

MAXIMUM INELASTIC BENDING MOMENTS AND END LOADS (IN FIXED RATIO) FOR BEAM SECTIONS OF GENERAL POLYGONAL SHAPE

1. APPLICATION AND DERIVATION OF PROGRAM

1.1 Introduction

This Data Item, together with ESDUpac A9004 (see Section 2), presents a Fortran computer program for calculating the combination of bending moments and end load in a beam that are associated with a defined maximum allowable extreme fibre stress or strain. The program accepts values for the bending moments and end load in the required proportions to one another and provides values in the same ratio that meet the criterion of the allowable stress or strain. The beam is of general solid or hollow polygonal section and the allowable stress or strain may be above the limit of proportionality of the beam material. It is assumed that the behaviour of the beam material above the limit of proportionality can be represented by the generalised form of stress-strain curve given in Data Item No. 76016*. In addition to the bending moments and end load, the elastic/inelastic stresses and strains are given at each of the vertices of the cross section together with the location of the neutral axis, the area, the position of the centroid and the second moments of area of the cross section.

This Data Item is arranged in three sections. Section 1 gives information on the application of the program, examples of its application and a summary of the theory from which the program is derived. Section 2 gives the instructions necessary for operating the program, guidance in interpreting the output and examples of input to and output from the program. Section 3 provides notes on the Fortran program with a description of the program structure and details of the tests that have been performed on the program.

The analysis assumes that the beam material is homogeneous. The strain is considered to vary linearly across the section with plane sections remaining plane, but the beam may be stressed beyond the limit of proportionality to an extent that may be defined by a maximum strain or stress. The stress-strain curve of the material is represented by the generalised form given in Data Item No. 76016 and is assumed to be the same in tension and compression. The curve is defined by the modulus and a choice of two proof stresses or the material coefficients m and f_n or two values of total strain and their corresponding stresses. Details of the theoretical basis of the program are given in Section 1.3.

The beam cross section may be subject to orthogonal moments about two arbitrarily chosen axes, x and y , and an end load acting at any point. The analysis takes account of the movement of the neutral axis from the elastic position resulting from the influence of the inelastic stress-strain properties for the particular loading condition. An option allows the beam to be constrained so that deflection in the x -direction is prevented (the neutral axis as a result remaining parallel to the x -axis). The elastic stresses and strains output are based on the assumption that the strain is directly proportional to the stress through the modulus of elasticity. Examples on the application of the program are given in Section 1.4.

No account has been taken of any type of instability, the possibility of which must also be considered. It is assumed that all the loads are applied simultaneously maintaining their ratios.

* Data Item No. 76016 "Generalisation of smooth continuous stress-strain curves for metallic materials".

1.2 NOTATION

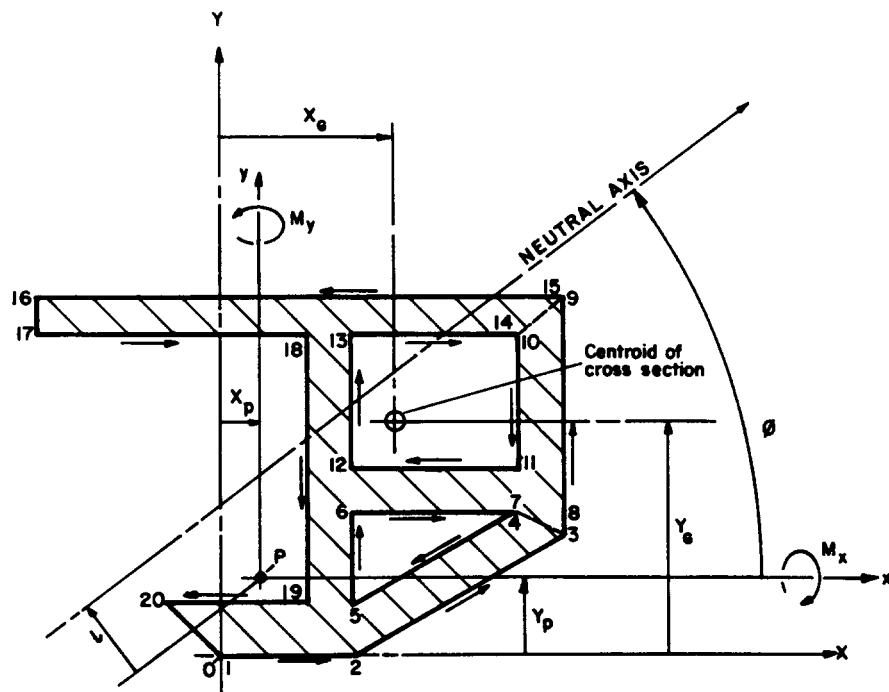
Both S.I. and British Units are quoted but any coherent system of units may be used.

A	area of cross section of beam	m^2	in^2
E	modulus of elasticity of material	N/m^2	lbf/in^2
E_t	tangent modulus of material	N/m^2	lbf/in^2
f	stress	N/m^2	lbf/in^2
f_{max}	maximum allowable stress	N/m^2	lbf/in^2
f_n	stress at which $E_t = E/2$	N/m^2	lbf/in^2
f_t	specification ultimate tensile stress of beam material*	N/m^2	lbf/in^2
f_1, f_2	stresses provided at any two points on inelastic region of stress-strain curve	N/m^2	lbf/in^2
I_x	second moment of area of beam cross section about centroid parallel to x-axis	m^4	in^4
I_y	second moment of area of beam cross section about centroid parallel to y-axis	m^4	in^4
I_{xy}	product of inertia of beam cross section about centroid	m^4	in^4
I_1, I_2, I_3	integrals used in beam analysis (see Section 1.3)	N	lbf
m	material characteristic (see Data Item No. 76016)		
l	perpendicular distance from point of application of end load to neutral axis (see Sketch 1.1)	m	in
M_x, M_y	applied moments about x- and y-axes, respectively	N m	lbf in
M_{xmax}	moment about x-axis with specified stress or strain at extreme fibre	N m	lbf in
M_{ymax}	moment about y-axis with specified stress or strain at extreme fibre	N m	lbf in
N	total number of vertex coordinate sets given for polygonal beam cross section		
P	applied end load acting at any point on cross section	N	lbf
P_{max}	end load on section with specified stress or strain at extreme fibre	N	lbf
R	radius of curvature of deflected beam	m	in

For footnote see end of Notation.

t_A, t_B	proof stresses, $t_A < t_B$		N/m ²	lbf/in ²
t_1	0.1 per cent tensile proof stress	} related to stress-strain curve representing beam material	N/m ²	lbf/in ²
t_2	0.2 per cent tensile proof stress		N/m ²	lbf/in ²
t_{2spec}	specification value of 0.2 per cent proof stress of beam material*		N/m ²	lbf/in ²
U_1, U_2, U_3	non-dimensional forms used in program, related to l, R and ϕ			
$O(X, Y)$	coordinate system for geometry	} parallel, see Sketch 1.1		
$P(x, y)$	coordinate system for loading			
α, β	coefficients in strain equation (see Section 1.3)		m ⁻¹	in ⁻¹
γ	coefficient in strain equation (see Section 1.3)			
ϵ	strain			
ϵ_{max}	maximum total strain in beam material			
ϵ_v	strain assigned to a vertex during iteration (function of ϵ_{max})			
$\epsilon_{pA}, \epsilon_{pB}$	permanent (proof) strains corresponding to t_A and t_B , respectively			
$\epsilon_{p1}, \epsilon_{p2}$	permanent (proof) strains corresponding to t_1 and t_2 , respectively			
ϵ_1, ϵ_2	total strains appropriate to stresses f_1 and f_2 (not necessarily t_1 and t_2)			
ϕ	angle between neutral axis and x-axis		degree	degree
X_G, Y_G	coordinates of position of centroid		m	in
X_P, Y_P	coordinates of point of application of end load		m	in

* Alternatively, a statistically derived value of design allowable stress such as a value from Reference 4 or 5 may be used.



Sketch 1.1 Coordinate system, loading sign convention and typical identification of vertices.
All quantities are shown positive.

Moments M_x and M_y are positive as shown. Both give tension stress in the positive xy quadrant. End load P is positive in tension, negative in compression.

The cross section is defined by the coordinates of continuously connected vertices numbered in sequence in the direction that keeps the material of the beam to the left. Holes are identified by departing from and rejoining the external geometry through the same adjacent vertices (see vertices 3 and 4, 7 and 8). Arrows show the direction of the numbering sequence.

The arrow on the neutral axis indicates the direction of viewing for the sign of l . If l is positive, point P (coordinates X_p , Y_p) lies on the right-hand side of the neutral axis when looking along the neutral axis in the direction of the arrow shown. The angle ϕ lies in the range -90 to $+90$ degrees. The value of R is defined as positive when the stress changes from positive to negative on passing from the left-hand to the right-hand side of the neutral axis.

1.3 Derivation

1.3.1 Sources

The non-linear inelastic analysis on which the program of ESDUpac A9004 is based was developed with the assistance, in particular, of the communications of Derivation 1.

1.3.2 Theory

The analysis is based on first order beam theory with non-linear material behaviour. Bending moments and end load are determined that give the maximum allowable extreme fibre stress or strain specified by the user. It is assumed that the strain distribution under the action of the bending moment is linear across the section and plane sections remain plane, but that the stress distribution may be non-linear.

The cross section is defined by the coordinates of continuously connected vertices numbered in sequence in the direction that keeps the material of the beam to the left. Holes should be numbered from a convenient vertex in the external geometry to which the sequence must return (see Sketch 1.1).

The end load P and moments M_x, M_y about axes P_x, P_y , are

$$P = \int_A f dA, \quad (1.1)$$

$$M_x = \int_A fy dA \quad (1.2)$$

and
$$M_y = \int_A fx dA, \quad (1.3)$$

where f is the stress at (x, y) and integration is over the cross-sectional area, A .

The strain is expressed as

$$\varepsilon = \alpha x + \beta y + \gamma. \quad (1.4)$$

The stresses are obtained from the strains using the inelastic stress-strain relationship

$$\varepsilon = \frac{f}{E} \left[1 + \frac{1}{m} \left(\left[\frac{f}{f_n} \right]^2 \right)^{\frac{m-1}{2}} \right] \quad (1.5)$$

which is derived from Equation (3.1) of Data Item No. 76016.

The analytical method used replaces Equations (1.1) to (1.3) by the relations

$$P = I_1, \quad (1.6)$$

$$\alpha M_y + \beta M_x + \gamma P = I_2 \quad (1.7)$$

and
$$\beta M_y - \alpha M_x = I_3, \quad (1.8)$$

where I_1, I_2 and I_3 are integrals that can be evaluated in closed form using Equation (1.5) and the cross-section geometry (that is, the vertex coordinates). For a side along which there is relatively small change in stress, the closed form evaluation fails to provide sufficient numerical accuracy. In such cases this is automatically replaced by an alternative formulation that enables computing accuracies to be maintained but requires numerical integration by Simpson's rule.

The position of the neutral axis is found by an iterative process. Starting values for the iteration are obtained from the elastic solution. The analysis has been used to provide solutions to the equations for two alternative conditions. In the first the beam is assumed to be free to deflect in both the x - and y -directions, that is the neutral axis is completely free to move under the influence of the loading and the stress-strain relationship. In this case P, M_x and M_y are always in the same ratio to each other as the given values. In the second condition the beam is assumed to be constrained so that deflection in the x -direction is prevented, that is, the neutral axis is parallel to the x -axis. In this case P and M_x are always in the same

ratio to each other as the given values. The value of M_{ymax} (corresponding to P_{max} and M_{xmax}) required to constrain the beam in this condition is given in the output.

Within the program, calculations are performed relative to the centroid rather than the point of application of load P .

1.3.3 Derivation and references

Derivation

This section lists selected sources that have assisted in the preparation of this Data Item.

1. MITCHELMORE, P.J. Private communications, British Aerospace plc, Civil Aircraft Division, Hatfield, October 1988.

References

The references given are recommended sources of information supplementary to that in this Item.

2. COZZONE, F.P. Bending strength in the plastic range. *Jl aero. Sci.*, Vol. 10, No. 5, pp. 137-151, May 1943.
3. BARRETT, A.J. The bending of some common beam sections into the plastic range. *Jl R. aeronaut. Soc.*, Vol. 57, pp. 110-115, February 1953.
4. ESDU Metallic Materials Data Handbook. Defence Standard 00-932, ESDU International plc.
5. USA, DoD Military Standardization Handbook. Metallic materials and elements for aerospace vehicle structures. MIL-HDBK-5. Dep. of Defense, Washington, D.C. USA.

1.4 Application of the Program Of ESDUpac A9004

The program can be used to obtain the maximum moments and end load that may be applied to a beam of any polygonal cross section. The combination of moments, M_{xmax} , and M_{ymax} , with the end load, P_{max} , is that appropriate to the maximum stress or strain, as defined by the user, at the extreme fibre of the cross section. The program also gives the location of the neutral axis, the radius of curvature of the deflected beam and the stress and strain at each vertex under the maximum loads. The program provides the corresponding data for the elastic solution appropriate to t_{2spec} at the extreme fibre. Additionally, values are given for the area of the cross section, its second moments of area and the position of the centroid.

The maximum stress or strain at the extreme fibre may be defined directly by the user who may alternatively choose to use the default value equal to the 0.5 per cent proof strain at the extreme fibre. Values higher than this should be used with caution since there may be problems when designing against proof stress requirements; also high shear stresses can occur.

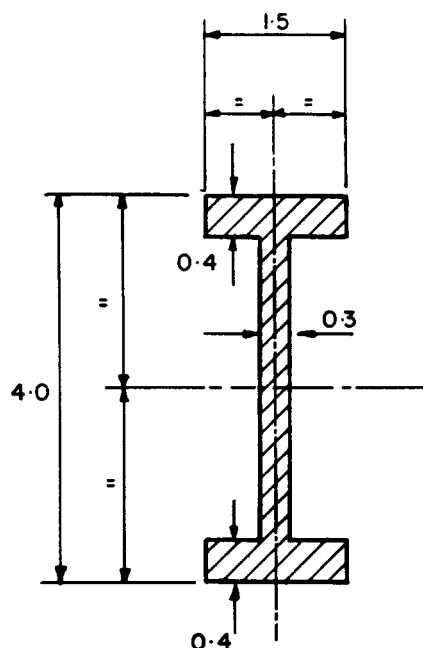
The program is applicable to any beam cross section that can be defined by line-linked points and for which the material modulus of elasticity and any two points on or characteristics of the material stress-strain curve are known. The program provides the choice between two analyses. In Option 1 the beam is completely free to move in the x- and y-directions and the combination of P_{max} , M_{xmax} and M_{ymax} is output

in the same ratios as the input of P , M_x and M_y . In Option 2 the beam is constrained so that deflection in the x -direction is prevented (the neutral axis as a result remaining parallel to the x -axis) and the combination of P_{max} and M_{xmax} are output in the same ratios as input values of P and M_x together with the constraining moment M_{ymax} .

1.4.1 Application to an I-section beam (moment only)

The program has been used to analyse an I-section beam subjected to moment only for comparison with Data Item Nos. Struct. 01.06.02* and 88035†. Data Item No. Struct. 01.06.02 assumes that the material stress-strain curve can be idealised as an elastic-perfectly plastic curve with a maximum stress equal to t_{2spec} . Data Item No. 88035 and this program, A9004A, have been used with a shallow non-linear stress-strain curve based on $m = 100$, $f_n = 38\,450 \text{ lbf/in}^2$ and $E = 10 \times 10^6 \text{ lbf/in}^2$. The curve yields a value of $t_2 = 40\,000 \text{ lbf/in}^2$ appropriate to the value of $t_{2spec}/E = 0.004$ on which Data Item No. Struct. 01.06.02 is based. It is assumed that $f_t = 41\,000 \text{ lbf/in}^2$. The default value of 0.5 per cent proof strain is used.

The I-section beam used in the comparison is shown in Sketch 1.2.



Sketch 1.2 All dimensions in inches

The parameters required for use with Data Item No. Struct. 01.06.02, derived from the above information, are:

$$\frac{t}{h} = 0.10, \frac{b}{d} = 0.20 \text{ and } \frac{Ph}{M_x} = 0,$$

where, in the notation of Data Item No. Struct. 01.06.02, b is the web thickness, d is the flange width, h is the section depth and t is the flange thickness.

The values of the maximum inelastic moments, M_{xmax} , obtained from the three sources are as follows.

* Data Item No. Struct. 01.06.02 "Form factors for flanged sections under combined bending and axial load".

† Data Item No. 88035 "Maximum inelastic bending moments and form factors for symmetric bending of beams".

Data Item No. Struct. 01.06.02	Data Item No. 88035	Program of ESDUpac A9004
114 300 lbf in	114 700 lbf in	114 500 lbf in

1.4.2 Application to a hollow circular beam (moment and end load)

While the program is strictly only applicable to polygonal cross sections its capacity to accommodate cross sections with up to 50 vertices enables it to be used to analyse a polygonal approximation to the circular section of a hollow beam as shown in Sketch 1.3. The 50 vertices occur at 15° intervals around the inner and outer circumference of the tube.

A comparison has been made with Example 1 of Data Item No. Struct. 01.06.01* which analyses a hollow circular section beam under a moment and an end load at the centroid. In Data Item No. Struct. 01.06.01 it is assumed that the material stress-strain curve is elastic-perfectly plastic and that $t_{2spec} = 50\,000 \text{ lbf/in}^2$. To adopt a close approximation to the bi-linear stress-strain curve of the Item it is assumed for program input that $m = 100$ and $f_n = 48\,060 \text{ lbf/in}^2$. In order to maintain the relationship that $t_2/E = 0.004$, for which the curves of Data Item No. Struct. 01.06.01 are calculated, the corresponding value of the modulus of elasticity of the material is taken here to be $E = 12.5 \times 10^6 \text{ lbf/in}^2$. The ultimate tensile stress is taken to be $f_t = 51\,000 \text{ lbf/in}^2$ and the default value of 0.5 per cent proof strain is used.

The values of the parameters required for use with Data Item No. Struct. 01.06.01 are specified in that Item as follows.

$$\frac{t}{D} = 0.2, \frac{M}{PD} = 2.5 \text{ and } Z = 0.684 \text{ in}^3,$$

where, in the notation of Data Item No. Struct. 01.06.01, D is the external diameter, t is the wall thickness, $Z = 2 I/D$ and the ratio M/PD is specified as shown above.

The assumption is made for the program analysis that the applied loading will be in the same ratio as that in Example 1 of Data Item No. Struct. 01.06.01 and that:

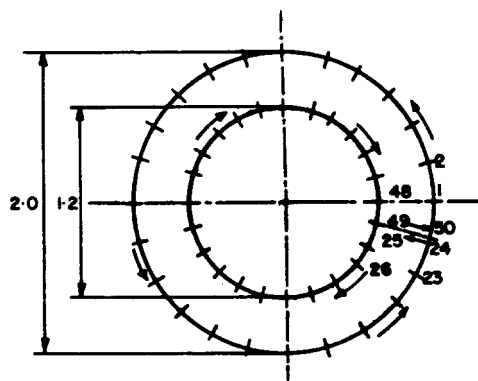
$$P = 4800 \text{ lbf}, M_x = 24\,000 \text{ lbf in and } M_y = 0 \text{ lbf in}.$$

The values of the maximum inelastic loads and moments and the second moments of area obtained from both sources are as follows.

	P_{max}	M_{xmax}	M_{ymax}	I_x
Data Item No. Struct. 01.06.01	9600 lbf	48 000 lbf in	0 lbf in	0.684 in ⁴
Program of ESDUpac A9004	9451 lbf	47 260 lbf in	0 lbf in	0.6681 in ⁴

In addition the program gives the stresses and strains at the 50 vertices around the inner and outer circumferences of the tube.

* Data Item No. Struct. 01.06.01 "Form factors for circular sections under combined bending and axial load".



Sketch 1.3 All dimensions in inches

2. ESDUpac A9004

2.1 Introduction

ESDUpac A9004 contains a Fortran computer program that accepts values that define the beam geometry, materials properties and loading conditions and outputs the combination of end load and bending moments associated with the allowable extreme fibre stress or strain. This section of the Data Item gives the instructions necessary to operate the program. Section 2.2 details the form of input required for the program. Section 2.3 describes the form of output to be expected and the way in which it should be interpreted. Examples of both input and output are given in Section 2.4.

The program is available in two formats: (a) a Fortran source code with example input and output files, and (b) an executable program for PC's. Both versions can be found on the Compact Disc and on disk in the Sub-series Software Volume.

2.2 Data Input Information

2.2.1 Data input form

To perform an analysis the program requires data to be input defining the form and extent of the material stress-strain curve and the geometry of the cross section and the form and values of loading applied. Table 2.1 details the input data and the form in which they are entered.

The first three lines of the input data files are reserved for the title of the user's analysis. Any lines that are not used here should be left as blank lines in the input data file.

To define the material stress-strain curve it is necessary to specify the modulus of elasticity and a limiting (maximum) stress or strain and either two points on the curve or two characteristic quantities. A choice of three forms of entry is given for the definition of the curve as follows.

- (1) Any two proof stresses, which are entered by quoting the permanent strain (proof strain) followed by the related proof stress in each case. Note that the permanent strains are not total strains but the offset values (see Data Item No. 76016).
- (2) The generalised stress-strain characteristic values m and f_n as defined in Data Item No. 76016.

- (3) Any two values of total strain with their related stresses.

The modulus of elasticity is required next, to define the elastic region of the curve. This is followed by values for t_{2spec} and f_t .

The extent of the inelastic stress-strain curve is then defined by choosing one of the following entries:

- (1) The maximum strain.
- (2) The maximum stress.
- (3) A default value of 0.5 per cent proof strain.

When the last is selected 0 is entered and the program will work to the true strain equivalent of the 0.5 per cent proof strain.

When the stress-strain curve has been defined, the data for the beam cross section geometry are entered. The vertices of the polygonal cross section are identified and numbered in sequence with the material to the left (see Sketch 1.1). Where the section has internal holes, the vertices of the holes (no matter how many) are numbered in a sequence following on from any vertex in the external geometry, to which the numbering must return (see Sketch 1.1). Note this means that the related points in the external and hole geometries where the change occurs will both appear twice in the numbering sequence. Further, the effect of this left hand rule is to move round the external geometry in an anticlockwise direction and to move round holes in a clockwise direction. The vertices are specified with respect to any conveniently chosen coordinates X and Y as shown in Sketch 1.1. The geometry is specified by entering first the total (N) of vertex numbers and then their X , Y coordinates in order. Examples given in Section 2.4.1 illustrate this system.

The form and magnitude of the loading must be entered next. This takes the form of specifying one of the two options available, Option 1, for which the beam is assumed to be free to deflect in both x - and y -directions (the neutral axis being completely free to move) and Option 2, for which the beam is assumed to be constrained so that deflection in the x -direction is prevented (the neutral axis as a result being parallel to the x -axis). Then values for the quantities P , M_x and M_y (Option 1) or P and M_x (Option 2) are entered to establish loading ratios.

Values of the loading P , M_x and M_y entered should use the sign convention given below Sketch 1.1. Specification of units is not required for any input values. It is assumed that all values entered will be based on a coherent system of units. The output values will be compatible with the system chosen. Examples using British Units and the S.I. system are given in Section 2.4.2.

2.2.2 Program limitations

The values of the generalised material characteristic m have been limited to the range $2 \leq m \leq 100$. This should embrace all practical values.

The highest vertex number (N) is limited to 50.

The maximum allowable stress may not exceed the ultimate tensile stress so that $f_{max} \leq f_t$. If $f_{max} > f_t$, either from input stress values or implied by ϵ_{max} , the analysis will be terminated.

The maximum total strain to which the program will work is 15 per cent, that is $\epsilon_{max} \leq 0.15$.

When selecting the vertex with the maximum stress the program works to a maximum accuracy of 0.1 per cent difference in the strains at the compared vertices.