

ENERGY HEIGHT METHOD FOR FLIGHT PATH OPTIMISATION

1. NOTATION AND UNITS

		<i>SI</i>	<i>British</i>
D	airframe drag	N	lbf
$D_L = W$	airframe drag at lift = weight	N	lbf
D_M	intake momentum drag (per aircraft)	N	lbf
F_G	engine gross thrust (per aircraft)	N	lbf
F_N	engine net thrust (per aircraft)	N	lbf
f	fuel used	kg	slug
g	acceleration due to gravity	m/s ²	ft/s ²
g_0	standard sea-level value* of g	m/s ²	ft/s ²
H	geopotential height*	m	ft
H_e	energy height ($= H + V^2/2g_0$)	m	ft
H_p	pressure height*	m	ft
L	aerodynamic lift	N	lbf
M	Mach number		
m	mass of aircraft	kg	slug
Q	fuel-flow (mass-flow basis)	kg/s	slug/s
s	distance travelled in horizontal plane	m	ft
T	temperature	K	K
t	time	s	s
V	true airspeed (air-path velocity or resultant velocity with respect to air)	m/s	ft/s
W	aircraft weight ($= mg$)	N	lbf
x_g	Earth-axis co-ordinate in horizontal plane	m	ft

For footnote refer to end of Notation Section

Z	geometric height* ($= -z_g$)	m	ft
z_g	Earth-axis co-ordinate in direction of local vertical, positive downwards	m	ft
α	angle of attack, <i>i.e.</i> angle between projection of true airspeed vector on to aircraft plane of symmetry and longitudinal datum or body axis	rad	rad
γ	angle of climb (flight-path inclination angle)	rad	rad
λ_f	constant (Lagrangian) multiplier in Equation (4.7)	kg/m	slug/ft
λ_t	constant (Lagrangian) multiplier in Equation (4.6)	s/m	s/ft
τ	angle between projection of engine gross thrust vector on to aircraft plane of symmetry and longitudinal datum or body axis; positive when F_G contributes to lift	rad	rad

Subscripts

max	denotes maximum value
std	denotes conditions in International Standard Atmosphere, ISA
1	denotes initial point of flight path to be optimised
2	denotes final point of flight path to be optimised

* See Data Item No. 77022 (Reference 4) for definitions.

1.1 Exact Conversion Factors Used or Implied (Bold Type Denotes Exact Values)

1 ft = 0.3048 m, **1 n.mile = 1852 m**,
1 knot (kn) = 1 n.mile per hour = 0.5144 m/s = 1.688 ft/s

2. INTRODUCTION

Data Item No. 89015 (Reference 6) gives a broad introduction to Flight-Path Optimisation Methods. The present Data Item deals with the Energy Height Method, in which the equations of motion are modified by various simplifying approximations and by the combination of potential energy ($mg_0 = H$) and kinetic energy ($mV^2/2$) in the form of an equivalent potential energy (mg_0H_e), where H_e is "Energy Height". The optimisation problem is thereby reduced to the maximisation of dH_e/dt at each value of H_e .

It is possible to consider the total energy of an aeroplane as being controlled by the throttles, and the distribution between kinetic and potential energy as controlled by the elevators. The Energy Height Method is concerned only with the total energy state of the vehicle and not with details of converting from one energy form to the other.

The energy height model began (in the 1940s) as an ad hoc approximation and subsequently found a theoretical basis in the singular-perturbation theory of ordinary differential equations*. It has been found that singular-perturbation methods are useful for extending Energy Height modelling approaches to more general problem formulations. It is intended that these methods will be dealt with in later Data Items. The present Data Item is concerned only with the implementation of the Energy Height Method and with its limitations.

3. ENERGY HEIGHT MODEL

In the case of zero wind† and for problems essentially concerned with motion in the vertical plane, Equations (4.1) to (4.8) of Data Item No. 89015 reduce to the following

$$\frac{dV}{dt} = \frac{F_G \cos(\alpha + \tau) - D_M - D - mg \sin \gamma}{m}, \quad \dagger(3.1)$$

$$\frac{d\gamma}{dt} = \frac{L + F_G \sin(\alpha + \tau) - mg \cos \gamma}{mV}, \quad \dagger(3.2)$$

$$\frac{dx_g}{dt} = V \cos \gamma, \quad \dagger(3.3)$$

$$-\frac{dz_g}{dt} = V \sin \gamma \quad \dagger(3.4)$$

$$\text{or} \quad -\frac{dz_g}{dt} = \frac{dZ}{dt} = \frac{dH}{dt} \frac{g_0}{g} = \frac{dH_p}{dt} \frac{g_0}{g} \frac{T}{T_{std}}, \quad (3.5)$$

$$\text{and} \quad \frac{dm}{dt} = -Q. \quad (3.6)$$

* A singular-perturbation problem (Reference 5) is one that depends on a parameter in such a way that solutions behave non-uniformly as the parameter tends towards some limiting value of interest. The nature of the non-uniformity varies from problem to problem. In practice, a uniformly valid, easily interpretable approximation to the non-uniformly behaving solution is sought.

† For still air conditions the distinctions between flight-path and air-path angles of climb and between flight-path and air-path velocities disappear and it is acceptable to represent these quantities by γ and V (see Reference 6).

Energy height is defined as

$$H_e = H + \frac{V^2}{2g_0} \quad (3.7)$$

Differentiating with respect to time gives

$$\frac{dH_e}{dt} = \frac{dH}{dt} + \frac{V}{g_0} \frac{dV}{dt} \quad (3.8)$$

and substitution from Equations (3.1), (3.4) and (3.5) gives

$$\frac{dH_e}{dt} = V \left[\frac{F_G \cos(\alpha + \tau) - D_M - D}{mg_0} \right] \quad (3.9)$$

Energy height has been used as a variable in many recently published investigations of flight-path optimisation, applying modern developments of the calculus of variations to the equations of motion with various degrees of simplification. In the **Energy Height Method** this simplification is taken to the stage at which it is possible to avoid the complexity of solution by the calculus of variations altogether. To this end, it is assumed that Equation (3.9) may be replaced by the following equation

$$\frac{dH_e}{dt} = \frac{V(F_N - D_{L=W})}{mg_0} \quad (3.10)$$

where,

$D_{L=W}$ represents the value of the aerodynamic drag when the lift is equal to the weight, and

F_N is the net thrust at the appropriate maximum thrust rating and is assumed not to vary with angle of attack.

For a given atmosphere,* the values H_e and V determine the values of H and M . It follows, therefore, from Equation (3.10) that for a given aircraft mass, dH_e/dt is a function of H_e and V only. This simplifies the problem to such an extent that straightforward routines are available for obtaining a solution.

The approximations involved and the consequent limitations of the Energy Height Method are discussed in Section 5 after consideration, in Section 4, of how the model is applied in the solution of various flight path optimisation problems.

* For ISA conditions the variation of T with H is identical with that of T with H_p (by definition). For "off-standard atmospheres", i.e. those obtained by adding a constant increment to the temperature in the Standard Atmosphere, at constant pressure height, the relationship between H and H_p is given in Data Item No. 77022. For other atmospheres (with T defined as a function of H or H_p) the relationships between H and H_p are given in Data Item No. 78012.

4. APPLICATIONS

4.1 Minimum Time (or Fuel) Climb

The main problem to which the Energy Height Method is applied is the determination of the flight path (*i.e.* the “trajectory” in the velocity-height plane) that minimises time or fuel used in transferring from one (velocity, height) point to another. Additional constraints may be imposed, some in the form of structural or aerodynamic limits (for example, maximum Mach number, equivalent air speed, or lift coefficient), others in the form of operational limits (for example, horizontal distance covered during the climb).

In the simplest case, *i.e.* the minimum time climb problem, it is required to find the flight path that will achieve a minimum value of the integral

$$t = \int_{H_{e_2}}^{H_{e_1}} \frac{dt}{dH_e} dH_e \quad (4.1)$$

where dt/dH_e is a function of H_e and V only (as noted in Section 3).

Mathematical theory (see, for instance, Reference 2) shown that an integral of this form has a stationary value if, and only if, along the entire path from H_{e_1} to H_{e_2} ,

$$\frac{\partial}{\partial V} \left(\frac{dt}{dH_e} \right) - \frac{d}{dH_e} \left[\frac{\partial}{\partial V'} \left(\frac{dt}{dH_e} \right) \right] = 0. \quad (4.2)$$

In Equation (4.2), V' represents the value of dV/dH_e along the flight path and, during the partial differentiation in the second term, is treated as an independent variable. Since for the Energy Height Method it has been seen that dV/dH_e is a function of H_e and V only, the second term in Equation (4.2) is identically zero. Hence, the condition for a stationary value of the integral in Equation (4.1) becomes that

$$\frac{\partial}{\partial V} \left(\frac{dt}{dH_e} \right) = 0 \quad (4.3)$$

along the entire path from H_{e_1} to H_{e_2} . This condition is equivalent to the definition of an envelope of curves of dt/dH_e against H_e , allowing a simple graphical approach to generation of the optimum path.

Similar equations may be obtained for the minimum fuel climb problem by replacing dt/dH_e by $Q dt/dH_e$ (or df/dH_e).

Examples of the determination of minimum time and minimum fuel climbs are given in detail in Appendices A and B respectively.

4.2 Minimum Time (or Fuel) Climb to Fixed Distance

The horizontal distance travelled during a climb may be expressed as follows,

$$s = \int_{H_{e_2}}^{H_{e_1}} V \cos \gamma \frac{dt}{dH_e} dH_e. \quad (4.4)$$

If this distance is constrained to a particular value (s_c , say) the optimum path has also to satisfy

$$s = s_c. \quad (4.5)$$

In such cases, mathematical theory shows that the following conditions must be met at all points along the optimum flight path

(for minimum time problems),

$$\frac{\partial}{\partial V} \left[(1 - \lambda_t V \cos \gamma) \frac{dt}{dH_e} \right] = 0, \quad (4.6)$$

or (for minimum fuel problems)

$$\frac{\partial}{\partial V} \left[(Q - \lambda_f V \cos \gamma) \frac{dt}{dH_e} \right] = 0. \quad (4.7)$$

It is then necessary to determine (for example, by making calculations for a range of values, followed by interpolation) values for the constant λ_t or λ_f that allow Equation (4.5) to be satisfied.

Examples of the determination of minimum time and minimum fuel climbs with distance constraints are given in detail in Appendices C and D respectively and include values of λ_t and λ_f which are representative for the classes of aircraft considered.

5. LIMITATIONS OF THE ENERGY HEIGHT METHOD

The limitations of the Energy Height Method for flight path optimisation result primarily from the simplifying approximations made to the equations of motion. The assumptions made in moving from Equation (3.9) to (3.10) are as follows,

$$\cos(\alpha + \tau) = 1.0 \text{ (which results in use of } F_N \text{ in place of } [F_G \cos(\alpha + \tau) - D_M]),$$

$$D = D_L = W,$$

$$F_N \text{ does not vary with angle of attack.}$$

Simple estimates may be made of the magnitudes of the approximations involved in the first two of these assumptions. Fortunately they have counteracting effects, on the one hand increasing the component of the thrust force along the flight direction whilst on the other increasing the drag. For a typical airliner, where the maximum climb angle is likely to be less than 10° (5° is typical), and where aircraft attitudes and accelerations are limited by passenger acceptability, the net effect is an overestimate of (thrust minus drag)

of about 0.5 per cent. For military aircraft, with much greater thrust/weight ratios and consequently greater climb angle potential, and where aircraft attitudes and accelerations are limited only by aircraft capability, the effect can be considerably larger. For a typical aircraft, the net effect in steady flight is found to vary from about 0.5 per cent underestimate at low altitude, to 10 per cent overestimate at 40 000 ft and Mach 0.9 and 2.5 per cent overestimate at 40 000 ft and Mach 1.6. In manoeuvring flight the effect can be expected to be greater.

It would appear, therefore, that the errors in the Energy Height Method may be fairly small along the optimum path in those cases where only the approximations in Equation (3.10) are involved.

However, in relying on Equation (3.10) alone to produce the optimum flight path the Energy Height Method disregards Equation (3.2). Two aspects are of concern,

1. whether the equations of motion are violated along the various phases of the predicted flight path,
2. what happens in transferring from one flight path phase to the next.

5.1 Validity of Solution

The first aspect may be considered by analysing the predicted flight path in order to obtain the history of velocity, height, and flight path inclination (γ). It is then possible to use Equations (3.1) and (3.2) to determine the values of thrust and lift actually required to achieve the predicted flight path. This has been done for the Example in Appendix A and the results are shown in Figure A7. It is found that the predicted flight path is valid where it is reasonably smooth, but becomes invalid where there are rapid changes in velocity and height. However, it should be noted that the region where the equations are violated is not relevant to the optimum path derived by the Energy Height Method.

5.2 Transfer Between Flight Path Phases

The second aspect is concerned with the practical difficulties inherent in the predicted solution. In particular, the method does not distinguish between two different (V, H) combinations, both of which correspond to the same value of energy height, and it does not take account of relatively high (or low) drag conditions involved in the transfer from one flight phase to another.

In early papers on the Energy Height Method (see, for instance, Rutowski, Reference 1) it is assumed that conversion from one form of energy to the other could take place in zero time. To take the Example in Appendix A as an illustration, it is seen that the optimum flight path consists of two separate branches, one at subsonic speeds and the other supersonic. The assumption that transfer between these two branches can be at constant H_e and instantaneous is unrealistic and optimistic. Even in a vertical dive, transfer at constant H_e (without any consideration of the manoeuvres necessary to enter and recover from the dive), would take about 7.7 seconds, during which time some 23 kg of fuel would be burnt. There would, in fact, be difficulty in achieving a transfer at constant H_e , since this would require thrust to be held equal to drag throughout the manoeuvre (since, for constant H_e , $dH_e/dt = 0$ in Equation (3.9). In the Example of Appendix A the most rapid transition would be in a vertical dive at full thrust, taking about 5.9 seconds and an expenditure of some 40 kg of fuel while energy height would increase by about 640 m (2100 ft). This time interval could be considered negligible in comparison with total flight time. However, there would be no guarantee that the transfer was optimal, particularly because of the lack of consideration of the entry and recovery phases. Operational considerations, such as passenger comfort or weapon system requirements, may limit the aircraft attitude during the transfer, thereby further increasing time and fuel expended. For instance, transfer in a 10° dive at full thrust would take about 23 seconds and 150 kg of fuel, with an increase of energy height of about 1880 m (6168 ft).

In general, an optimum path consists of several phases. For instance, in the Example in Appendix A, these are

- sea level acceleration
- subsonic climb
- transfer in a dive from subsonic to supersonic climb
- supersonic climb to about 34 500 ft
- dive to about 33 500 ft
- climb to 36 089 ft
- level acceleration at 36 089 ft
- climb to 40 000 ft
- level acceleration to the required flight condition.

Progress from each one of these phases to the next would, in reality, involve normal acceleration manoeuvres (positive or negative) which are ignored by the Energy Height Method and are precluded by the assumptions underlying the method. The more phases that there are in an optimum flight path, the more important it becomes to give due consideration to these transfers between phases in determining the optimum flight path. More detailed studies have been made elsewhere of methods to introduce more realistic transfers between phases in the climb and also between the horizontal acceleration phase at low level and the subsonic climb (see, for instance, Reference 3). This is, however, outside the range of the subject of this Data Item.

6. REFERENCES

1. RUTOWSKI, E.S. Energy approach to the general aircraft performance problem. *J. Aeronautical Sciences, Sci*, March 1954.
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3. BREAKWELL, J.V. Optimum flight-path-angle transitions in minimum-time airplane climbs. *J. Aircraft*, Vol.14, No.8, August 1977.
4. ESDU Equations for calculation of International Standard Atmosphere and associated off-standard atmospheres. ESDU Data Item No. 77022, November 1977 (with Amendments A and B, February 1986).
5. SMITH, D.R. *Singular perturbation theory: an introduction with applications*. Cambridge University Press, 1985.
6. ESDU Introduction to Data Items on flight path optimisation. ESDU Data Item No. 89015, August 1989.

APPENDIX A MINIMUM TIME TO CLIMB TO A GIVEN HEIGHT AND SPEED

A1. MINIMUM TIME TO CLIMB TO A GIVEN HEIGHT AND SPEED*

Aircraft “A” is a fighter aircraft with a wing loading of 5884 N/m^2 (122.9 lbf/ft^2) and a thrust/weight ratio of 0.75 (sea level, static, maximum reheat). The thrust/weight and drag/weight characteristics in the Standard Atmosphere are shown in Figure A1. Use the Energy Height Method to determine,

- the optimum path for minimum time,
- the time taken by means of the optimum path to reach $M = 1.98$ at $H = 40\,000 \text{ ft}$ starting from 200 kn at sea level, with level accelerations to and from the optimum path,
- the loss in time incurred by replacing the optimum path by a more practicable path, for example one made up of constant Mach number and constant calibrated airspeed (CAS) phases.

It is shown in Section 4.1 that the solution of the equations within the Energy Height Method approximations corresponds to a definition of the optimum path as the envelope of curves of dt/dH_e against H_e . This leads to simple graphical methods for the determination of the optimum path – presented here as methods (a) and (b).

Method (a)

Calculate the value of dH_e/dt (and, hence, dt/dH_e), over the relevant speed/Mach number range for a number of selected heights using Equation (3.10), i.e.

$$\frac{dH_e}{dt} = \frac{V(F_N - D_L - W)}{mg_0}. \quad (\text{A1.1})$$

Because climb characteristics are dependent on the instantaneous mass of the aircraft (particularly as it approaches its ceiling), some allowance should be made for fuel expended in reaching each height. This may be calculated completely by an iterative procedure. Alternatively, or as an initial value in the iterations, allowance may be made approximately by means of some appropriate simple formula – in the present example the mass at any height is calculated from the expression,

$$\text{mass} = \text{mass at start} (1 - 0.704 \times 10^{-6} H), \quad \text{for } H \text{ in feet.}$$

Plot dt/dH_e at each height against H_e as shown in Figure A2. The time between any two energy heights is then obtained from the area beneath the curve corresponding to the path connecting the two points. The optimum path to any energy height is obtained from the envelope BC, which is obviously the curve with the minimum area between it and the H_e axis. The broken lines, AB and CD, represent the level accelerations at the beginning and end of the flight path.

The main disadvantage with this method is that it does not give a clear indication of the choice being made for the path and, therefore, ignores difficulties which might be inherent in the chosen path. Details of the optimum path are more evident from the following method.

* Heights and speeds are defined in terms of operational units of feet and knots and all graphical data are presented in this way. However, all calculations have been performed using a consistent (SI) set of units. See Section 1.1 for conversion factors.

Method (b)

The value of dH_e/dt over the relevant speed/Mach number range is calculated for a number of selected heights (again at appropriate values for the aircraft mass) and plotted against speed, as in Figure A3. This figure may be used to construct contours of dH_e/dt on a V, H chart, as in Figure A4*. If curves of constant energy height are then superimposed on this chart, it can be seen that Equation (4.3) is satisfied at points where the contour curves of dH_e/dt and H_e have common tangents. It is, therefore, possible to determine the optimum path as the curve joining these points of common tangency.

In the present example, the locus of the points of tangency has three distinct legs, corresponding to two maxima separated by a minimum. The “minimum” leg is of no relevance to an optimum path, which would normally be expected to follow the first (maximum) leg until an energy height is reached where dH_e/dt is higher on the second maximum leg.

The complex shape of this second maximum leg needs some explanation. To obtain the detailed shape shown, it is necessary to construct the contour charts to a greater scale in two distinct regions, one from 30 000 ft to 40 000 ft and the other from 15 000 ft to 25 000 ft. These enlarged sections of the charts are shown in Figure A5, from which the following observations are made.

- (a) Sharp corners in the optimum path between 33 000 ft and 35 000 ft are related to discontinuities of slope in the dH_e/dt contours (which occur as different engine control parameters come into play). Engine-related kinks also occur between 43 000 ft and 44 000 ft, whilst similar effects result from the slope discontinuities in the atmospheric parameters at the Tropopause.
- (b) The highly-curved section of the locus between 650 and 720 kn results from the closeness of the dH_e/dt contours to the H_e contours over an extended length. This would, of course, make the exact shape of the locus very sensitive to the assumed drag and thrust characteristics.

The predicted flight path has been analysed to check its validity with respect to the equations of motion. For this, the time gradients of velocity, height, and flight path inclination have been used in Equations (3.1) and (3.2) to determine the values of thrust and lift actually required to achieve the predicted flight path. The results are shown in Figure A6 (a detailed definition of the trajectory) and Figure A7 (the resulting values of thrust/maximum thrust available, drag/(drag at lift equals weight), and lift/weight). It should be noted that it proved impossible to obtain a solution to the equations of motion for values of energy height from 11 700 to 11 900 m. This is indicated in Figure A7 by the fact that the curves disappear off the top of the chart. Elsewhere, it will be seen that

lift is significantly less than the weight (because of the component of the gross thrust in the lift direction),
 drag is consequently less than drag at lift = weight,
 F_G only exceeds $F_{G_{max}}$ slightly

except in the highly-curved sections of the locus between 650 and 720 kn (energy height from 11 200 to 12 500 m). Hence, any real doubts about the validity of the predicted flight path are restricted to this region. It has already been noted that the central part of this region is completely invalid but it appears unlikely

* In fact, the original for the contour chart in Figure A4 (and A5) was produced directly (and simply) by computer, leading to a contour map containing far greater detail than it is practicable to achieve by manual means. However, for such detail to be meaningful, it is essential that the aircraft and engine characteristics are represented as realistically as possible. Linear interpolation within tabulated characteristics leads to unrepresentative discontinuities of slope in the contours, with the possibility of incorrect indication of the optimum path. Representation of the characteristics by some form of analytical expression is to be preferred. In either case, the representation must take account of slope discontinuities in the given data. In the example shown, discontinuities in slope of the dH_e/dt contour lines occur for various specific reasons, notably at the Tropopause, and where the engine characteristics are affected by the onset of spillage drag and high altitude cut-offs.