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Force analysis of planar linkages

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VIEWpac 9022A**

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THE PREPARATION OF THIS DATA ITEM

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The Mechanisms Committee, which first met in 1975, is continuing the work that previously was the responsibility of the former Gearing and Mechanisms Committee, which first met in November 1964. Like the earlier Committee it has the co-operation of many engineers and scientists in industry, research establishments and universities from whom much assistance and information is being received.

FORCE ANALYSIS OF PLANAR LINKAGES

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FORCE ANALYSIS OF PLANAR LINKAGES

1. NOTATION AND UNITS

Units are given in SI. However, any coherent system of units may be used.

		Units
a_{ix}, a_{iy}	components of linear acceleration of centre of mass of link i in x - and y -directions, respectively	m/s^2
$\underline{A}$	square matrix of known coefficients of $\underline{x}$ in equations of motion	—
$\underline{B}$	column matrix in equations of motion	—
E_i	resultant of external forces acting on link i	N
E_{ix}, E_{iy}	components of resultant of external forces in x - and y - directions, respectively	N
f	general arbitrary function	—
F_s	frame shaking force	N
F_{sx}, F_{sy}	components of frame shaking force in x - and y - directions, respectively	N
g	acceleration due to gravity	m/s^2
h	interval of time (time step)	s
I_{ik}	mass-moment of inertia of link i about axis perpendicular to plane of motion and passing through point k	kg m^2
L_{kl}	length of line joining points k and l (see subscripts)	m
$\underline{L}_{kl}$	vector defining length and direction of line joining points k and l	—
m_i	mass of link i	kg
M_s	frame shaking moment	N m
n	total number of moving links	—
P_{ij}	force exerted by link i on link j	N
P_{xij}, P_{yij}	components of joint force P_{ij} in x - and y - directions, respectively	N
r_{ik}	distance between centre of mass of link i and point k	m
R_{ix}, R_{iy}	coordinates of position of centre of mass of link i in x - and y - directions, respectively (with respect to origin of x, y axes)	m
S	substitution in differential equation	—
t	time	s

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T_{ik}	external torque acting on link i about point k	N m
v_{ix}, v_{iy}	components of linear velocity of centre of mass of link i in x - and y -directions, respectively	m/s
x, y	axes fixed in machine frame	—
$\underline{x}$	column matrix of unknowns in equations of motion	—
$Z1 \dots$	intermediate calculation parameters	—
α_i	angular acceleration of link i (anticlockwise positive)	rad/s ²
γ_{ik}	angle between datum line and line between point k and centre of mass of link i (anticlockwise positive)	degree
n_{ik}	angle between datum line and line between point k and point of action of resultant of external forces on link i	degree
θ_i	angle between x - direction and datum line of link i (anticlockwise positive)	degree
ω_i	angular velocity of link i (anticlockwise positive)	rad/s

Subscripts

f	refers to final (termination) condition
i, j	refer to link identification numbers
k, l	refer to identification letters of points on links (possibly joints)
max	refers to maximum
0	refers to initial condition

A dot above a variable denotes differentiation of that variable with respect to time.

2. INTRODUCTION

This Data Item presents the analytical methods used to estimate numerical values of reaction forces at joints that connect links of single degree-of-freedom, planar, multibar linkage mechanisms. Force analysis is an essential feature of the design of linkages used in industrial machinery because it can lead to improved reliability combined with higher operating speeds and loads.

It is advisable, during the design of a new mechanism, to obtain estimates of the joint forces[†] once the link lengths have been determined. Provisional values of cross-sectional geometry and a suitable material should be chosen for each moving link. Force analysis gives provisional values of joint forces which are then used to estimate the stresses in individual links and the deflections of important points on those links. The proposed cross-sectional geometry and material may then be altered and the joint forces re-determined in an iterative process until an acceptable design is obtained. Also estimates of joint forces are used to determine appropriate bearing types and sizes. Values of frame shaking force and frame shaking moment

[†] See Section 2.2 for definitions

are essential when designing the fixed machine frame and when deciding whether it is necessary to balance a linkage. Force values are used to compare the performance of alternative linkage configurations and to evaluate the effect of introducing different balancing techniques (see References 9, 10 and 11).

Traditionally, values of the necessary kinematic and dynamic quantities are obtained by a laborious graphical approach. Modern methods are based on the derivation of numerous equations describing the kinematic relationships and the equilibrium conditions followed by numerical solution of these equations using a computer. Three general strategies for estimating joint forces are described in this Item.

(a) Static force analysis

Inertia is ignored and the reactions at the joints due only to the external force are determined. Each joint force is expressed as a function of the position of the input link and the linkage is assumed to be stationary at each position. Static force analysis is suitable only for the analysis of very slow moving linkages.

(b) Kinetostatic force analysis

The reactions at the joints due to inertia and external forces are determined. Each joint force is expressed as a function of the position of the input link and the motion of the input link is assumed to be known. A commonly-used assumption is that the mechanism is driven via an input crank rotating at constant angular velocity.

(c) Dynamic force analysis

The reactions at the joints due to inertia and external forces are determined. Each joint force is expressed as a function of time and the driving force or torque is assumed to be known as a function of time, input link position or input link velocity.

The development of rigid-body mathematical models of linkages using these strategies is demonstrated in this Item. The range of application and the advantages and limitations of each method are stated. The wide variety of linkage configurations used in industrial machinery and the different loading conditions of each machine make the presentation of generic design data impractical. Consequently the aim of this Data Item is to provide a simple framework for the development of computer-based methods for analysis of linkages in particular applications. Section 11 contains the equations required to determine the joint forces by kinetostatic force analysis for commonly-used four-bar linkages. A computer program implementing kinetostatic force analysis of four-bar planar linkages is included in Section 12.

2.1 Use of This Data Item

Section 3 provides guidance on the choice of an appropriate strategy. Flowcharts (see Sketches 4.1, 5.1 and 6.1) in Sections 4, 5 and 6 illustrate the steps to be taken in static, kinetostatic and dynamic force analyses, respectively. Two important steps, the derivation of the kinematic relationships and the compilation of the equilibrium equations, are common to each strategy. The offset slider-crank linkage is used to illustrate the application of these steps. The derivation of the equations that describe the kinematics of this linkage in terms of the motion of the input link is given in Section 7 and the compilation of the equilibrium equations for the moving links is given in Section 8.

2.2 Definitions and Terminology

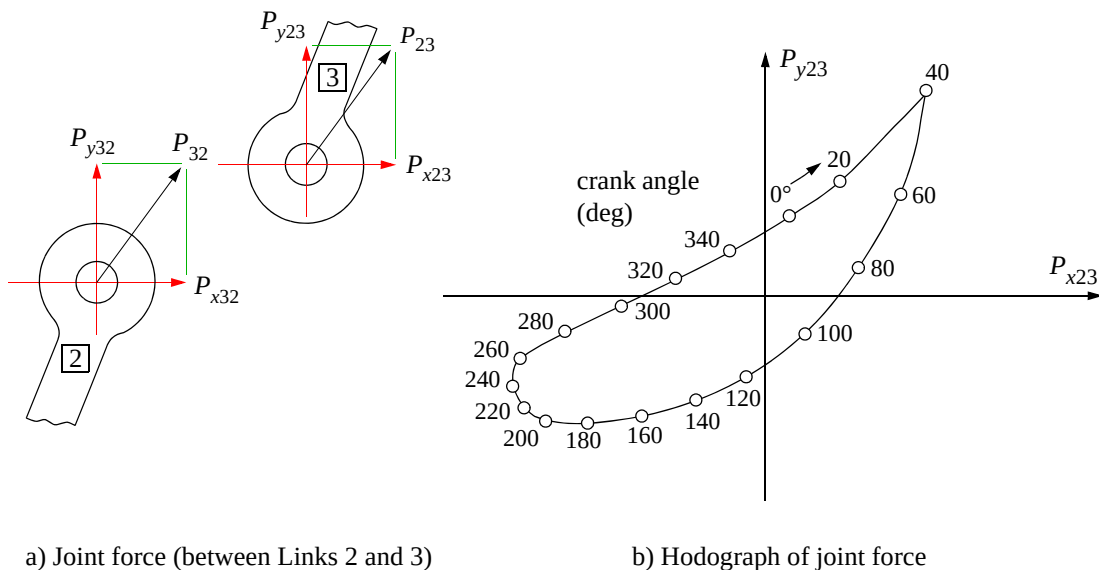
The following terms require definition and explanation since they are used throughout this Item. In addition, some guidelines for presentation of results are suggested.

2.2.1 Joint force

A joint force is the internal force of interaction between two moving links or between a moving link and the fixed frame at a bearing. These forces occur as a result of:

- (a) inertia effects associated with the accelerations of the moving links and possible impacts due to loss of contact in the bearings,
 - (b) gravitational forces associated with the weights of the moving links
- and (c) external forces and torques (see Section 2.2.2).

In this Item, for calculation purposes, the components of each joint force in the x- and y-axis directions are considered separately. The polar plot or hodograph (see Sketch 2.1(b)) is the recommended form of graphical presentation because it shows the variation in magnitude and direction of the force during the cycle and may be helpful in predicting loss of contact in bearings. For use in design, the magnitude of each joint force is usually characterised by a peak value and an r.m.s. value. The notation adopted for force components is shown in Sketch 2.1(a), where, for example, P_{x23} represents the component of force in the x-axis direction exerted by Link 2 on Link 3. Thus $P_{x23} = -P_{x32}$.



Sketch 2.1 Typical joint force data

2.2.2 External forces and torques

External forces or torques arise due to interactions between moving links and other machine elements. In practical applications external interactions may include:

- (a) input torque - for example the electromagnetic torque acting between the rotor and stator in an electric motor driving a sewing machine,
- (b) process force - for example the shearing force between tool and workpiece in a punching tool or the force exerted by a gas pressure on the piston in a reciprocating compressor,
- (c) friction - for example the friction in the main crankshaft bearings of an automobile engine,

- (d) forces from springs, dampers or other force elements fixed to moving links to modify the dynamic performance of the linkage

and (e) aerodynamic (windage) force.

Values of significant external forces and torques are required as inputs to the methods presented in this Item. Guidance on the choice of functions for generating these quantities is given in Section 8.2.

2.2.3 Frame shaking force and frame shaking moment

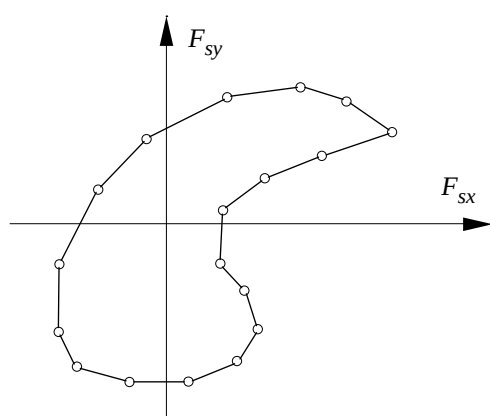
The vector sum of the joint forces that act at the interface between the moving links and the fixed machine frame at any instant can be represented by a resultant force at a specified point and a moment about that point. The chosen point is conventionally taken to be the rotational centre of the input shaft. The instantaneous values of this force and moment, which vary in both magnitude and direction throughout the machine cycle, are known as the frame shaking force and the frame shaking moment. These parameters are the most commonly used indicators of lack of balance of a linkage.

Values of frame shaking force in the x - and y -directions may be calculated by summing the components of force at the fixed frame joints (see Section 11). Alternatively the components of frame shaking force may be calculated directly by finding the sum of the products of mass and acceleration of the centre of mass of each moving link as given by

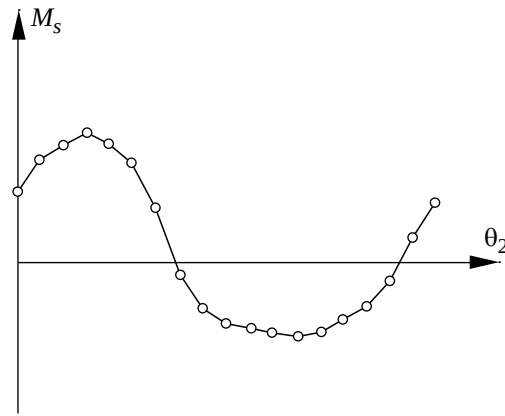
$$F_{sx} = - \sum_{i=2}^n m_i a_{ix}$$

and
$$F_{sy} = - \sum_{i=2}^n m_i a_{iy} .$$

By convention the fixed frame is known as Link 1. The frame shaking moment may be calculated by summing the moments of the joint forces at the fixed frame joints (see Section 11). The recommended form of presentation of the frame shaking force is the polar plot or hodograph (see Sketch 2.2(a)) but it may be useful to display the x - and y -components against time or input link position. Similarly, the frame shaking moment is usually displayed graphically against time or input link position (see Sketch 2.2(b)). Both quantities may be characterised by a peak value and an r.m.s. value of magnitude.



(a) Frame shaking force hodograph



(b) Frame shaking moment graph

Sketch 2.2 Typical shaking force and shaking moment data

3. SELECTION OF AN APPROPRIATE FORCE ANALYSIS STRATEGY

Three strategies for estimating joint forces are presented in this Item. Although much of the intermediate analysis required to implement each strategy is common, it is wise to choose at the outset the strategy that is most appropriate for a particular application. Each method of force analysis is founded on a set of assumptions and so the accuracy of the results obtained by the use of each method will reflect the degree to which the behaviour of the linkage matches the assumed behaviour. The methods require different input information and involve mathematical techniques of varying complexity.

The following general notes are applicable to all three strategies.

- (a) Estimates of the magnitude, direction and point of application of all significant external forces and moments must be known in terms of the position or velocity of the input link.
- (b) It is assumed that all links are stiff enough to be considered rigid and that clearances in bearings are small enough to be neglected.
- (c) Estimates of the mass and inertia properties of each link are required.

The choice of the most appropriate force analysis strategy is not usually obvious and may be dependent on the available computing facilities and expertise. The following guidelines may be helpful.

- (i) Static force analysis is simple to use and is adequate when a linkage is stationary or moving very slowly. It is not suitable for analysis of high-speed machinery.
- (ii) Kinetostatic force analysis gives accurate estimates of joint forces for linkages that are designed with a large crank inertia (*e.g.* flywheel) to ensure constant input angular velocity. In practice kinetostatic force analysis is adequate for application to many industrial machines.
- (iii) Dynamic force analysis gives (theoretically) the most correct and reliable results and is recommended for all linkages that operate at high-speed. In practice this accuracy depends on the availability of a realistic approximation to the actual input force or torque acting on the linkage. Computer software is essential for this method.

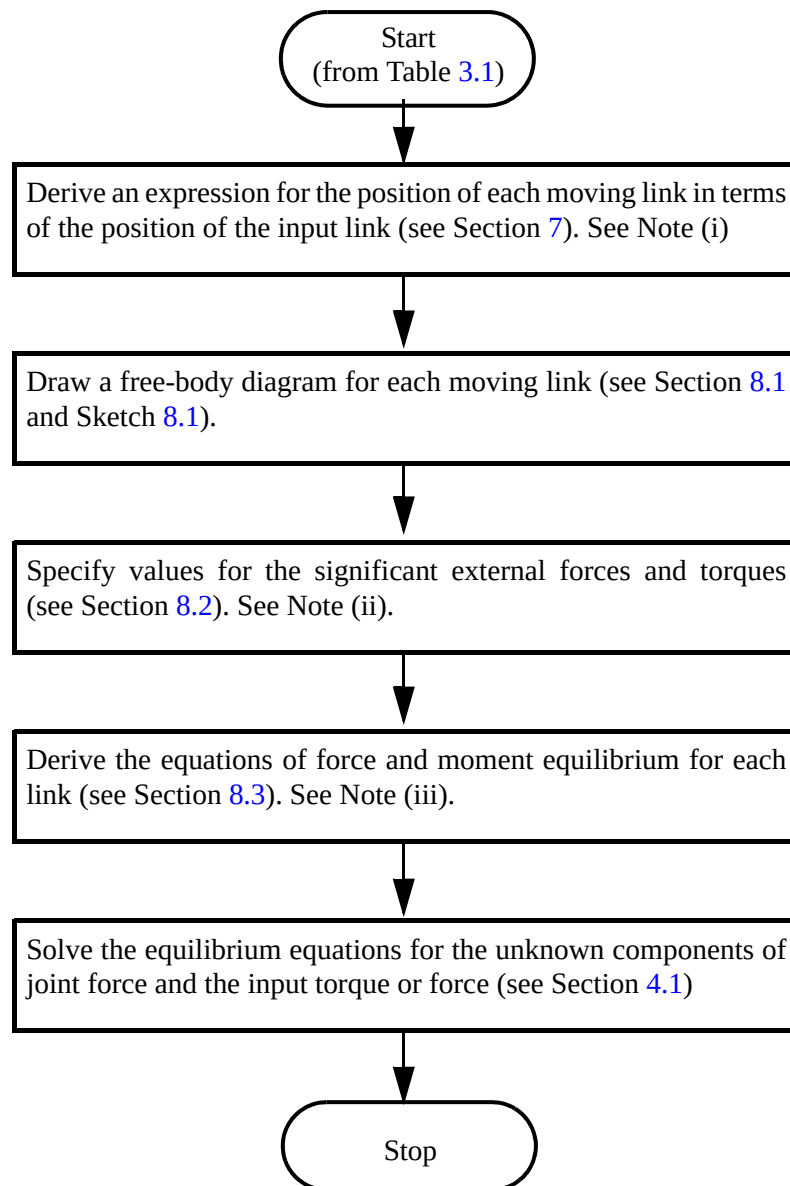
Table 3.1 presents a comparison of salient features of the recommended force analysis strategies.

TABLE 3.1 Comparison of force analysis strategies

Force analysis strategy	Assumptions	Prerequisite information	Output information	Advantages	Limitations	Typical valid applications
Static force analysis	Rigid links Zero clearances Zero inertia forces	Length, mass and location of centre of mass of each link External forces and moments Position of input link	Joint forces One external force or torque	Hand calculation of forces may be feasible	Not suitable for continuously moving linkages or high-speed linkages	Jacks, cranes, clamps and robot grippers
Kinetostatic force analysis	Rigid links Zero clearances Known input motion	Length, mass, mass-moment of inertia and location of centre of mass of each link External forces and moments Position, velocity and acceleration of input link	Joint forces One external force or torque	Equations of motion are quickly derived	Matrix algebra is usually required to solve equations Assumed input motion may not be realistic	Many industrial machines with "constant" input speed
Dynamic force analysis	Rigid links Zero clearances Known input(driving) force or torque	Length, mass, mass-moment of inertia and location of centre of mass of each link External forces and moments Input force and torque Position and velocity of input link at start of motion	Joint forces Variation of input, position, velocity and acceleration with time t	Potentially accurate results Time is the independent variable	Numerical integration of non-linear differential equation may be difficult and time consuming Matrix algebra is usually required to solve equations	Many industrial machines with variable input speed High-speed switches

4. STATIC FORCE ANALYSIS

In static force analysis the linkage is viewed as a stationary structure at a series of positions of the input link. The forces due to inertia of the moving links are ignored and so static force analysis is not suitable for determination of joint forces in high-speed linkages. The procedure is summarised in the flowchart of Sketch 4.1. Most of the steps are explained in Sections 7 and 8 and illustrated by reference to the offset slider-crank linkage shown in Sketch 7.1.



Sketch 4.1 Static force analysis procedure

Notes

- (i) It is necessary only to derive kinematic equations for the position of each moving link (see Equations (7.2c) and (7.2d) in Section 7). The expressions for velocity and acceleration are not required because inertia forces are assumed to be zero.

- (ii) The analysis procedure permits one of the components of one external force or the external torque to be calculated together with the joint forces. All other significant external forces and torques must be specified (see Section 8.2).
- (iii) In the equilibrium equations (see Equations (8.1) to (8.8) in Section 8) it is necessary only to include terms for the external forces, external torques and gravitational forces. The inertia terms (those including accelerations of the moving links) are assumed to be zero.

4.1 Solution of Equations

The number of equations must equal the number of unknowns. Each of the joint forces is unknown and one component of external force or one external torque acting on any link may also be unknown. The equilibrium equations are then regarded as a set of linear simultaneous equations which may be solved by any of the standard matrix methods such as Gaussian elimination or triangular decomposition given in References 19 and 20. Alternatively the equations may be of a form that can be manipulated algebraically by hand to give explicit expressions for the unknowns.

To avoid repetition the equilibrium equations (Equations (8.1) to (8.8)) are derived in Section 8. In this section it is assumed the unknowns are the joint forces and the torque, T_{2A} , required at the input link for equilibrium. It is also assumed that friction is negligible, gravitational force acts on each link and the only significant external force, E_{4x} , acts on Link 4. Equations (8.1) to (8.8) are simplified using these assumptions and the motion assumptions of static force analysis to give

$$P_{x12} - P_{x23} = 0, \quad (4.1)$$

$$P_{y12} - P_{y23} - m_2 g = 0, \quad (4.2)$$

$$P_{x23} L_{AB} \sin \theta_2 - P_{y23} L_{AB} \cos \theta_2 - m_2 g r_{2A} \cos(\theta_2 + \gamma_{2A}) + T_{2A} = 0, \quad (4.3)$$

$$P_{x23} - P_{x34} = 0, \quad (4.4)$$

$$P_{y23} - P_{y34} - m_3 g = 0, \quad (4.5)$$

$$P_{x23} r_{3B} \sin(\theta_3 + \gamma_{3B}) - P_{y23} r_{3B} \cos(\theta_3 + \gamma_{3B}) - P_{x34} r_{3C} \sin(\theta_3 + \gamma_{3C}) + P_{y34} r_{3C} \cos(\theta_3 + \gamma_{3C}) = 0, \quad (4.6)$$

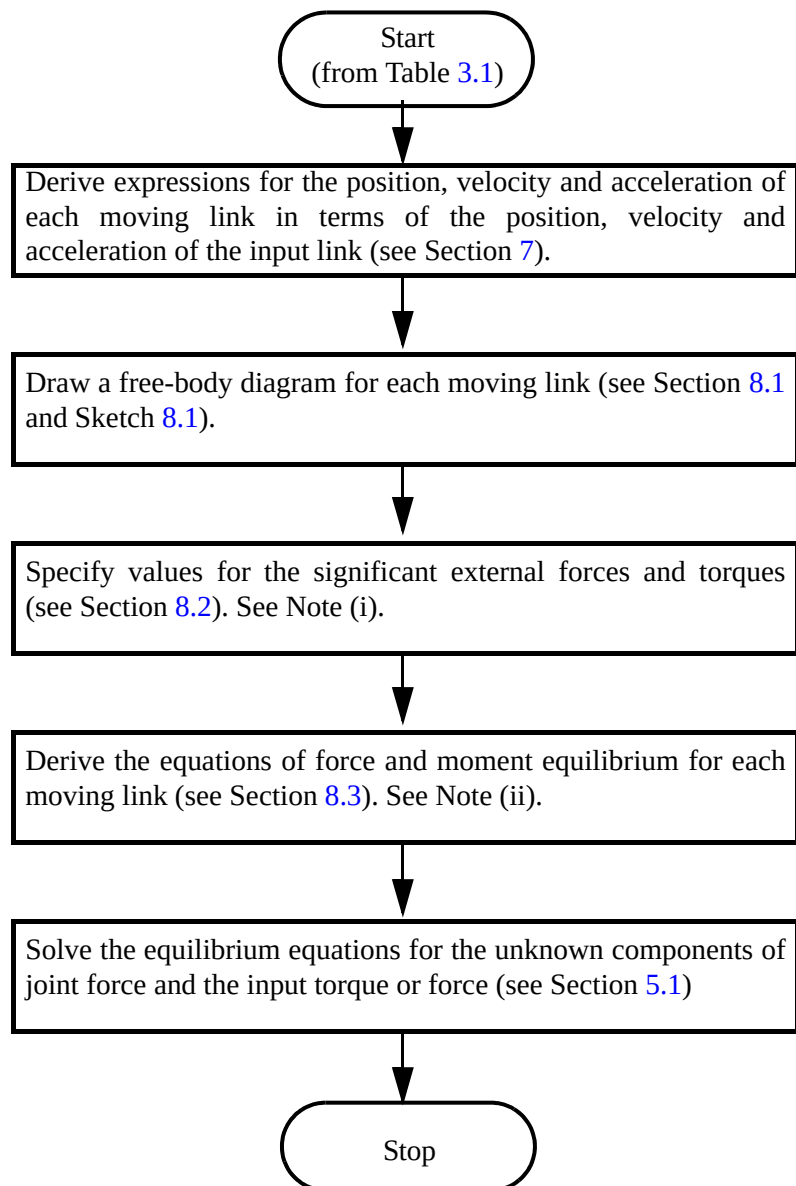
$$P_{x34} + E_{4x} = 0 \quad (4.7)$$

and $P_{y34} - P_{y41} - m_4 g = 0. \quad (4.8)$

These equations may be rearranged to give explicit expressions for the unknown joint forces and input torque.

5. KINETOSTATIC FORCE ANALYSIS

In kinetostatic force analysis the linkage is viewed at a series of positions of the input link. Inertia effects of the moving links are included together with external forces, external torques and gravitational forces. Kinetostatic force analysis is suitable for the determination of joint forces in high-speed linkages only when the motion (position, velocity and acceleration) of the input link is completely known. One commonly-used assumption is that the angular velocity of a rotating input link is constant. However, instantaneous values of joint forces can be found if the acceleration of the input link is assumed to be known at a given position of the input link. The analysis procedure is summarised in the flowchart of Sketch 5.1. Most of the steps are explained in Sections 7 and 8 and are illustrated by reference to the offset slider-crank linkage shown in Sketch 7.1.



Sketch 5.1 Kinetostatic force analysis procedure

Notes

- (i) The analysis procedure permits one of the components of one external force or the external torque to be calculated with the joint forces. All other significant external forces and torques must be specified (see Section 8.2).
- (ii) In the equilibrium equations (see Equations (8.1) to (8.8) in Section 8) it is necessary to include terms for the external forces, external torques, gravitational forces and inertia effects. The inertia terms (those including accelerations of the moving links) are calculated using the expressions derived in Section 7.

5.1 Solution of Equations

The number of equations must equal the number of unknowns. Each of the joint forces is unknown and one component of external force or one external torque acting on any link may also be unknown. The equilibrium equations are then regarded as a set of linear simultaneous equations which may be solved by any of the standard matrix methods such as Gaussian elimination or triangular decomposition given in References 19 and 20. Alternatively, for simple linkages, the equations are sometimes of a form that can be manipulated algebraically by hand to give explicit expressions for the unknowns.

To avoid repetition the equilibrium equations (Equations (8.1) to (8.8)) are derived in Section 8. In this section it is assumed that the unknowns are the joint force and the torque, T_{2A} , required at the input link for equilibrium. It is also assumed that friction is negligible, gravitational force acts on each link and the only significant external force, E_{4x} , acts on Link 4. Equations (8.1) to (8.8) are simplified using these assumptions and the motion assumptions of kinetostatic force analysis to give

$$P_{x12} - P_{x23} = m_2 a_{2x} , \quad (5.1)$$

$$P_{y12} - P_{y23} - m_2 g = m_2 a_{2y} , \quad (5.2)$$

$$P_{x23} L_{AB} \sin \theta_2 - P_{y23} L_{AB} \cos \theta_2 - m_2 g r_{2A} \cos(\theta_2 + \gamma_{2A}) + T_{2A} = I_{2A} a_2 , \quad (5.3)$$

$$P_{x23} - P_{x34} = m_3 a_{3x} , \quad (5.4)$$

$$P_{y23} - P_{y34} - m_3 g = m_3 a_{3y} , \quad (5.5)$$

$$P_{x23} r_{3B} \sin(\theta_3 + \gamma_{3B}) - P_{y23} r_{3B} \cos(\theta_3 + \gamma_{3B}) - P_{x34} r_{3C} \sin(\theta_3 + \gamma_{3C}) + P_{y34} r_{3C} \cos(\theta_3 + \gamma_{3C}) = I_{3G} a_3 , \quad (5.6)$$

$$P_{x34} + E_{4x} = m_4 a_{4x} \quad (5.7)$$

and $P_{y34} - P_{y41} - m_4 g = 0 . \quad (5.8)$

These equations may be rearranged to give explicit expressions for the unknown joint forces and input torque.

6. DYNAMIC FORCE ANALYSIS

In dynamic force analysis the linkage motion is evaluated numerically at a series of successive instants. Inertia effects of the moving links are included together with external forces, external torques and gravitational forces. The external driving force or torque is assumed to be a known function of position or velocity of the input link. Starting values of position and velocity of the input link are also assumed known. Dynamic force analysis is suitable for the determination of joint forces in high-speed linkages when the driving force or torque is known with confidence. However, dynamic force analysis is time consuming and requires the use of advanced numerical methods implemented in computer software. The procedure presented in this Item is based on a Newtonian formulation of the equilibrium equations. Methods using an energy approach based on a Lagrangian formulation of the equilibrium equations (see References 1, 3, 16, 17 and 18) offer computational advantages, especially for the analysis of complex linkages. The procedure followed in this Item is summarised in the flowchart of Sketch 6.1. Some steps are explained in Sections 7 and 8 and are illustrated by reference to the offset slider-crank linkage shown in Sketch 7.1.

The procedure used to find the joint forces also enables the determination of the time response of critical points on any moving link and the fluctuations in velocity of the input link during the operating cycle.

Notes:

- (i) Every external force and external torque must be known in order to determine the motion of the moving links. Values for these forces and torques are not always readily available (see Section 8.2).
- (ii) In the equilibrium equations (see Equations (8.1) to (8.8) in Section 8) it is necessary to include terms for the external forces, external torques, gravitational forces and inertia.

6.1 Solution of Equations

6.1.1 Prediction of values of position and velocity of input link

The motion of the input link, and consequently all other links, at any instant is dependent on the motion history of the complete linkage mechanism and is the result of the application of all known input forces and torques. Knowing the motion, i.e. acceleration, velocity and position, of the input link at a particular time, say t , it is possible with the use of numerical techniques to evaluate the motion at some future time, $t + h$. The step, h , is chosen initially by the user although it is altered by the algorithm to provide a stepping procedure through the problem that is numerically stable. The present problem, if it could be expressed explicitly, would be described by a single nonlinear second-order differential equation with variable coefficients. However, tied up with the unknown motion characteristics of the input, α_2 , ω_2 and θ_2 , are the unknown joint forces, P_{x12} , P_{y12} , P_{x23} , P_{y23} , P_{x34} , P_{y34} and P_{y41} , which need to be evaluated simultaneously with the motion characteristics. The equations involving the eight unknowns are Equations (6.7) to (6.14) which may be expressed in matrix form as

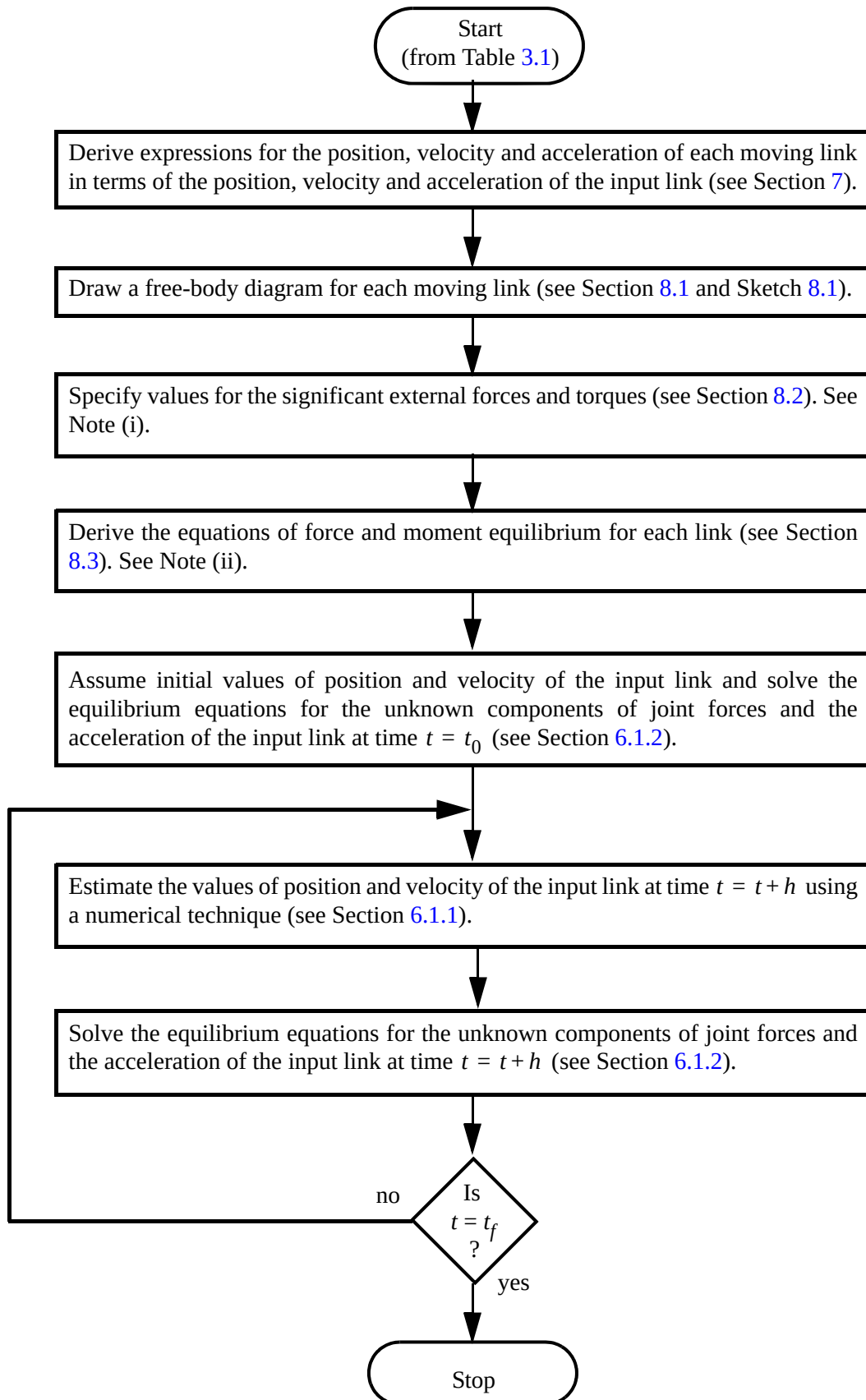
$$\underline{A} \underline{x} = \underline{B} \quad , \quad (6.1)$$

where the column matrix $\underline{x}$ contains the set of unknowns given by

$$\underline{x} = [\alpha_2, P_{x12}, P_{y12}, P_{x23}, P_{y23}, P_{x34}, P_{y34}, P_{y41}]^T \quad (6.2)$$

and matrices $\underline{A}$ and $\underline{B}$ contain elements that are functions of the position, θ_2 , and velocity, ω_2 , of the input link. The matrix Equation (6.1), may be solved (see Section 6.1.2) for the eight unknowns, at a particular time, once the variables θ_2 and ω_2 have been obtained from the numerical integration routine.

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Sketch 6.1 Dynamic force analysis procedure

For engineering applications single-step fourth-order numerical methods, specifically those referred to as Runge-Kutta (see References 2, 19 and 20), are used since, as implied, they extrapolate from one time point to the next, simply from known information at the earlier time. They are reasonably simple to apply, they are fairly robust and they adequately provide the accuracy required. An often used variation of the fourth-order Runge-Kutta method worthy of particular mention is that due to Gill (see Reference 2) which was developed for efficient use on a digital computer. This method can deal only with one or more first-order equations but, providing it is possible to isolate the highest derivative of the dependent variable, any high-order equation may be reduced to a set of first-order equations.

The present problem may be expressed by the following general second-order differential equation

$$\ddot{\theta}_2 = f(t, \theta_2, \dot{\theta}_2) \quad (6.3)$$

with known initial conditions $(\theta_2)_0$ and $(\dot{\theta}_2)_0$ at $t = 0$,

where $\ddot{\theta}_2 = \alpha_2$
and $\dot{\theta}_2 = \omega_2$.

This equation may be separated into two first-order equations by the use of the following substitution

$$S = \dot{\theta}_2, \quad (6.4)$$

giving the following equations and initial conditions

$$\dot{S} = f(t, \theta_2, \dot{\theta}_2), \quad \dot{\theta}_2 = (\dot{\theta}_2)_0 \text{ at } t = t_0 \quad (6.5)$$

$$\text{and } \dot{\theta}_2 = S, \quad \theta_2 = (\theta_2)_0 \text{ at } t = t_0. \quad (6.6)$$

The first step in the process is the evaluation of the derivative $\dot{S} (= \ddot{\theta}_2)$ at $t = t_0$. Since θ_2 and $\dot{\theta}_2$ are known at $t = t_0$, Equation (6.1) is then simply a set of eight simultaneous equations in the unknowns $\underline{x}$. The next procedural step is the evaluation of the dependent variables, θ_2 and $\dot{\theta}_2$, at the new time step, $t_0 + h$, which is evaluated from function values, f , at the previous time step, t_0 . (The fourth-order Runge-Kutta process actually requires function estimates at an intermediate time step, $t + h/2$, but this is carried out in exactly the same fashion as the full step procedure.) At this new time step, $t_0 + h$, since θ_2 and $\dot{\theta}_2$ are now known, the matrix Equation (6.1) may once again be called to evaluate $\dot{S} (= \ddot{\theta}_2)$ and the joint forces. This step process is repeated until the predetermined termination condition (final time), t_f is met.

6.1.2 Solution of equilibrium equations

The number of equations must equal the number of unknowns. Each of the joint forces (P_{x12} , P_{y12} , P_{x23} , P_{y23} , P_{x34} , P_{y34} and P_{y41}) is unknown and the acceleration, α_2 , of the input link is unknown. Those acceleration terms in Equations (8.1) to (8.8) must be expressed in terms of acceleration of the input link, α_2 using Equations (7.4a), (7.4b), (7.4c) and (7.4d), (7.7a), (7.7b), (7.7c), (7.7d), (7.7e) and (7.7f). This results in the required set of linear simultaneous equations which are solved at time, $t + h$, once values of position and velocity of the input link at that time have been estimated (see Section 6.1.1). For the offset slider-crank linkage it is assumed that an external force, E_{4x} , acts on Link 4, an external drive torque, T_{2A} , acts on Link 2 and that gravitational and friction forces are negligible. Equations (8.1) to (8.8) are rearranged