

PARAMETER ESTIMATION OF LINEAR SYSTEMS FROM FREQUENCY RESPONSE MEASUREMENTS

1. INTRODUCTION

This is the fourth Data Item in a series on the parameter estimation of multi degree-of-freedom linear systems. Earlier Items (References 7 to 9) dealt with systems modelled by the general state space representation and analysed using computer-based optimisation techniques. In these Items two classes of solution were identified, those that led to a direct evaluation of the unknowns and those that led to a nonlinear set of equations in the unknowns and consequently could realistically only be solved using an iterative procedure. These iterative methods require initial estimates of all the unknowns which were assumed available prior to the implementation of the procedure.

The present Data Item, in contrast to the earlier Items, assumes that the system being studied is modelled by a set of coupled second-order differential equations with constant coefficients. It is also assumed that a set of frequency response measurements are available from forced harmonic excitations of the system at a range of frequencies that adequately encompasses the significant resonant modes of the system response. Like the latter methods of the earlier Items the parameter estimation procedure results in a set of nonlinear equations that are solved iteratively from a starting point defined by initial estimates of all the parameters.

Two classes of system are identified, those whose modes of response are all lightly-damped, $\zeta < 0.2$, and those systems containing one or more resonant modes with damping ratios higher than this. For the former class, the Item shows how estimates for the natural undamped frequencies and damping ratios of all the system modes can be found directly from a Nyquist plot of the system measurements, even for those difficult cases in which resonant frequencies are very close together. It is also shown how a sufficient number of initial estimates of the unknown parameters can be evaluated directly from these system characteristics in order to implement the parameter estimation procedure. For the latter class of system, $\zeta > 0.2$, the Nyquist approach is seen not to be suitable and reference is made to other methods for system characteristic evaluation.

The method used is based on Derivations 1 and 2 and, although it can conceivably be implemented as a fully automatic computer-based procedure, it is described in a way that allows the analysis engineer to make various decisions at the early stages in the analysis based on the actual measurement data. The method is most familiar for its use in the experimental study of structural vibrations, such as the response of aircraft wings, turbine blades or even suspension bridges, but is widely applicable to all branches of engineering.

The model and response characteristics of a single degree-of-freedom system are analysed. The methods used are shown to apply to a two degree-of-freedom system and consequently to any higher-order system. It is shown how initial estimates of the parameters of a system are obtained from a set of measurements and how these are used in an iterative procedure to improve the estimates.

An example featuring a two degree-of-freedom lightly-damped system illustrates the evaluation of the resonant mode characteristics of the system and the implementation of the parameter estimation procedure.

2. NOTATION

Any coherent system of units may be used.

A, B	coefficients in denominator of receptance functions
$\text{Adj}\{\dots\}$	adjoint of $\{\dots\}$
$\underline{B}$	coefficient matrix of $\dot{\underline{z}}(t)$
b_{ij}	element of $\underline{B}$
c	arbitrary constant (see Appendix A)
$\text{Det}\{\dots\}$	determinant of $\{\dots\}$
E, F	real and complex parts, respectively, of numerator factors of receptance function
G, H	as E and F but for second receptance function
$\underline{I}$	unit matrix
i	complex operator
J	Objective Function
$\underline{K}$	coefficient matrix of $\underline{z}(t)$
k_{ij}	element of $\underline{K}$
$\underline{M}$	coefficient matrix of $\ddot{\underline{z}}(t)$
m_{ij}	element of $\underline{M}$
N	number of measurements
n	number of degrees-of-freedom
R	receptance
$\text{Re}\{\dots\}$	real part of $\{\dots\}$
r	radius
t	time
$\underline{\mathbf{U}}(\omega)$	column matrix of input excitations as functions of frequency
$U(\omega)$	element of $\underline{\mathbf{U}}(\omega)$
$\underline{\mathbf{u}}(t)$	column matrix of input excitations as functions of time
$u(t)$	element of $\underline{\mathbf{u}}(t)$
$\underline{\mathbf{Z}}(\omega)$	column matrix of displacements as functions of frequency

$Z_1(\omega), Z_2(\omega)$	elements of $\underline{Z}(\omega)$
$\underline{z}(t)$	column matrix of displacements as functions of time
$z_1(t), z_2(t)$	elements of $\underline{z}(t)$
γ	unknown parameter
$\underline{\gamma}$	row matrix containing set of unknown parameters
Δ	denotes increment
δ	phase angle of measurement origin to true force reference direction
ε	error in total response
ζ	damping coefficient
η	structural damping loss factor
ϕ	phase angle
ω	frequency
ω_n, ω_r	natural undamped frequency and resonant frequency, respectively

Superscripts

*	denotes complex conjugate
T	denotes transform

Subscripts

c	corrected
e	denotes estimate
I	denotes initial estimate
i, j, k, s	arbitrary integers
m	denotes measurement
r	denotes resonance
x, y	denotes real and imaginary parts, respectively
0	denotes displacement origin

A bar below a quantity denotes a matrix quantity.

One or two dots above a quantity denotes the first or second derivative, respectively, of that quantity with respect to time.

3. BACKGROUND DESCRIPTION

Continuous dynamical systems possess, in principle, an infinite number of degrees-of-freedom, but in practice damping reduces this infinite set to a finite but possibly large number. Mathematical approximations of such systems necessarily restrict attention to those response modes that are seen to cover adequately the resulting response, determined most often from direct examination of the measurements obtained from a forced excitation of the system. There also exist certain other systems that involve a given number of degrees-of-freedom in which the particular elements of the system are known to contribute in a particular way to the overall behaviour of the system.

The response of any multi degree-of-freedom coupled system, which behaves linearly over a certain range of motions of interest, is known to be equivalent to the sum of the responses of a set of uncoupled single degree-of-freedom systems. Consequently, provided the contributions to the total response from each single degree-of-freedom can be separately identified, those methods of analysis used for the estimation of the characteristics of a single degree-of-freedom system can be applied to each of the response modes of the original system.

This Item shows that for systems whose response modes are all lightly damped, $\zeta < 0.2$, estimates for all the system characteristics, namely the natural undamped frequency and damping ratio of each response mode, can be found directly from a Nyquist plot of the response measurements. This is accomplished with the use of the "circle-fit" method in which circles are fitted to those measurement points in the vicinity of each resonance. The method is shown to be suitable even for systems that contain resonant frequencies that are close together.

This essentially graphical approach, however, suffers from three main disadvantages, particularly when the measurements are acknowledged to contain errors of measurement, namely:

- (i) identification of a resonant frequency (as the frequency at which there is a maximum rate of change of the position of the vector tip locus with frequency) cannot always be made with accuracy,
- (ii) construction of a circle that is a best fit for only those plotted points close to a resonant frequency results in many of the test data being ignored, and
- (iii) the geometrical constructions necessary to evaluate damping are inaccurate and rely on the previously estimated resonant frequency.

These inherent inaccuracies are, in part, overcome by treating the mode characteristic evaluations as starting values in an iterative parameter estimation method. In this method an Objective Function, formed from the sum of the squares of the differences between measurement and model prediction, is minimised with respect to each of the unknown parameters in turn. The set of equations that results from this process is found to be nonlinear in the set of unknowns and is solved iteratively using the Newton-Gauss procedure. On the assumption that initial estimates can be obtained for the natural frequency and damping ratio of each resonant mode of the system the iterative parameter estimation procedure can be applied to any system whatever the level of damping present.

The parameter estimation method presented in this Item is most familiar for its use in the experimental study of structural vibrations and is usually associated with the theory of modal testing where test data are used to postulate a mathematical model description including the number of degrees-of-freedom. In these circumstances the response model determined from the parameter estimates is sufficient for prediction purposes. However, care must be exercised in the use of these estimates to predict parameters of the equivalent coupled system because this approach does not allow good parameter prediction unless the plant equations of the coupled system have been derived from a sound theoretical basis independent of the measurement data.

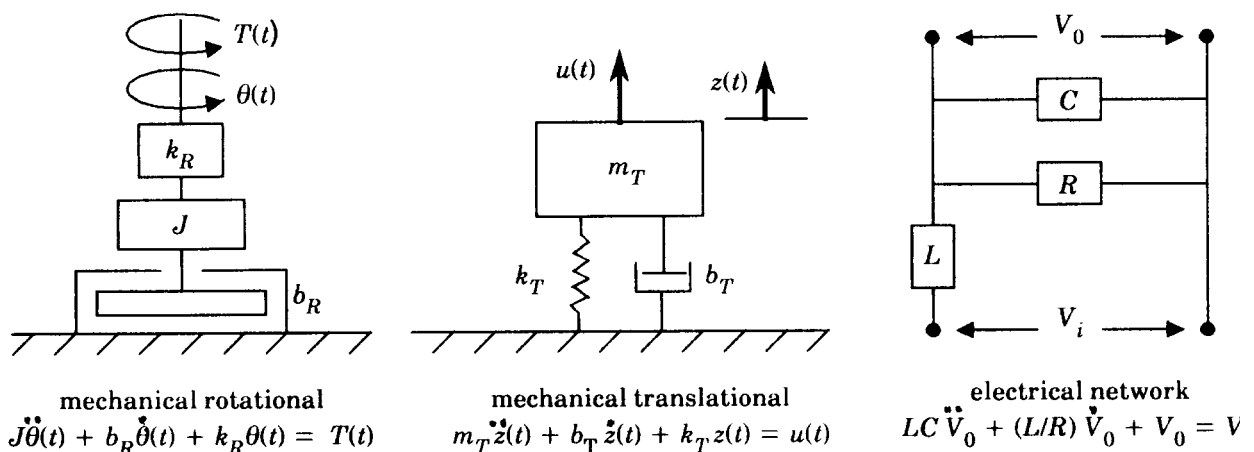
4. SINGLE DEGREE-OF-FREEDOM SYSTEMS

4.1 System Model

A single degree-of-freedom, second-order linear system may be generally represented by the following differential equation

$$m \ddot{z}(t) + b \dot{z}(t) + k z(t) = u(t), \quad (4.1)$$

where $z(t)$ is the state variable, $u(t)$ is the input variable and m , b and k are constant coefficients. The unknown parameters of the system to be found usually consist of one or all of the coefficients m , b and k . The symbols chosen for the coefficients in Equation (4.1) are usually used to represent a translational mechanical system in which m , b and k stand for mass, damping constant and spring stiffness, respectively, and $z(t)$ relates to the displacement of the system mass. However any one degree-of-freedom system may be represented by Equation (4.1) and three examples are illustrated in Sketch 4.1.



Sketch 4.1 Schematic examples of one degree-of-freedom systems

When a system is excited by a harmonic input of the form $u(t) = U(\omega) e^{i\omega t}$, where $U(\omega)$ is chosen as real (i.e. the phase angles of the input and the chosen datum coincide), a steady-state solution results after the transient has decayed. That steady-state solution takes the form $z(t) = Z(\omega) e^{i\omega t}$, where the amplitude of the response, $Z(\omega)$, is given by

$$Z(\omega) = \frac{U(\omega) / m}{(i\omega)^2 + i(2\zeta\omega\omega_n) + \omega_n^2}, \quad (4.2)$$

where $\omega_n = \text{natural undamped frequency} \equiv \sqrt{\{k / m\}}$ (4.3a)

and $\zeta = \text{damping ratio} \equiv b / (2\sqrt{\{km\}})$. (4.3b)

The amplitude of response, $Z(\omega)$, in Equation (4.2) is a complex quantity which contains information on both the magnitude and phase of the output motion.

Equation (4.2) may be expressed as a frequency response function, $R(\omega)$, given by

$$R(\omega) = \frac{Z(\omega)}{U(\omega)} = \frac{1/m}{(i\omega)^2 + i(2\zeta\omega\omega_n) + \omega_n^2}, \quad (4.4)$$

which is independent of the amplitude of the input. In modal analysis when $Z(\omega)$ represents displacement, the term $R(\omega)$ is known as the system receptance. Similarly the terms mobility and inertance (or accelerance) are used when velocity and acceleration, respectively, are used in place of $Z(\omega)$.

The modulus and phase of a complex quantity, of say $c(\omega) = a + ib$, are used throughout this Item and are defined as

$$\text{modulus of } c(\omega) = \sqrt{\{a^2 + b^2\}}$$

$$\text{and the phase of } c(\omega) = \tan^{-1}\{b/a\}.$$

The original set of unknown parameters of the system, m , b and k , are seen in Equation (4.4) to have been replaced by a new parameter set, $1/m$, ω_n and ζ , since, as will be seen, estimates for the latter two parameters can be obtained directly from the system response measurements, at least for lightly damped oscillatory systems.

The viscous damping assumption used in the above model representation of a system is invariably chosen for mathematical convenience. However, hysteretic damping is acknowledged to be more representative of the damping mechanism observed in structures where the damping rate varies inversely with frequency. The equivalent frequency response function is

$$R(\omega) = \frac{Z(\omega)}{U(\omega)} = \frac{1/m}{(i\omega)^2 + i(\eta\omega_n) + \omega_n^2}, \quad (4.5)$$

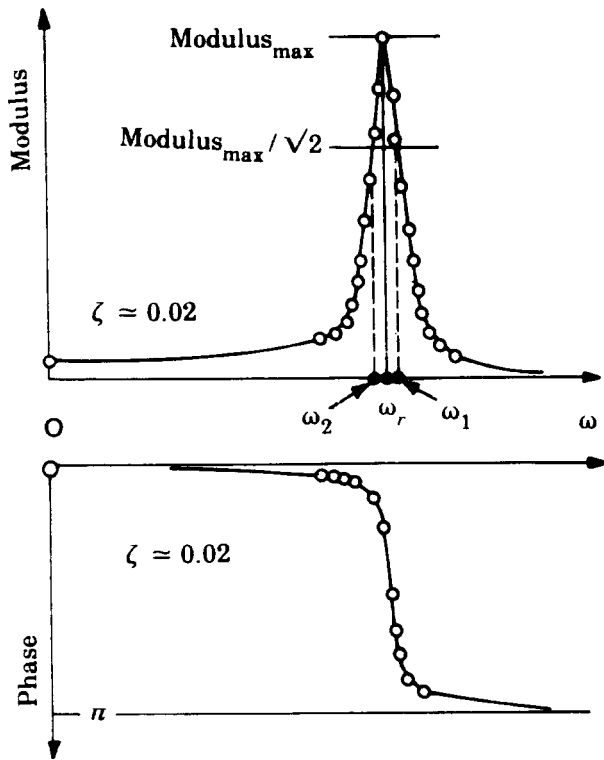
where η is referred to as the structural damping loss factor. If the damping is small, as is often the case for structures, then the difference between the shapes of the response curves in the region of resonance due to the assumption of either type of damping mechanism is negligible and so in this region $\eta \equiv 2\zeta$.

4.2 System Characteristics

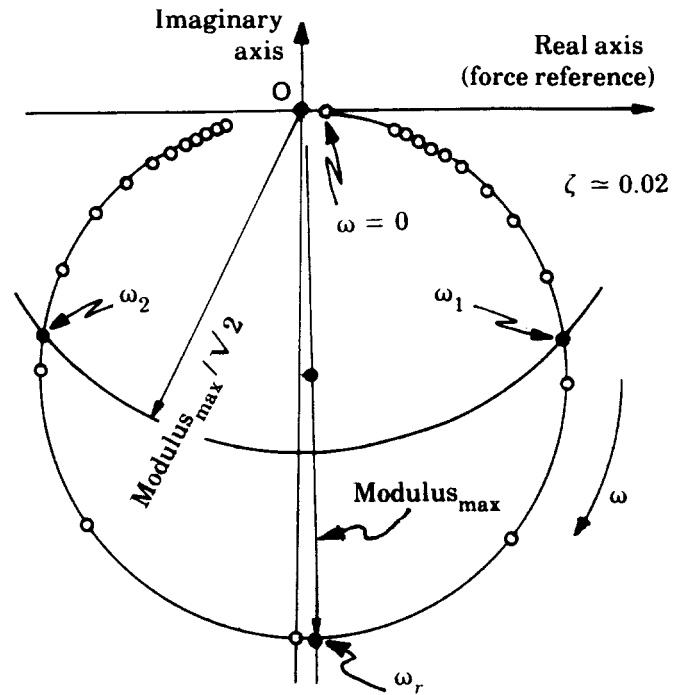
This section will show how estimates for the resonant frequency and damping ratio for systems that are lightly damped, $\zeta < 0.2$, may be obtained directly from measurements of the frequency response function plotted in the form of a Nyquist diagram. For systems with moderate damping, $0.2 < \zeta < 0.7$, or large damping, $\zeta > 0.7$, this method is shown to be unsuitable and estimation techniques using the Bode diagram are recommended.

Consider a set of measurements of both the response of a system, $Z(\omega)$, and the amplitudes of input, $U(\omega)$, used to produce the responses, from which equivalent measurements of the frequency response function have been obtained. This latter function may be plotted as separate modulus and phase diagrams versus frequency, either linear or logarithmic (Bode diagram), or as a Nyquist diagram combining the modulus and phase in a single diagram.

Sketch 4.2 shows typical plots for a lightly damped system and Sketch 4.3 shows plots for two systems with moderate and large damping having damping ratios approximately equal to 0.5 and 1.2. Each sketch shows separate modulus and phase plots and an equivalent Nyquist diagram.

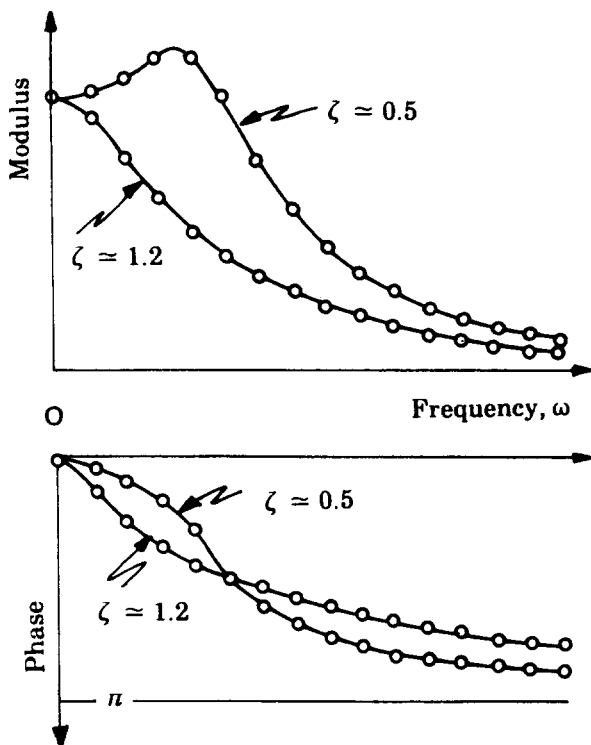


(a) Modulus and phase

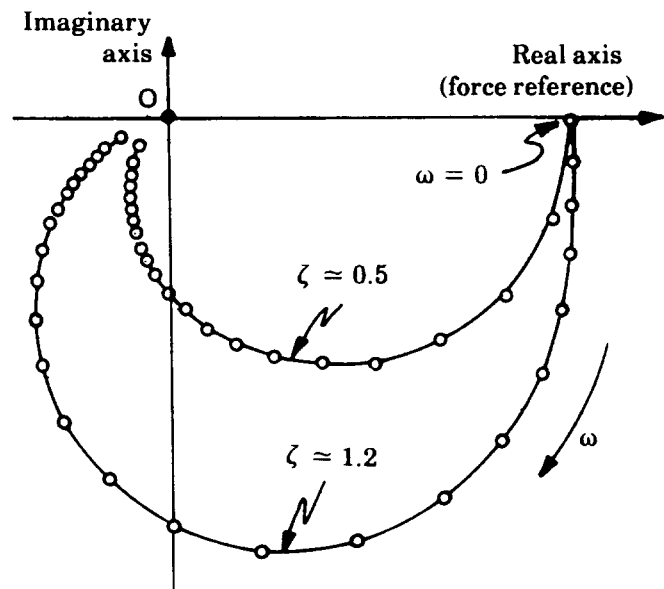


(b) Nyquist diagram

Sketch 4.2 Frequency response function for a lightly damped system



(a) Modulus and phase



(b) Nyquist diagram

Sketch 4.3 Frequency response function for moderate and heavily damped systems

4.2.1 Separate modulus and phase diagrams

In Sketch 4.2(a) the resonant frequency, ω_r , is the frequency at which the maximum response occurs and in terms of the natural undamped frequency of the system is given by

$$\omega_r = \omega_n(1 - 2\zeta^2)^{1/2}. \quad (4.6)$$

When the damping is small, $\zeta \approx 0$, this resonant frequency is assumed to be a satisfactory initial estimate of the natural undamped frequency of the system and may be used as such in subsequent calculations.

The damping ratio, ζ , is determined by measuring the frequency bandwidth on the modulus versus frequency plot, between points on the response curve for which the response is $1/\sqrt{2}$ times the maximum response. The damping ratio is then given by

$$\zeta = (\omega_2 - \omega_1)/(2\omega_n) \approx (\omega_2 - \omega_1)/(2\omega_r), \quad (4.7)$$

where $(\omega_2 - \omega_1)$ is known as the “half-power bandwidth” or “3 dB bandwidth” since the amplitude ratio of $(1/\sqrt{2})$ corresponds to a reduction of amplitude measured in decibels of $20 \log_{10} \{1/\sqrt{2}\} = -3.01$ dB. This method is known as the peak amplitude method. However, for lightly-damped systems it is clearly difficult to measure the peak accurately unless a measurement is taken at precisely the resonant frequency. It is noted that measured phase information remains unused in this method.

The systems of Sketch 4.3(a) show the modulus and phase diagrams for cases of $\zeta = 0.5$ and $\zeta = 1.2$. The method used for the low damping case cannot be used for these cases since the resonant frequency is now not a satisfactory estimate of the natural frequency of the system. Estimates for the natural undamped frequency and damping ratio may be obtained with the use of the Bode diagram which has been described in previously issued Data Items, Nos. 74021 and 80002, References 5 and 6, respectively. The response of a second-order system to a sinusoidal input is also treated in Data Item No. 69005 (see Reference 4).

4.2.2 Nyquist diagram

More accurate information may be obtained from a Nyquist diagram or vector plot, an example of which is shown in Sketch 4.2(b). For a lightly damped system in which the damping is viscous a displacement plot is approximately circular in shape and the resonant frequency, ω_r , is that point on the curve where the rate of change of frequency with arc length is a maximum. This maximum is usually obtained numerically using a finite difference method since in practice the frequency is known only at the points of measurement. The circle-fit method usually produces an estimate for ω_n to an accuracy of about 10% of the frequency increments between points.

This is in direct contrast to the systems shown in Sketch 4.3(b) from which no good estimate of ω_r or ω_n can be obtained. It is however theoretically possible to obtain estimates of the zeros of Equation (4.4) from the Nyquist diagram [see Appendix D of Data Item No. 74021 (Reference 5)] which works to an accuracy that improves with the proximity of the locus to the point $(-1, 0)$. It is not surprising to discover that good estimates can usually only be obtained for lightly-damped systems when the circle-fit method can be used to produce equally good estimates.

Returning to the Nyquist diagram for the lightly-damped system, the damping ratio may be obtained from Sketch 4.2(b) in the following way. As for the peak amplitude method the -3 dB points may be found on the vector plot as the intersections of a circle whose radius is $(\text{Modulus}_{\max}/\sqrt{2})$ with centre at the origin of the vector plot. These intersections are the frequencies, ω_1 and ω_2 , as indicated in the sketch, and may be used in Equation (4.7) to estimate the damping ratio.

Two further more generalised methods are available with the vector plot to obtain the damping ratio.

The first is obtained with the use of the following equation

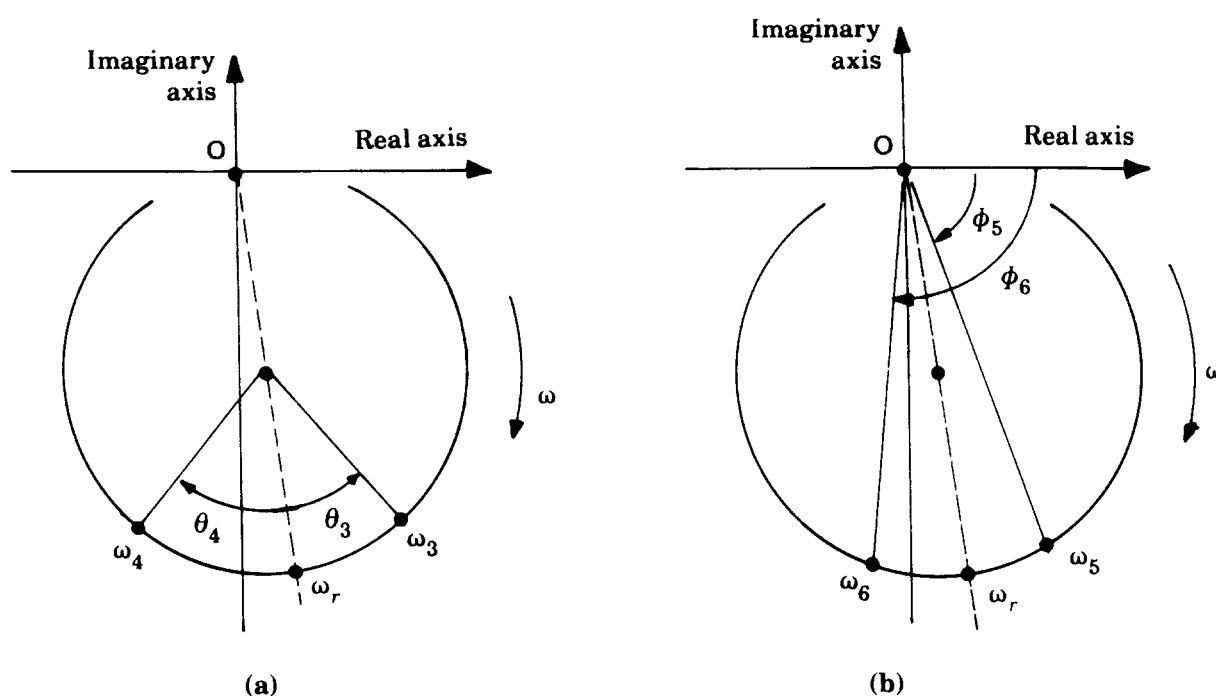
$$\zeta \approx (\omega_4 - \omega_3) / [\omega_n (\tan(\theta_3/2) + \tan(\theta_4/2))], \quad (4.8)$$

where ω_3 , ω_4 , θ_3 and θ_4 are defined in Sketch 4.4(a) and ω_r is used for ω_n . In theory the points either side of the resonant frequency, defining ω_3 and ω_4 , are chosen arbitrarily but in practice they are fixed by the measurements. It should be noted that the angles θ_3 and θ_4 are not the phase angles at the points denoted by ω_3 and ω_4 , respectively. When $\theta_3 = \theta_4 = 90^\circ$, Equation (4.8) reduces to that given by Equation (4.7).

The second method, termed by Mead (see Reference 11) the phase change method, uses the phase difference between two plotted points *close to the maximum response* in the following equation

$$\zeta \approx (\omega_6 - \omega_5) / [\omega_n |\phi_6 - \phi_5|], \quad (4.9)$$

where ϕ_5 and ϕ_6 are given in radians and all the quantities in Equation (4.9) are defined in Sketch 4.4(b). Once again ω_r is used in Equation (4.9) as a satisfactory estimate for ω_n .



Sketch 4.4 Damping ratio estimation

4.3 System Objective and Parameter Estimation

Section 4.2 has shown how estimates for the natural undamped frequency, ω_n (or a good approximation to it, ω_r), and the damping ratio, ζ , of a lightly-damped system may be found from the measurements of the frequency response function. This has been accomplished with the use of the circle-fit to the measurements plotted as a Nyquist, or vector, diagram. To complete the evaluation of the unknowns of the system receptance it remains to find the numerator of Equation (4.4) which contains the unknown parameter m . This may be achieved by fitting Equation (4.4) to the complete set of measurements, thus minimising the difference between the model and measurements using a least-squares procedure. During this process,

which will be fully described for a two degree-of-freedom system in Section 5, the natural undamped frequency and the damping ratio are kept fixed at the previously found estimates. This least-squares process may be implemented for any system whatever the level of damping inherent in the system but does rely on known estimates for ω_n and ζ which would have to be obtained from other sources unless the damping ratio was less than 0.2.

A further significant improvement in parameter estimates can now be made by treating all of the parameters, ω_r , ζ and m , as unknowns in a complete estimation process, see Section 5, using the obtained estimates as starting values. Since this latter estimation process uses all the data in the vector diagram a number of the disadvantages mentioned in Section 3 are partially overcome.

With these known parameter estimates and known input values to the system, $U(\omega)$, the amplitude of response, $Z(\omega)$, may be found. With this system response representation, prediction of the response from other input excitations, such as shocks or impulses, is possible. Such an approximation is, consequently, often sufficient since it is a model of the system that is known to behave as the original system although it is not necessarily related to it in a physical way.

It is possible at this stage of the analysis to return to the original plant description, Equation (4.1), and evaluate the coefficients b and k with the use of Equations (4.3a) and (4.3b). The accuracy of these system elements will depend largely upon the extent to which the model of the system represents the true system.

5. TWO DEGREE-OF-FREEDOM SYSTEMS

The two degree-of-freedom system and characteristics considered in this section can be readily extended to cover systems involving any number of degrees-of-freedom.

5.1 System Model

Consider the general two degree-of-freedom coupled system whose equations of motion are

$$m_{11}\ddot{z}_1(t) + m_{12}\ddot{z}_2(t) + b_{11}\dot{z}_1(t) + b_{12}\dot{z}_2(t) + k_{11}z_1(t) + k_{12}z_2(t) = 0 \quad (5.1a)$$

$$\text{and} \quad m_{21}\ddot{z}_1(t) + m_{22}\ddot{z}_2(t) + b_{21}\dot{z}_1(t) + b_{22}\dot{z}_2(t) + k_{21}z_1(t) + k_{22}z_2(t) = u(t). \quad (5.1b)$$

Here m_{ij} , b_{ij} and k_{ij} , $i = 1, 2, j = 1, 2$ are the constant coefficients, $z_1(t)$ and $z_2(t)$ are the state variables and $u(t)$ is the input forcing function. The unknown parameters to be found consist of some or all of the twelve constant coefficients.

Equations (5.1a) and (5.1b) may be rewritten in matrix form as

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{z}_1(t) \\ \ddot{z}_2(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ u(t) \end{bmatrix} \quad (5.2a)$$

$$\text{or} \quad \underline{M} \ddot{\underline{z}}(t) + \underline{B} \dot{\underline{z}}(t) + \underline{K} \underline{z}(t) = \underline{u}(t), \quad (5.2b)$$

where the off-diagonal terms of the matrices $\underline{M}$, $\underline{B}$ and $\underline{K}$ are the elements that couple the two motions, $z_1(t)$ and $z_2(t)$.

Equation (5.2a) can also be written in the equally general (more usually quoted) form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{z}_3(t) \\ \ddot{z}_4(t) \end{bmatrix} + \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} \dot{z}_3(t) \\ \dot{z}_4(t) \end{bmatrix} + \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \begin{bmatrix} z_3(t) \\ z_4(t) \end{bmatrix} = \begin{bmatrix} 0 \\ u_1(t) \end{bmatrix} \quad (5.2c)$$

by using transformations of the form

$$z_3(t) = z_1(t) + (m_{12}/m_{11})z_2(t) \quad \text{and} \quad z_4(t) = z_2(t) + (m_{21}/m_{22})z_1(t)$$

where $u_1(t) = u(t)/m_{22}$

and ϕ_{ij} and ψ_{ij} , $i = 1, 2, j = 1, 2$, are functions of b_{ij} and m_{ij} and of k_{ij} and m_{ij} , respectively.

Assuming a harmonic excitation of the form $u(t) = U(\omega)e^{i\omega t}$, where $U(\omega)$ is chosen as real (*i.e.* the phase angle datum is chosen to coincide with the phase angle of the input) and steady-state solutions $z_1(t) = Z_1(\omega)e^{i\omega t}$ and $z_2(t) = Z_2(\omega)e^{i\omega t}$, then Equation (5.2b) becomes

$$[\underline{M}(\omega)^2 + \underline{B}(\omega) + \underline{K}]\underline{Z}(\omega) = \underline{U}(\omega), \quad (5.3)$$

where $\underline{Z}(\omega) = [Z_1(\omega) \ Z_2(\omega)]^T$,

and $\underline{U}(\omega) = [0 \ U(\omega)]^T$.

It may be noted that only a single forcing function is used to excite the system even though mathematically such a restriction is unnecessary. This restriction is imposed for the purely practical reason that it is almost impossible (and unnecessary) to drive a system with two independent harmonic forces exactly matched in phase angle at all times.

Since the matrices $\underline{M}$, $\underline{B}$ and $\underline{K}$ are square, Equation (5.3) may be re-expressed as

$$\underline{Z}(\omega) = \frac{\text{Adj}\{\underline{M}(\omega)^2 + \underline{B}(\omega) + \underline{K}\}\underline{U}(\omega)}{\text{Det}\{\underline{M}(\omega)^2 + \underline{B}(\omega) + \underline{K}\}}, \quad (5.4)$$

where the terms Adj and Det refer to Adjoint[†] and Determinant, respectively. Equation (5.4) may be expanded to show more clearly the form of the two equations which are

$$Z_1(\omega) = \frac{-[m_{12}(i\omega)^2 + b_{12}i\omega + k_{12}]U(\omega)}{\text{DENOM}} \quad (5.5a)$$

$$\text{and} \quad Z_2(\omega) = \frac{[m_{11}(i\omega)^2 + b_{11}i\omega + k_{11}]U(\omega)}{\text{DENOM}}, \quad (5.5b)$$

where $\text{DENOM} = [m_{11}(i\omega)^2 + b_{11}i\omega + k_{11}][m_{22}(i\omega)^2 + b_{22}i\omega + k_{22}] - [m_{12}(i\omega)^2 + b_{12}i\omega + k_{12}][m_{21}(i\omega)^2 + b_{21}i\omega + k_{21}]$.

[†] If $\underline{\Lambda}$ is a square matrix and $\text{Det}(\underline{\Lambda})_{ij}$ is a cofactor of an element, Λ_{ij} , then the transpose of the matrix of cofactors is the adjoint of $\underline{\Lambda}$.