ$ it is seen that $H_{11} = 1917 \text{ N/mm}$ and $H_{22} = 958 \text{ N/mm}$. These values can be compared with the stiffnesses obtained from the simple formulae $hG_{\alpha z}$ and $hG_{\beta z}$ (which assume constant shear stress over the plate thickness) giving values of 2300 N/mm and 1150 N/mm respectively. If the commonly used shear reduction factor, of $5/6$, for isotropic plates is used, then these numbers become equal to those obtained from Figure 1. The location of maximum and minimum values for H_{11} and H_{12} could be of significance when designing such plates for minimum deflections, where the deflection is dominated by shear.

Figure 2 shows the variation of H_{11} and H_{22} with total number of layers for a $[0/90]_s$ plate of constant thickness. Thus, the layer thicknesses are reduced as the number is increased. The stiffnesses gradually converge to about $0.095 \times 10^6 \text{ lbf/in}$. This is because the plate becomes more homogeneous as the number of layers is increased. The results are in close agreement with those given in Reference 3 for the same problem. It might be expected that a converged value equal to $h \times (\text{mean value for } G) \times 5/6$ would be achieved. However, this value works out to be $0.117 \times 10^6 \text{ lbf/in}$, which is 23 per cent higher than the value obtained by this improved calculation.

Figure 3 shows the results for a two layer plate, where the top layer fibre angle is held at 0° and the bottom layer fibre angle is varied from 0° to 180° . The layer properties are the same as those given in Figure 1. Both H_{11} and H_{22} are symmetrical about $\psi_2 = 90^\circ$ and are equal at this point, as expected. Also, when $\psi_2 = 0^\circ$ and 180° they are equal to the values given in Figure 1 at $\psi_k = 0^\circ$. However, the main feature